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Unit group of some finite semisimple group algebras

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Abstract
We provide the structure of the unit group of $\mathbb{F}_{p^k}S_n$, where $p > n$ is a prime and S_n denotes the symmetric group on n letters. We also provide the complete characterization of the unit group of the group algebra $\mathbb{F}_{p^k}A_6$ for $p \geq 7$, where A_6 is the alternating group on 6 letters.

Keywords: Unit group, Group algebra, Finite field, Wedderburn decomposition

Mathematics Subject Classification: Primary 16S34, Secondary 16U60, 20C05

Introduction

Let $q = p^k$ for some prime p and $k \in \mathbb{N}$. Let $\mathbb{F}_q$ denote the finite field of cardinality q. For any group G, let $\mathbb{F}_qG$ denote the group algebra of G over $\mathbb{F}_q$. We will follow [14] for basic notations. The group of units of $\mathbb{F}_qG$ has applications in different areas, including the construction of convolutional codes (see [5–7]) and solving problems in combinatorial number theory (see [3]) et cetera. This necessitates finding the explicit structure of the group of units of $\mathbb{F}_qG$.

In [15], the author has described units of $\mathbb{F}_qG$, where G is a p-group. The authors of [13] complete the study of unit groups of semisimple group algebras of all groups up to order 120, except that of the symmetric group S_5 and groups of order 96. In Remark 3.7 of this article, we complete the characterization for the group S_5. However, the complexity of the problem increases if the group has a larger size, as that requires solving an equation of type $\sum_{i=1}^{n} k_i n_i^2 = |G|$. The most commonly used tricks include identifying a normal subgroup H of G and considering the algebra $\mathbb{F}_q(G/H)$ inside $\mathbb{F}_qG$. This is not possible if the group G does not have any normal subgroup. This is why we will be using representations of the group A_6 to solve the problem in case of the same. See [16, 17] et cetera for more exposition about other groups.

The objective of this article is twofold. We start by investigation of $\mathbb{F}_qS_n$ where $p > n$. This is mainly a consequence of the representation theory of S_n over $\mathbb{C}$ and the connection between the Brauer characters of the group when $p > n$ and the ordinary characters over $\mathbb{C}$. We state the characterization in this case in Theorem 3.6. The group of units of the semisimple algebras $\mathbb{F}_qA_5$ and $\mathbb{F}_q\mathrm{SL}(3,2)$ have been characterized in [1, 12], respectively. In this article, we look at the next non-Abelian simple group A_6, the alternating group on six letters. We give a complete characterization of $\mathbb{F}_q A_6$ for the case $p \geq 7$ in Theorem 4.8.

The rest of the article is organized as follows: in "Preliminaries" section, we give some basic definitions and results. "Units of $\mathbb{F}_{p^k} S_n$ for $p \nmid n$3" section is about the general description of representations of S_n over an arbitrary field of characteristic $p > n$ and deducing the structure of $\mathcal{U}(\mathbb{F}_{p^k} S_n)$ for $p > n$. In section 4, we present the result about $\mathbb{F}_{p^k} A_6$ where $p \geq 7$.

Preliminaries

We start by fixing some notations. Already mentioned notations from section 1 are adopted. For a field extension $E/\mathbb{F}_q$, $\mathrm{Gal}(E/\mathbb{F}_q)$ will denote the Galois group of the extension. For $m \in \mathbb{N}$, the notation $\mathrm{M}(m, R)$ denotes the ring of $m \times m$ matrices over R and $\mathrm{GL}(m, R)$ will denote the set of all invertible matrices in $\mathrm{M}(m, R)$. For a ring R, the set of units of R will be denoted by $R^\times$. Let $Z(R)$ and $J(R)$ denote the center and the Jacobson radical, respectively. If G is a group and $x \in G$, then $[x]$ will denote the conjugacy class of x in G. For the group ring $\mathbb{F}_q G$, the group of units will be denoted as $\mathcal{U}(\mathbb{F}_q G)$. For the notations on projective spaces, we follow [4].

We say an element $g \in G$ is a p'-element if the order of g is not divisible by p. Let e be the exponent of the group G and η be a primitive rth root of unity, where $e = p^f r$ and $p \nmid r$. Let

$$I_{\mathbb{F}_q} = \left\{ l \pmod{e} : \text{ there exists } \sigma \in \mathrm{Gal}(\mathbb{F}_q(\eta)/\mathbb{F}_q) \text{ satisfying } \sigma(\eta) = \eta^l \right\}.$$

Definition 2.1 For a p'-element $g \in G$, the cyclotomic $\mathbb{F}_q$-class of g, denoted by $S_{\mathbb{F}_q}(\gamma_g)$, is defined as $\left\{ \gamma_{g^l} : l \in I_{\mathbb{F}_q} \right\}$, where $\gamma_{g^l} \in \mathbb{F}_q G$ is the sum of all conjugates of g^l in G.

Then, we have the following results, which are crucial in determining the Artin–Wedderburn decomposition of $\mathbb{F}_q G$.

Lemma 2.2 [2, *Proposition* 1.2] *The number of simple components of $\mathbb{F}_q G/J(\mathbb{F}_q G)$ is equal to the number of cyclotomic $\mathbb{F}_q$-classes in G.*

Definition 2.3 Let π be a representation of a group G over a field F. π is said to be absolutely irreducible if π^E is irreducible for every field $F \subseteq E$, where π^E is the representation $\pi \otimes E$ over E.

Definition 2.4 A field F is a splitting field for G if every irreducible representation of G over F is absolutely irreducible.

Lemma 2.5 [2, *Theorem* 1.3] *Let n be the number of cyclotomic $\mathbb{F}_q$-classes in G. If $L_1, L_2, \cdots, L_n$ are the simple components of $Z(\mathbb{F}_q G/J(\mathbb{F}_q G))$ and $S_1, S_2, \cdots, S_n$ are the cyclotomic $\mathbb{F}_q$-classes of G, then with a suitable reordering of the indices,*

$|S_i| = [L_i : \mathbb{F}_q]$.

Lemma 2.6 [11, *Lemma* 2.5] *Let K be a field of characteristic p and let A_1, A_2 be two finite dimensional K-algebras. Assume A_1 to be semisimple. If $g : A_2 \longrightarrow A_1$ is a surjective homomorphism of K-algebras, then there exists a semisimple K-algebra l such that $A_2/J(A_2) = l \oplus A_1$.*

We need the following lemmas from our previous work to compute some components of the Artin–Wedderburn decomposition of $\mathbb{F}_qG$, for a finite group G under consideration.

Lemma 2.7 [1, *Lemma* 3.1] *Let G be a group of order n and $\mathbb{F}$ be a field of characteristic $p > 0$. Let G acts on $\{1, 2, \cdots, k\}$ doubly transitively. Set $G_i = \{g \in G : g \cdot i = i\}$ and $G_{i,j} = \{g \in G : g \cdot i = i, g \cdot j = j\}$. Then, the $\mathbb{F}G$ module*

$$W = \left\{ x \in \mathbb{F}^k : \sum_{i=1}^{k} x_i = 0 \right\}$$

is an irreducible $\mathbb{F}G$ module if $p \not| k, p \not| |G_{1,2}|$.

Lemma 2.8 [1, *Corollary* 3.8] *Let G be a finite group, K be a finite field of characteristic $p > 0$, $p \not| |G|$. Suppose there exists an n dimensional irreducible representations of G over k. Then, $M(n, k)$ appears as one of the components of the Artin-Wedderburn decomposition of the semisimple algebra $\mathbb{F}_qG$.*

Units of $\mathbb{F}_{p^k}S_n$ for $p \not| n$

We start the section by talking about representations of S_n over a finite field. We define the Brauer character and state some important results about representations over an arbitrary field. See [8] for further details.

Let E be a field of characteristic p. We choose a ring of algebraic integers A in $\mathbb{C}$ such that $E = A/M$, where M is a maximal ideal of A containing pA. Take f to be the natural map $A \longrightarrow E$. Take $W = \{z \in \mathbb{C} | z^m = 1 \text{ for some } m \in \mathbb{Z} \text{ with } p \not| m\}$ (note that $W \subseteq A$). Now let π be a representation of a finite group G over E. Let S be the set of p' elements of G. For $\alpha \in S$, let $\epsilon_1, \epsilon_2, \ldots, \epsilon_l \in E^\times$ be the eigenvalues of $\pi(\alpha)$ with multiplicities. Then, for every i, there exists a unique $u_i \in W$ such that $f(u_i) = \epsilon_i$. Define $\phi : S \longrightarrow \mathbb{C}$ as $\phi(\alpha) = \Sigma u_i$. Then, ϕ is called the Brauer character of G afforded by π.

Remark 3.1

The description of Brauer character comes along with a choice of a maximal ideal M of A.

Suppose $\pi_1, \pi_2, \ldots, \pi_k$ are all the non-isomorphic irreducible representations of G over E up to isomorphism. Let ϕ_i be the Brauer character afforded by π_i. Then, $\phi_i's$ are called irreducible Brauer characters and we denote by $IBr(G)$ the set $\{\phi_i\}$. We denote by $Irr(G)$ the set of irreducible characters of G over $\mathbb{C}$. We have the following results.

Lemma 3.2 [8, *Theorem* 15.13] *We have* $IBr(G) = Irr(G)$ *whenever* $p \nmid |G|$.

For the rest of this section, take $G = S_n$, the symmetric group on n letters. We say a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ of n is p-singular if for some j we have $\lambda_{j+1} = \lambda_{j+2} = \ldots = \lambda_{j+p}$. If a partition is not p-singular, it is called p-regular. Then we have the following.

Lemma 3.3 [9, *Theorem* 11.5] *If* F *is a field of characteristic* p, *then as* λ *varies over the* p*-regular partitions,* D^λ *varies over the complete set of inequivalent irreducible* FS_n*-modules, where* $D^\lambda = \dfrac{S^\lambda}{S^\lambda \cap (S^\lambda)^\perp}$ *and* S^λ *denotes the Specht module corresponding to the partition* λ. *Moreover, every field is a splitting field for* S_n.

Proof
The proof follows immediately from the fact that every partition of n *is a p-regular partition.* □

Lemma 3.4 *The dimensions of non-isomorphic irreducible representations of* S_n *over* E *coincide with the dimensions of non-isomorphic irreducible representations of* S_n *over* $\mathbb{C}$ *when characteristic of the field* E *is greater than* n.

Proof
Since the dimension of a representation is as same as the value of the corresponding character χ *at the identity element of the group, the result follows from Lemma 3.2.* □

Proposition 3.5 *Let* S_n *denote the symmetric group on* n *letters and* $\mathbb{F}_{p^k}$ *be a finite field where* $p > n$. *Then,*

$$\mathbb{F}_{p^k} S_n \cong \bigoplus_{\chi \in Irr(G)} \mathrm{M}(\chi(1), \mathbb{F}_{p^k}).$$

Proof
Since being a semisimple algebra $\mathbb{C}S_n \cong \bigoplus_{\chi \in Irr(G)} \mathrm{M}(\chi(1), \mathbb{C})$, *the result follows from Lemmas 2.8, 3.2 and 3.4.* □

Theorem 3.6 *Let* S_n *denote the symmetric group on* n *letters and* $\mathbb{F}_{p^k}$ *be a finite field where* $p > n$. *Then,*

$$\mathcal{U}(\mathbb{F}_{p^k} S_n) \cong \bigoplus_{\chi \in Irr(G)} \mathrm{GL}(\chi(1), \mathbb{F}_{p^k}).$$

Proof
This follows immediately from Proposition 3.5 and the fact that given two rings R_1, R_2, *we have* $(R_1 \times R_2)^\times = R_1^\times \times R_2^\times$. □

Remark 3.7

Theorem 3.6 improves the result of [10] and proves that when $p > 5$, unit group of $\mathbb{F}_{p^k}S_5$ is $\mathcal{U}(\mathbb{F}_{p^k}S_5)$ given by

$$\mathbb{F}_{p^k}^{\times} \oplus \mathbb{F}_{p^k}^{\times} \oplus GL(4, \mathbb{F}_{p^k}) \oplus GL(4, \mathbb{F}_{p^k}) \oplus GL(5, \mathbb{F}_{p^k}) \oplus GL(5, \mathbb{F}_{p^k}) \oplus GL(6, \mathbb{F}_{p^k})$$

Remark 3.8

For an irreducible representation χ of S_n over a field of characteristic $p > n$, this is characterized by a partition λ of n. The value $\chi(1)$ can be calculated as the number of standard Young tableaux of shape λ.

Units of $\mathbb{F}_{p^k}A_6$ for $p \geq 7$

We start with the description of the conjugacy classes in A_6. Using [18], the group has 7 conjugacy classes, of which the representatives are given by $(1), a = (1,2)(3,4), b = (1,2,3), c = (1,2,3)(4,5,6), d = (1,2,3,4)(5,6), e = (1,2,3,4,5)$ and $f = (1,2,3,4,6)$. We have the following relations:

$$\text{for all } g \notin [e] \cup [f],\ [g] = [g^{-1}], \tag{4.1}$$

$$\text{and } [e] = [e^4], [e^2] = [e^3] = [f]. \tag{4.2}$$

Proposition 4.1 *Let $\mathbb{F}_q$ be a field of characteristic $p \geq 7$ and $G = A_6$. Then, the Artin–Wedderburn decomposition of $\mathbb{F}_qG$ is one of the following*:

$$\mathbb{F}_q \oplus \bigoplus_{i=1}^{6} M(n_i, \mathbb{F}_q),$$

$$\mathbb{F}_q \oplus \bigoplus_{i=1}^{4} M(n_i, \mathbb{F}_q) \oplus M(n_5, \mathbb{F}_{q^2})$$

Proof

Since $p \geq 7$, we have $p \not| |A_6|$; by Maschke's theorem we have $J(\mathbb{F}_qG) = 0$. Hence, Wedderburn decomposition of $\mathbb{F}_qG$ is isomorphic to $\bigoplus_{i=1}^{n} M(n_i, K_i)$, where for all $1 \leq i \leq n$, we have $n_i > 0$ and K_i is a finite extension of $\mathbb{F}_q$.

Firstly, from Lemma 2.6, we have

$$\mathbb{F}_qG \cong \mathbb{F}_q \bigoplus_{i=1}^{n-1} M(n_i, K_i), \tag{4.3}$$

taking g to be the map $g(\sum_{x\in A_6} \alpha_x x) = \sum_{x\in A_6} \alpha_x$. Now to compute these n_i's and K_i's we calculate the cyclotomic $\mathbb{F}_q$ classes of G. Note that $p^k \equiv \pm 1 \mod 4, p^k \equiv \pm 1 \mod 3$ for any prime p. Hence, $S_{\mathbb{F}_q}(\gamma_g) = \{\gamma_g\}$ whenever $g \notin [e] \cup [f]$ (by Equation 4.1). Hence, we have to consider $S_{\mathbb{F}_q}(\gamma_g)$ in the other cases.

When $p \equiv \pm 1 \mod 5$, $S_{\mathbb{F}_q}(\gamma_e) = \{\gamma_e\}$ and $S_{\mathbb{F}_q}(\gamma_f) = \{\gamma_f\}$, by Eq. 4.2 and the fact that $p^k \equiv \pm 1 \mod 5$. Thus, by Lemmas 2.2 and 2.5, there are seven cyclotomic $\mathbb{F}_q$-classes and $[K_i : \mathbb{F}_q] = 1$ for all $1 \le i \le 6$. This gives that in this case the Artin–Wedderburn decomposition is

$$\mathbb{F}_q \oplus \bigoplus_{i=1}^{6} M(n_i, \mathbb{F}_q).$$

When $p \equiv \pm 2 \mod 5$ and k is even, then $p^k \equiv -1 \mod 5$. Similarly, in this case the Artin–Wedderburn decomposition is

$$\mathbb{F}_q \oplus \bigoplus_{i=1}^{6} M(n_i, \mathbb{F}_q).$$

Lastly, when $p \equiv \pm 2 \mod 5$ and k is odd, then $p^k \equiv \pm 2 \mod 5$ and $S_{\mathbb{F}_q}(\gamma_e) = \{\gamma_e, \gamma_f\}$ by Eq. 4.2. Thus, by Lemmas 2.2 and 2.5, there are six cyclotomic $\mathbb{F}_q$-classes and $[K_i : \mathbb{F}_q] = 1$ for all $1 \le i \le 4, [K_5 : \mathbb{F}_q] = 2$. In this case, the Artin–Wedderburn decomposition is

$$\mathbb{F}_q \oplus \bigoplus_{i=1}^{4} M(n_i, \mathbb{F}_q) \oplus M(n_5, \mathbb{F}_{q^2}).$$

□

Since $\dim \mathbb{F}_q A_6 = |A_6| = 360$, Proposition 4.1 gives that the n_i's should satisfy $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n_6^2 = 359$ or $n_1^2 + n_2^2 + n_3^2 + n_4^2 + 2n_5^2 = 359$. Since these equations do not have a unique solution, we find some of the n_i's using representations of A_6 over $\mathbb{F}_q$ and invoke Lemma 2.7 to reach a unique solution for the mentioned equations. We have the following results.

Lemma 4.2 *The group S_6 has four inequivalent irreducible representations of degree 5, which on restriction on A_6 give two inequivalent irreducible representations of A_6 over $\mathbb{F}_{p^k}$ for $p \ge 7$. Moreover, these irreducible representations are obtained from two non-isomorphic doubly transitive actions on a set of 6 points.*

Proof

Note that S_6 acts on $T = \{1, 2, 3, 4, 5, 6\}$ doubly transitively. Hence, by Lemma 2.7, we get an irreducible representation of degree 5. Since tensoring with sign representation gives irreducible representations, we get two inequivalent irreducible representations of degree 5 of S_6, say π_1 and π_2.

For the other two irreducible representations of dimension 5, we consider the outer automorphism of S_6, say φ, given on generators as follows:

$$\varphi((1,2)) = (1,2)(3,4)(5,6)$$
$$\varphi((2,3)) = (1,3)(2,5)(4,6)$$
$$\varphi((3,4)) = (1,5)(2,6)(3,4)$$
$$\varphi((4,5)) = (1,3)(2,4)(5,6)$$
$$\varphi((5,6)) = (1,6)(2,5)(3,4).$$

This gives another doubly transitive action on T, which is not isomorphic to the previous action. Thus, we get another 5 dimensional irreducible representation, say π_3. Tensoring π_3 with the sign representations, we get π_4 which is a 5-dimensional irreducible representation of S_6 different from π_3. By considering the characters of the corresponding representations, we see that π_1, π_2, π_3 and π_4 are all distinct.

Since A_6 acts doubly transitively on T via the restrictions of these two actions, we obtain two non-isomorphic 5-dimensional irreducible representations of A_6. □

Corollary 4.3 *The algebra* $\mathbb{F}_q A_6$ *has two components to be* $\mathrm{M}(5, \mathbb{F}_q)$ *for* $p \geq 7$.

Proof
Immediately follows from Lemmas 4.2 and 2.7. □

Corollary 4.4 *There does not exist any* 4 *dimensional irreducible representations of* A_6 *over* $\mathbb{F}_{p^k}$ *for* $p \geq 7$.

Proof
From Lemma 3.3, we know that any field $\mathbb{F}_{p^k}, p \geq 7$ *is a splitting field of* S_6. *Hence, by Proposition 3.5, we have degrees of irreducible representations of* S_6 *are* $\{1, 5, 9, 10, 16\}$.

Recall that by Frobenius reciprocity we have the following bijection

$$\mathrm{Hom}_{\mathbb{F}_q S_6}(\mathrm{Ind} V, W) \cong \mathrm{Hom}_{\mathbb{F}_q A_6}(V, \mathrm{Res} W),$$

where Ind, Res denote the induction functor, restriction functor, respectively. Here V is an irreducible representation of A_6 and W is an irreducible representation of S_6. Suppose A_6 has an irreducible representation V with $\dim V = 4$. Since $[S_6 : A_6] = 2$, we have that $\dim \mathrm{Ind} V = 8$. Since S_6 does not have any irreducible representation of dimension 8, the induced representation splits. Being $\dim \mathrm{Ind} V = 8$, $\mathrm{Ind}(V)$ does not have any component of dimensions 9, 10 and 16. Now, let us assume that $\dim W = 5$, then by Lemma 4.2, $\mathrm{Res} W$ is an irreducible representation. Hence $\mathrm{Hom}_{\mathbb{F}_q A_6}(V, \mathrm{Res} W) = 0$, which implies that $\mathrm{Ind} V$ does not have any irreducible component of dimension 5. Similarly, $\mathrm{Ind} V$ does not have any irreducible component of dimension 1. This completes the proof. □

Corollary 4.5 *The algebra* $\mathbb{F}_q A_6$ *has one component to be* $\mathrm{M}(9, \mathbb{F}_q)$ *for* $p \geq 7$.

Proof
The group A_6 *being isomorphic to* $\mathrm{PSL}(2, \mathbb{F}_9)$ *acts doubly transitively on a set with 10 points (see [4]), hence the conclusion.* □

Corollary 4.6 *We have* $(n_1, n_2, n_3, n_4, n_5, n_6) = (5, 5, 9, 8, 8, 10)$ *or* $(n_1, n_2, n_3, n_4, n_5) = (5, 5, 9, 10, 8)$ *up to permutation.*

Proof
Since A_6 *has one 1-dimensional, two 5-dimensional and one 9-dimensional irreducible representations, we can assume that* $n_1 = 5, n_2 = 5, n_3 = 9$. *Hence, we are left with the equation*

$$n_4^2 + n_5^2 + n_6^2 = 228 \text{ or } n_4^2 + 2n_5^2 = 228.$$

Then, $(n_4, n_5, n_6) \in \{(4, 4, 14), (8, 8, 10)\}$, $(n_4, n_5) \in \{(14, 4), (10, 8)\}$. Hence, the result is obvious from Corollary 4.4. □

Proposition 4.7 *Let* $\mathbb{F}_{p^k}$ *be a field of characteristic* $p \geq 7$ *and* A_6 *denote the alternating group on six letters. Then, the Artin–Wedderburn decomposition of* $\mathbb{F}_{p^k} A_6$ *is*

$$\mathbb{F}_q \oplus \mathrm{M}(5, \mathbb{F}_q) \oplus \mathrm{M}(5, \mathbb{F}_q) \oplus \mathrm{M}(9, \mathbb{F}_q) \oplus \mathrm{M}(10, \mathbb{F}_q) \oplus \mathrm{M}(8, \mathbb{F}_{q^2}),$$

when $p \equiv \pm 2 \mod 5, k \equiv 1 \mod 2$ *and*

$$\mathbb{F}_q \oplus \mathrm{M}(5, \mathbb{F}_q) \oplus \mathrm{M}(5, \mathbb{F}_q) \oplus \mathrm{M}(8, \mathbb{F}_q) \oplus \mathrm{M}(8, \mathbb{F}_q) \oplus \mathrm{M}(9, \mathbb{F}_q) \oplus \mathrm{M}(10, \mathbb{F}_q),$$

otherwise.

Proof
Follows from Proposition 4.1 and Corollary 4.6. □

Theorem 4.8 *Let* $\mathbb{F}_{p^k}$ *be a field of characteristic* $p \geq 7$ *and* A_6 *denote the alternating group on six letters. Then, the unit group of the algebra,* $\mathcal{U}(\mathbb{F}_{p^k} A_6)$ *is*

$$\mathbb{F}_q^\times \oplus \mathrm{GL}(5, \mathbb{F}_q) \oplus \mathrm{GL}(5, \mathbb{F}_q) \oplus \mathrm{GL}(9, \mathbb{F}_q) \oplus \mathrm{GL}(10, \mathbb{F}_q) \oplus \mathrm{GL}(8, \mathbb{F}_{q^2}), \tag{4.4}$$

when $p \equiv \pm 2 \mod 5, k \equiv 1 \mod 2$ *and*

$$\mathbb{F}_q^\times \oplus \mathrm{GL}(5, \mathbb{F}_q) \oplus \mathrm{GL}(5, \mathbb{F}_q) \oplus \mathrm{GL}(8, \mathbb{F}_q) \oplus \mathrm{GL}(8, \mathbb{F}_q) \oplus \mathrm{GL}(9, \mathbb{F}_q) \oplus \mathrm{GL}(10, \mathbb{F}_q), \tag{4.5}$$

otherwise.

Proof

This follows immediately from Proposition 4.7 and the fact that given two rings R_1, R_2*, we have* $(R_1 \times R_2)^{\times} = R_1^{\times} \times R_2^{\times}$. □

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Competing interests
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