Journal of the Egyptian Mathematical Society 25 (2017) 445–451

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original Article

On a fractional-order model for HBV infection with cure of infected cells

Salman S.M.[a,*], Yousef A.M.[b]

[a] Faculty of Education, Alexandria University, Alexandria, Egypt
[b] Faculty of Science, South valley University, Qena, Egypt

ARTICLE INFO

Article history:
Received 15 April 2017
Revised 29 May 2017
Accepted 28 June 2017
Available online 11 July 2017

MSC:
39B05
37N30
37N20

Keywords:
Fractional calculus
Viral dynamics
Hepatitis B virus
Equilibrium points
Local stability

ABSTRACT

In this work, a fractional-order model for HBV infection with cure of infected cells is considered. Local asymptotic stability of equilibria is discussed. Numerical simulation using PECE method is presented to illustrate the theoretical analysis. It has been noticed that when introducing a fractional-order derivative to the model, the peak of the infection is reduced, however, the disease takes a longer time to be eradicated.

1. Introduction

Over recent years, mathematical models which mimic viral population dynamics in host cells have gained much interest [1,2]. These models improved our understanding of HIV-1 and different viruses, like HBV and HCV [3,4]. Mathematical analysis is important for these models to get a realistic view for the virus dynamics in vivo. Particularly, global stability of a fixed point can enhance our knowledge of virus dynamics. Many models were established for describing the dynamics of HBV infection [5,6]. Such models were modified in [7,8]. Recently, in an HBV model, the infected hepatocytes may be reverted to be uninfected by losing all cccDNA from their nucleus [7].

An HBV infection model of hepatocytes of infected cells which will be cured was introduced by Cruz Vargas–De–León [9] and is given by:

$$\begin{aligned} \frac{dx}{dt} &= \lambda - \mu x - \beta xz + \delta y, \\ \frac{dy}{dt} &= \beta xz - (\nu + \delta)y, \\ \frac{dz}{dt} &= \sigma y - \gamma z, \end{aligned} \tag{1.1}$$

where x, y and z refer to the densities of hepatocytes which are uninfected, infected, and virions which are disease-free, respectively. Susceptible hepatocytes are supposed to be produced at rate λ, die at μx, and infected at βxz, where β is the rate of infection. The rate of death of infected hepatocytes is νy, σy is the production rate of free virions from infected hepatocytes, and γz is the clearance rate of viral particles. The term δy appears in the first equation of (1.1), corresponds to the cure-created rate of uninfected hepatocytes.

2. Fractional calculus

Indeed, fractional calculus generalizes the classical integer-order differentiation and integration to arbitrary orders [10–12]. A large number of researchers have modeled many real processes with the aid of fractional calculus [13–15]. Due to the applicability of differ-

* Corresponding author.
E-mail addresses: samastars9@gmail.com (S. S.M.), ahmed.hussien1@sci.svu.edu.eg (Y. A.M.).

http://dx.doi.org/10.1016/j.joems.2017.06.003

ential equations with fractional-order in science and engineering, a lot of applied researchers have been attracted to it. Such equations have been implemented analytically and numerically. In fact, fractional calculus has the ability of performing both fractional-order integration and differentiation. Existence of solutions for differential equations with fractional-order can be found in [12,16].

Many contributions for the developing theorems of the solutions' existence of functional differential equations with fractional-order existed in [11,17,18].

In fractional calculus we can find many definitions of derivatives of fractional-order, however, we will consider the famous Caputo derivatives due its advantage on initial value problems which can be summarized in the flexibility of using classical initial conditions without encountering any problem during solvability.

Definition 1. The fractional-order integral of the function $f(t)$, $t > 0$ can be defined as

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds,$$

where $\beta \in \mathbb{R}^+$, and the fractional-order derivative of $f(t)$, $t > 0$ can be defined as

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t), \quad D = \frac{d}{dt},$$

where $\alpha \in (n-1, n)$.

In addition, consider also the following basic properties. Let $\beta, \gamma \in \mathbb{R}^+$, $\alpha \in (0, 1)$,

- $I_a^{\beta} : L^1 \to L^1$, and if $f(x) \in L^1$, then $I_a^{\gamma} I_a^{\beta} f(x) = I_a^{\gamma+\beta} f(x)$.
- $\lim_{\beta \to n} I_a^{\beta} f(x) = I_a^n f(x)$ uniformly on $[a, b]$, $n = 1, 2, 3, \ldots$, where $I_a^1 f(x) = \int_a^x f(s)ds$.
- $\lim_{\beta \to 0} I_a^{\beta} f(x) = f(x)$ weakly.
- If $f(x)$ is absolutely continuous on $[a, b]$, then $\lim_{\alpha \to 1} D_a^{\alpha} f(x) = \frac{df(x)}{dx}$.

For the basic features of fractional-order derivatives and integrals, one can see [19–21].

Actually, to find the solution of fractional-order differential equations there are two familiar methods: frequency domain [22] and time domain [23]. Since the frequency domain method is not efficient in observing chaos, the second method is more usable because it is more effective (see [24,25]).

As a matter of fact, difficulties arise when trying to find an analytic solution of a differential equation, that is why one searches for numerical methods.

Indeed, the predictor-correctors scheme has been used to simulate many fractional-order differential equations with Caputo fractional derivative such as the fractional-order Chua system, the fractional-order Chen system, the fractional-order Lorenz system, etc. [26].

Recently, it has turned out that applying fractional calculus, rather than integer calculus, to mathematical epidemiology provides a good tool for the description of memory feature which is a hallmark in many biological systems [27]. The property of memory does not take into consideration the involved process history in such models but it also has a great impact on the future of the process [28]. Actually, a differential operator with an integer-order is a local operator, whereas a differential operator with a fractional-order is non local in as it takes into account that the future state do not relay only on the present state but on all of the history of its previous states too.

Now introduce the fractional-order derivatives to the HBV infection model with cells which will be cured after infection given by (1.1) as follows:

$$\begin{aligned} D^{\alpha} x &= \lambda - \mu x - \beta xz + \delta y, \\ D^{\alpha} y &= \beta xz - (\nu + \delta) y, \\ D^{\alpha} z &= \sigma y - \gamma z, \end{aligned} \tag{2.1}$$

where $\alpha \in (0, 1)$. The major reason of considering a fractional-order model in this paper is that fractional-order differential equations have a relation to systems with memory like systems of the Immune system which creates memory T-cells and B-Cells. Such cells learn from their experience of fighting any threat. So, they can use their expertise to detect and fight the same threat later. On the contrary, integer-order models do not have any information about the memory of neither the hepatocytes nor the free virions in (1.1) (see [29–32]).

To summarize, mathematical models which use ordinary differential equations with integer order have been proven valuable in understanding the dynamics of biological systems. However, the behavior of most biological systems has memory or aftereffects. The modeling of these systems using fractional-order differential equations has more advantages than classical integer-order mathematical modeling, in which such effects are neglected. Consequently, the subject of fractional calculus has gained importance due to its demonstrated applications in numerous diverse and widespread fields of science and engineering. For application of fractional-order derivative, one can see [33–37].

3. Existence of uniformly stable, positive, and eventually bounded solution

Let $x_1(t) = x(t), x_2(t) = y(t), x_3(t) = z(t)$.

$$\begin{aligned} f_1(x_1, x_2, x_3) &= \lambda - \mu x_1 - \beta x_1 x_3 + \delta x_2, \\ f_2(x_1, x_2, x_3) &= \beta x_1 x_3 - (\nu + \delta) x_2, \\ f_3(x_1, x_2, x_3) &= \sigma x_2 - \gamma x_3. \end{aligned}$$

Let $D = \{x_1, x_2, x_3 \in \mathbb{R} : |x_i(t)| < a, t \in [0, T], i = 1, 2, 3\}$, then on D we have:

$|\frac{df_1}{dx_1}| < k_1$, $|\frac{df_1}{dx_2}| < k_2$, $|\frac{df_1}{dx_3}| < k_3$, $|\frac{df_2}{dx_1}| < k_4$, $|\frac{df_2}{dx_2}| < k_5$, $|\frac{df_2}{dx_3}| < k_6$, $|\frac{df_3}{dx_1}| < k_7$, $|\frac{df_3}{dx_2}| < k_8$, $|\frac{df_3}{dx_3}| < k_9$, where k_i, $i = 1, 2, \ldots, 9$ are positive constants. This implies that each of the functions f_1, f_2, f_3 satisfies Lipschitz conditions with three arguments x_1, x_2 and x_3. Then each of the functions f_1, f_2, f_3 are absolutely continuous w.r.t. three arguments x_1, x_2 and x_3.

Rewrite system (2.1) as follows:

$$\begin{aligned} D^{\alpha} x_1(t) &= f_1(x_1, x_2, x_3), t > 0, \quad \text{and} \quad x_1(0) = x_{01}, \\ D^{\alpha} x_2(t) &= f_2(x_1, x_2, x_3), t > 0, \quad \text{and} \quad x_2(0) = x_{02}, \\ D^{\alpha} x_3(t) &= f_3(x_1, x_2, x_3), t > 0, \quad \text{and} \quad x_3(0) = x_{03}. \end{aligned}$$

Note that by a solution of the fractional-order HBV model (2.1) we mean a column vector $(x_1, x_2, x_3)'$, $x_1, x_2, x_3 \in C[0, T]$, $T < \infty$, where $C[0, T]$ denotes the class of continuous functions defined on $[0, T]$ and the "dash" denotes the transpose of the vector.

Theorem 1. *The fractional-order model (2.1) has a unique uniformly Lyapunov stable solution.*

Proof. Rewrite system (2.1) in a matrix form:

$$D^{\alpha} X(t) = F(X(t)), \quad t > 0, X(0) = X_0,$$

where $X(t) = (x_1, x_2, x_3)'$,

$F(X(t)) = (f_1(x_1(t), x_2(t), x_3(t)), f_2(x_1(t), x_2(t), x_3(t)), f_3(x_1(t), x_2(t), x_3(t))'$.

Now applying Theorem 2.1 [38], we deduce that the fractional-order HBV model (2.1) has a unique solution. In addition, by Theorem 3.2 [38] this solution is uniformly Lyapunov stable. □

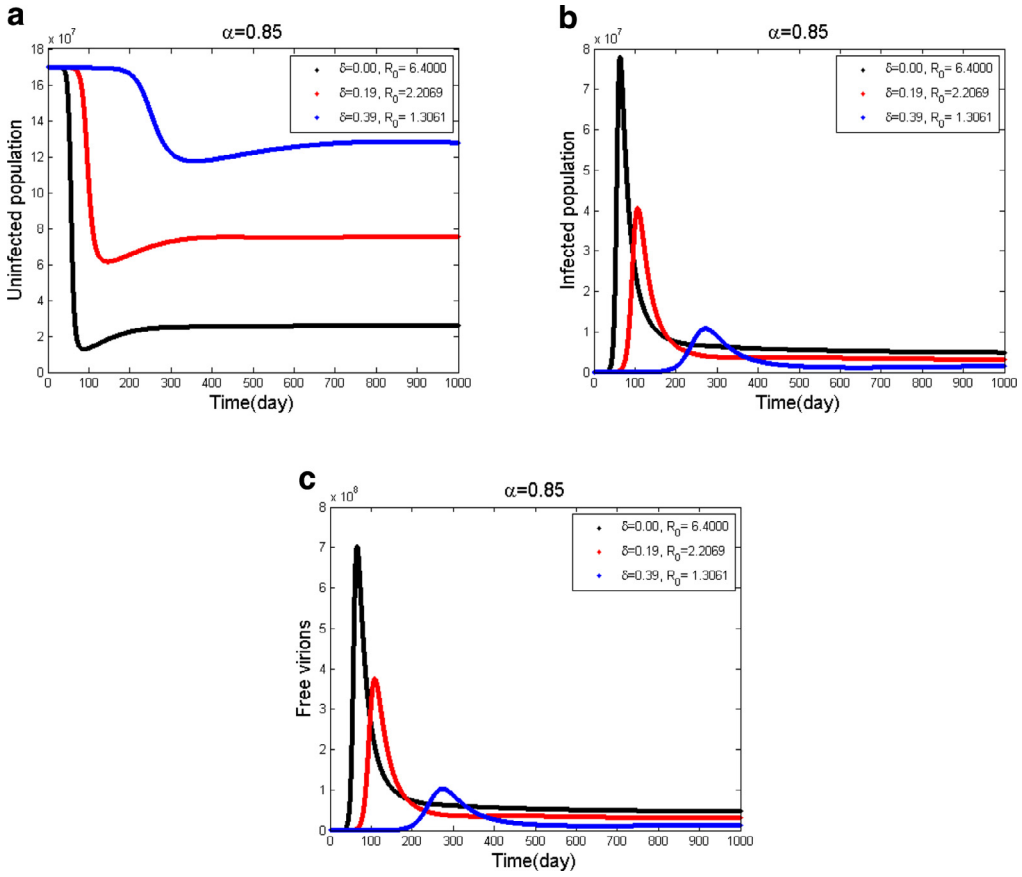

Fig. 1. Solution of system (2.1) with $\alpha = 0.85$.

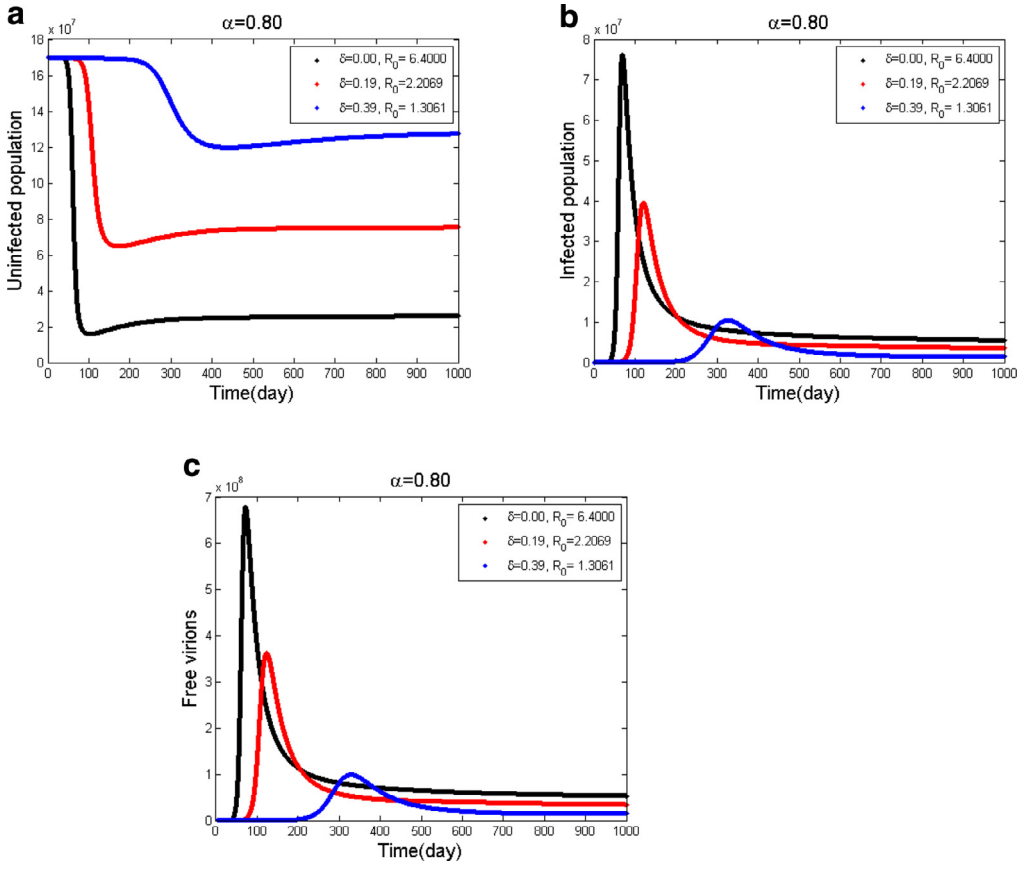

Fig. 2. Solution of system (2.1) with $\alpha = 0.80$.

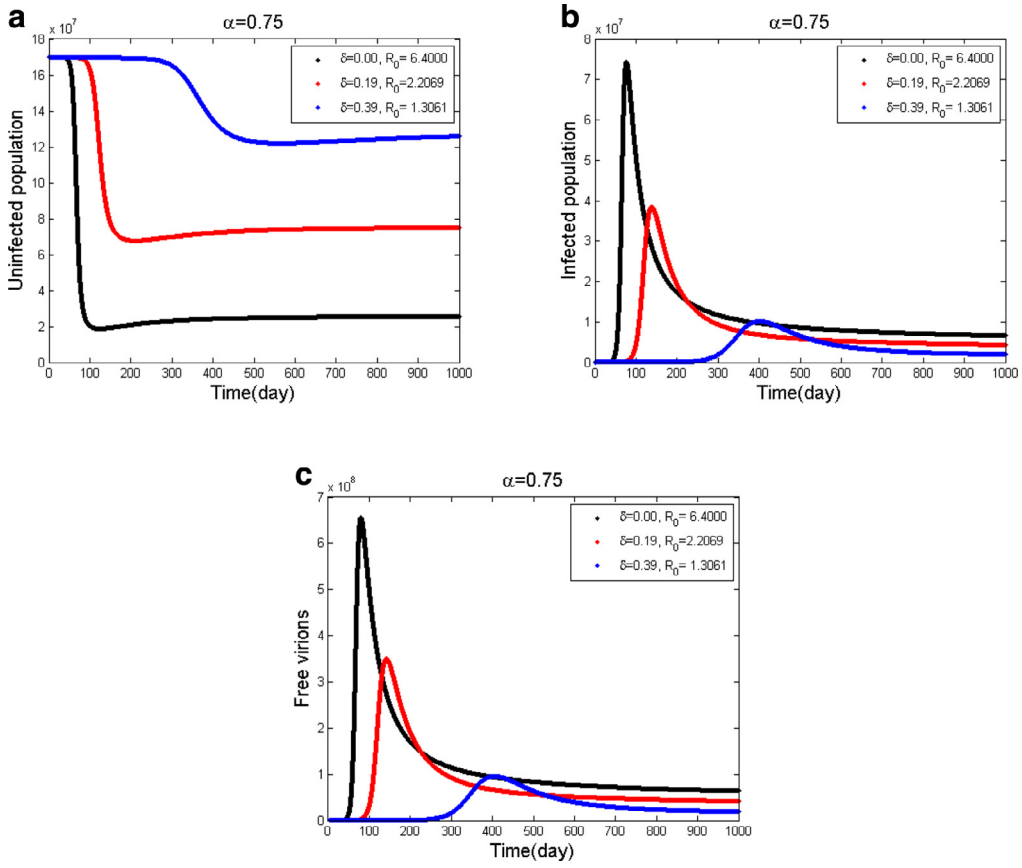

Fig. 3. Solution of system (2.1) with $\alpha = 0.75$.

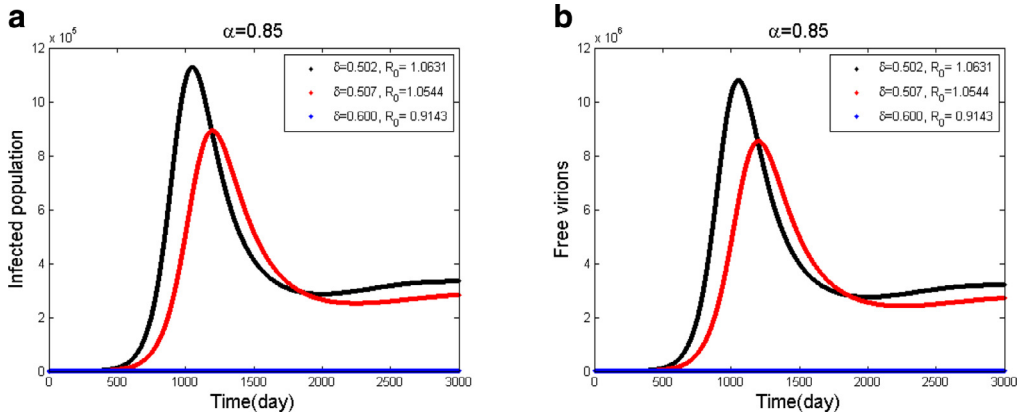

Fig. 4. Time series of system (2.1) with different δ, where $\alpha = 0.85$.

Since we are dealing with a biological model, it does not make sense to have negative solutions. The following theorem prove that the state variables $x(t)$, $y(t)$, and $z(t)$ are all positive.

Theorem 2. *Assume that the parameters λ, μ, β, δ, ν, σ, and γ are all positive and real and denote $\mathbb{R}^3_+ = \{X \in \mathbb{R}^3 | X \geq 0\}$ and let $X(t) = (x(t), y(t), z(t))'$. Then for any $X(0) > 0$, the solution $X(t)$ of (2.1) on $t \geq 0$ will remain in $\mathbb{R}^3_+$.*

Proof. We will prove this theorem by contradiction. Suppose there exists $t^* > 0$ at which one of the elements of the solution will be zero and until all elements of the solution will be positive. Let $\alpha \in (0, 1)$, there are three possibilities:

(i) If $x(t^*) = 0$ holds, then $y(t) > 0$, $z(t) > 0$ when $t \in [0, t^*)$ and $x(t) > 0$ on $[0, t^*]$.

Let $m_1 = \min_{t\in[0,t^*]} z(t)$, $c_1 = \mu + \beta m_1$. The first equation of (2.1) gives:

$$D^\alpha x(t) > c_1 x(t), \quad t \in [0, t^*],$$

hence

$$x(t) > x(0) E_\alpha(-c_1 t^\alpha), \quad t \in [0, t^*],$$

where $E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+1)}$ is the Mittag Leffler function. Since $x(0) > 0$, then $x(t^*) > 0$ which is a contradiction.

(ii) If $y(t^*) = 0$ holds, then $x(t) > 0$, $z(t) > 0$ when $t \in [0, t^*]$ and $y(t) > 0$ when $t \in [0, t^*)$. From the second equation of system (2.1) we have

$$D^\alpha y(t) > -(\nu + \delta) y(t), \quad t \in [0, t^*],$$

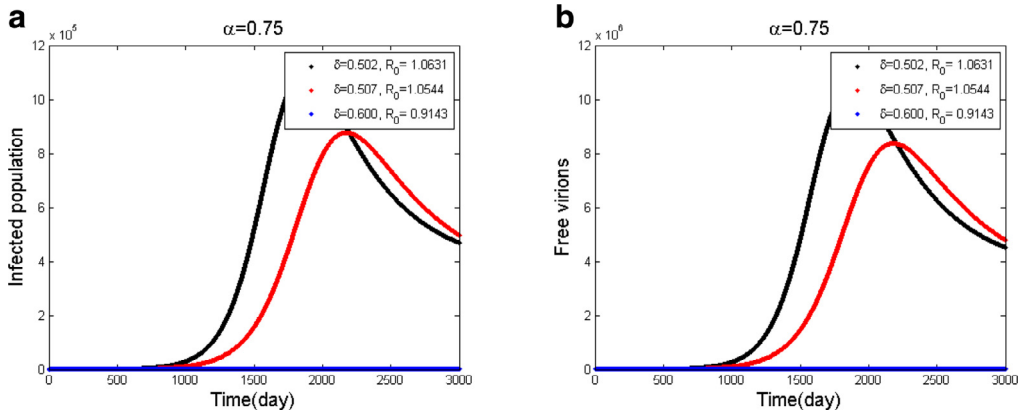

Fig. 5. Time series of system (2.1) with different δ, where $\alpha = 0.75$.

which implies that

$$y(t) > y(0)E_\alpha(-(\nu+\delta)t^\alpha), \quad t \in [0, t^*].$$

Since $y(0) > 0$, one has $y(t^*) > 0$ which is a contradiction.

(iii) If $z(t^*) = 0$ holds, then $x(t) > 0$, $y(t) > 0$ when $t \in [0, t^*]$ and $z(t) > 0$ when $t \in [0, t^*)$. The third equation of (2.1) gives

$$D^\alpha z(t) > -\gamma z(t), \quad t \in [0, t^*],$$

which implies

$$z(t) > z(0)E_\alpha(-\gamma t^\alpha), \quad t \in [0, t^*].$$

Since $z(0) > 0$, one has $z(t^*) > 0$ which is a contradiction. Therefore, the solution of system (2.1) will remain in $\mathbb{R}^3_+$.
Adding the first two equations in (2.1), for the biological justification, we suppose that $\mu \le \nu$.

$$D^\alpha\{x(t)+y(t)\} = \lambda - \mu x - \nu y \le \lambda - \mu(x+y), \tag{3.1}$$

which implies that

$$x(t)+y(t) \le [x(0)+y(0)]E_\alpha(-\mu t) + \frac{\lambda}{\mu}[1 - E_\alpha(-\mu t)]. \tag{3.2}$$

Hence, we have $\lim_{t\to\infty} sup\,[x(t)+y(t)] \le \frac{\lambda}{\mu}$. The last equation of (2.1) then leads to

$$D^\alpha z(t) \le \frac{\sigma\lambda}{\mu} - \gamma z, \tag{3.3}$$

which in turn implies $\lim_{t\to\infty} sup\, z(t) \le \frac{\sigma\lambda}{\mu\gamma}$ □

4. Equilibrium points and their asymptotic stability

For any infectious disease, the most important issue is its ability to invade a population. Many epidemiological models have a disease-free equilibrium (DFE) where the disease is not present in the population. These models usually have a parameter, which is the basic reproduction number $\Re_0$. If $\Re_0 < 1$, then the DFE is locally asymptotically stable, and the disease will not spread, but if $\Re_0 > 1$, then the DFE is unstable and the disease will spread all over the population. The basic reproduction number for system (2.1) is given by:

$$\Re_0 = \frac{\beta\lambda\sigma}{\gamma\mu(\nu+\delta)}.$$

System (2.1) has two equilibria, namely:
The disease-free equilibrium (DFE) $E^0 = (\frac{\lambda}{\mu}, 0, 0)$, and if $\Re_0 > 1$, there is an endemic equilibrium (EE) $E^* = (\frac{\gamma(\nu+\delta)}{\sigma\beta}, \frac{\lambda}{\nu} - \frac{\gamma\mu(\nu+\delta)}{\sigma\beta\nu}, \frac{\lambda\sigma}{\nu\gamma} - \frac{\mu(\nu+\delta)}{\beta\nu})$.

Next, we discuss local stability analysis of the two equilibria of (2.1).

Proposition 1. *E^0 is locally asymptotically stable if $0 < \Re_0 < 1$ and is unstable if $\Re_0 > 1$.*

Proof. Calculate the Jacobian matrix of (2.1) evaluated at E^0 to obtain:

$$J = \begin{pmatrix} -\mu & \delta & -\frac{\beta\lambda}{\mu} \\ 0 & -(\nu+\delta) & \frac{\beta\lambda}{\mu} \\ 0 & \sigma & -\gamma \end{pmatrix}.$$

The characteristic equation reads:

$$P(\Lambda) = \Lambda^3 + (\mu+\nu+\delta+\gamma)\Lambda^2 + \left(\mu(\nu+\delta+\gamma) + \gamma(\delta+\nu) - \frac{\sigma\beta}{\mu}\right)\Lambda + \gamma\mu(\delta+\nu) - \sigma\beta\lambda = 0, \tag{4.1}$$

which has three eigenvalues:

$$\Lambda_1 = -\mu,$$

$$\Lambda_2 = \frac{-(\gamma+\nu+\delta)\mu + \sqrt{\mu^2(\gamma+\nu+\delta)^2 - 4\mu(\gamma\delta\mu + \gamma\mu\nu - \beta\lambda\sigma)}}{2\mu},$$

$$\Lambda_3 = \frac{-(\gamma+\nu+\delta)\mu - \sqrt{\mu^2(\gamma+\nu+\delta)^2 - 4\mu(\gamma\delta\mu + \gamma\mu\nu - \beta\lambda\sigma)}}{2\mu}.$$

If $\mu(\gamma+\delta+\nu)^2 < 4(\gamma\delta\mu + \gamma\mu\nu - \beta\lambda\sigma)$, then Λ_2 and Λ_3 are two complex conjugates. The asymptotic stability of E^0 can be guaranteed if the following sufficient condition is satisfied [39]:

$$|arg(\Lambda_i)| > \alpha\frac{\pi}{2}, \quad i = 1, 2, 3. \tag{4.2}$$

If $0 < \Re_0 < 1$, then the above three characteristic roots will have negative real parts. Thus, E^0 is locally asymptotically stable.

If $\Re_0 > 1$, then at least one eigenvalue will be a positive real root. Thus, E^0 is unstable and the endemic equilibrium E^* emerges. Biologically, the local asymptotic stability of the DFE indicates that the infection by the virus of hepatitis B never persists. □

As a matter of fact, the region of stability of the fractional-order model is greater than the region of stability of the integer-order model.

The characteristic equation (4.1) can be rewritten as:

$$P(\Lambda) = \Lambda^3 + a_1\Lambda^2 + a_2\Lambda + a_3 = 0,$$

where:

$$a_1 = \mu+\nu+\delta+\gamma,$$

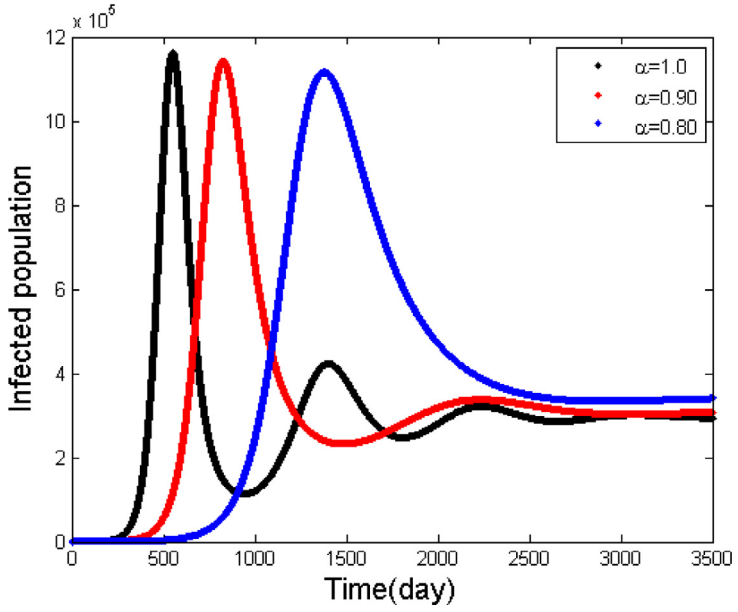

Fig. 6. The infected hepatocytes for different α with $\delta = 0.502$.

$$a_2 = \mu(\nu+\delta+\gamma)+\gamma(\delta+\nu)-\frac{\sigma\beta}{\mu},$$
$$a_3 = \gamma\mu(\delta+\nu)-\sigma\beta\lambda.$$

The discriminant of $P(\Lambda)$ is given by:

$$D(P) = 18a_1a_2a_3+(a_1a_2)^2-4a_3(a_1)^3-4(a_2)^3-27(a_3)^2.$$

If the discriminant of $P(\Lambda)D(P)$ is positive, then Routh–Hurwitz conditions [40,41]are the necessary and sufficient conditions for (4.3), i.e.,

$a_1 > 0, \quad a_3 > 0, \quad a_1a_2 > a_3$ if $D(P) > 0$.

Next, we discuss the stability of E^* for $\Re_0 > 1$. The Jacobian matrix at E^* is given by

$$J = \begin{pmatrix} \frac{-\beta\lambda\sigma}{\nu\gamma}+\frac{\mu\delta}{\nu} & \delta & -\frac{\gamma(\nu+\delta)}{\sigma} \\ \beta(\frac{\lambda\sigma}{\nu\gamma}-\frac{\mu(\nu+\delta)}{\beta\nu}) & -(\nu+\delta) & \frac{\beta\gamma(\nu+\delta)}{\sigma\beta} \\ 0 & \sigma & -\gamma \end{pmatrix}.$$

Then the characteristic equation of the linearized system of (2.1) at E^* is

$$G(s) = s^3+b_1s^2+b_2s+b_3,$$

where

$$b_1 = \frac{\beta\lambda\sigma}{\nu}-\frac{\delta\mu}{\nu}+\nu+\delta+\gamma,$$
$$b_2 = \beta\lambda\sigma-\frac{\delta\beta\sigma\lambda}{\gamma\nu}+\frac{\delta\beta\sigma\lambda}{\nu}+\frac{\gamma\beta\sigma\lambda}{\nu}-\frac{\mu\delta\gamma}{\nu},$$
$$b_3 = \gamma\mu\delta-\frac{\delta\beta\lambda\sigma}{\nu}+\frac{\mu\delta^2\gamma}{\nu}.$$

According to the stability conditions in [39] and [42] we have the following proposition.

Proposition 2. *E^* is locally asymptotic stable if all eigenvalues of $J(E^*)$ satisfy*

$$|arg(s_i)| > \alpha\frac{\pi}{2}, \quad i = 1,2,3. \tag{4.3}$$

The discriminant of $G(s)$ is given by

$$D(G) = -\begin{vmatrix} 1 & b_1 & b_2 & b_3 & 0 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 3 & 2b_1 & b_2 & 0 & 0 \\ 0 & 3 & 2b_1 & b_2 & 0 \\ 0 & 0 & 3 & 2b_1 & b_2 \end{vmatrix}.$$

That is, $D(G) = 18b_1b_2b_3+(b_1b_2)^2-4b_3b_1^3-4b_2^3-27b_3^2$.

Now considering the stability conditions in [43], the following proposition can be stated.

Proposition 3.

(i) *If $\Re_0 > 1$, then E^* is locally asymptotically stable.*
(ii) *If $D(G) > 0$, and the conditions of Routh–Hurwitz are satisfied, so,*

$$b_1 > 0, b_3 > 0, b_1b_2 > b_3,$$

then E^* is locally asymptotically stable.
(iii) *If $D(G) < 0$, $b_1 > 0, b_2 > 0, b_1b_2 = b_3$, $\alpha \in (0,1)$, then E^* is locally asymptotically stable.*
(iv) *If $D(G) < 0$, $b_1 \geq 0$, $b_2 \geq 0$, $b_3 > 0$, $\alpha \in (0.5, \frac{2}{3})$, then E^* is locally asymptotically stable.*
(v) *If $D(G) < 0$, $b_1 < 0$, $b_2 < 0$, $\alpha > \frac{2}{3}$, then E^* is unstable.*

5. Numerical methods and results

In this section, a numerical solution of system (2.1) is given to verify the correctness of the theoretical analysis in Section 3. The PECE method is applied [44]. We used the same values of the parameters mentioned in [9], that is, $\lambda = 5\times10^5 cells.(mL.d)^{-1}$, $\mu = 0.003d^{-1}$, $\beta = 4\times10^{-10}ml.(copies.d)^{-1}$, $\sigma = 6.24d^{-1}$, $\gamma = 0.65d^{-1}$, $\nu = 0.1d^{-1}$, $x(0) = 1.7\times10^8 cells.(ml)^{-1}$, $y(0) = 0$, $z(0) = 400$ copies.$(ml)^{-1}$. In order to illustrate the effect of the fractional-order parameter α on the infection, we considered different values of it, that is, $\alpha \in [0.75, 0.85]$.

The disease-free equilibrium $E^0 = (\frac{\lambda}{\mu}, 0, 0) = (1.6667\times10^8, 0, 0)$ is unstable when $\Re_0 > 1$ as seen in Figs. 1–3 for different δ and asymptotically stable when $\Re_0 < 1$ as seen in Figs. 4 and 5. As it is quite clear from Fig. 6, in the fractional-order case, the peak of the infection is reduced, however, the disease takes a longer time to be eradicated.

6. Conclusion and future work

In this work, a model for HBV infection with a fractional-order derivative is introduced as a generalization of an integer-order model. The positivity of the solution is proved. Local asymptotic stability of fixed point is discussed. Numerical simulations are performed to confirm the analysis by applying PECE method. Since fractional-order models possess memory, FODE gives us a more realistic way to model viral dynamics. Indeed, the global asymptotic behavior of FODE is still open since the chain rule is not valid there and this will be our future work.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments which improved the manuscript.

References

[1] M.A. Nowak, C.R.M. Bangham, Population dynamics of immune responses to persistent viruses, Science. 272 (1996) 74–79.
[2] K. Wang, W. Wang, Propagation of HBV with spatial dependence, Math. Biosci. 210 (2007) 78–95.
[3] S. Bonhoeffr, J.M. Coffin, M.A. Nowak, Human immunodeficiency virus drug therapy and virus load, J. Virol. 71 (1997) 3275–3278.
[4] M.Y. Li, H. Shu, Global dynamics of an in-host viral model with intracellular delay, Bull. Math. Biol. 72 (2010) 1492–1505.
[5] S.A. Gourley, Y. Kuang, J.D. Nagy, Dynamics of a delay differential equation model of hepatitis b virus infection, J. Biol. Dyn. 2 (2008) 140–153.
[6] K. Wang, W. Wang, S. Song, Dynamics of an HBV model with diffusion and delay, J. Theor. Biol. 253 (2008) 36–44.
[7] S.M. Ciupe, R.M. Ribeiro, P.W. Nelson, A.S. Perelson, Modeling the mechanisms of acute hepatitis b virus infection, J. Theor. Biol. 247 (2007) 23–35.
[8] Y. Ji, L. Min, Y. Ye, Global analysis of a viral infection model with application to HBV infection, J. Biol. Syst. 18 (2010) 325–337.
[9] C. Vargas-De-León, Stability analysis of a model for HBV infection with cure of infected cells and intracellular delay, Appl. Math. Comput. 219 (2012) 389–398.
[10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006.
[11] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, 198, Academic Press, New York, 1999.
[12] A.M.A. El-Sayed, A.E.M. El-Mesiry, H.A.A. EL-Saka, On the fractional-order logistic equation, Appl. Math. Lett. 20 (2007) 817–823.
[13] I.S. Jesus, J.A.T. Machado, J.B. Cunha, Fractional electrical impedances in botanical elements, J. Vib. Control 14 (2008) 1389–1402.
[14] I.S. Jesus, J.A.T. Machado, J.B. Cunha, Fractional-order electrical impedance of fruits and vegetables, in: Proceedings of the 25th IASTED International Conference Modeling, Identification, and Control, Spain, 2006, pp. 489–494.
[15] V.D. Djordjevic, J. Jaric, B. Fabry, J.J. Fredberg, D. Stamenovic, Fractional derivatives embody essential features of cell rheological behavior, Ann. Biomed. Eng. 31 (2003) 692–699.
[16] A.M.A. El-Sayed, Nonlinear functional-differential equations of arbitrary orders, Nonlinear Anal. 33 (1998) 181–186.
[17] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, Fract. Calc. Appl. Anal. 5 (2002) 367–386.
[18] S. Das, Functional Fractional Calculus for System Identification and Controls, Springer, 2007.
[19] A.M.A. El-Sayed, F.M. Gaafar, Fractional-order differential equations with memory and fractional-order relaxation oscillation model, Pure Math. Appl. 12 (2001) 296–310.
[20] A.M.A. El-Sayed, F.M. Gaafar, H.H. Hashem, On the mayimal and minimal solutions of arbitrary orders nonlinear functional integral and differential equations, Math. Sci. Res. J. 8 (2004) 336–348.
[21] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional-order, in: Fractals and Fractional Calculus in Continuum Mechanics, Springer, 1997, pp. 223–276.
[22] H. Sun, V. Abdelwahed, B. Onaral, Linear approximation for transfer function with a pole of fractional-order, IEEE Trans. Autom. Control 29 (1984) 441–444.
[23] K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, J. Nonlinear Dyn. 29 (2002) 3–22.
[24] M.S. Tavazoei, M. Haeri, Unreliability of frequency domain approximation in recognizing chaos in fractional-order systems, IET Signal. Proc. 1 (2007) 171–181.
[25] M.S. Tavazoei, M. Haeri, Limitation of frequency domain approximation for detecting chaos in fractional-order systems, J. Nonlinear Anal. 69 (2008) 1299–1320.
[26] V.D. Gejji, S. Bhalekar, P. Gade, Dynamics of fractional-order Chen systems with delays, Pramana J. Phys. 79 (2012) 61–69.
[27] R. Hilfer, Applications of fractional calculus in physics, World Sci. 128 (2000) 1021–1032.
[28] A.A. Stanislavsky, Memory effects and macroscopic manifestation of randomness, Phys. Rev. E. 61 (2008) 4752–4759.
[29] A.A.M. Arafa, S.Z. Rida, M. Khalil, The effect of anti-viral drug treatment of human immunodeficiency, Appl. Math. Model. 37 (2013) 2189–2196.
[30] A.A.M. Arafa, S.Z. Rida, M. Khalil, Fractional modeling dynamics of HIV and CD4+ t-cells during primary infection, Nonlinear Biomed. Phys. 6 (1) (2012).
[31] A.A.M. Arafa, S.Z. Rida, M. Khalil, A fractional-order model of HIV infection: numerical solution and comparisons with data of patients, Int. J. Biomath. 7 (2014).
[32] S. Crotty, R. Ahmed, Immunological memory in humans, Semin. Immunol. 16 (2004) 197–203.
[33] A.A. Elsadany, A.E. Matouk, Dynamical behaviors of fractional-order lotka?volterra predator?prey model and its discretization, J. Appl. Math. Comput. 49 (2015) 269–283.
[34] A.E. Matouk, Dynamical behaviors, linear feedback control and synchronization of the fractional order liu system, J. Nonlinear Syst. Appl. 1 (2010) 135–140.
[35] A.S. Hegazi, E. Ahmed, A.E. Matouk, The effect of fractional order on synchronization of two fractional order chaotic and hyperchaotic systems, J. Fractional Calc.Appl. 1 (2011) 1–15.
[36] A.E. Matouk, Chaos synchronization of a fractional-order modified van der pol?duffing system via new linear control, backstepping control and takagi?sugeno fuzzy approaches, Complexity 21 (2015) 116–124.
[37] A.E. Matouk, A.A. Elsadany, Dynamical analysis, stabilization and discretization of a chaotic fractional-order GLV model, Nonlinear Dyn. 85 (2016) 1597–1612.
[38] A.M.A. El-Sayed, On the existence and stability of positive solution for a nonlinear fractional-order differential equation and some applications, Alex. J. Math. 1 (2010) 1–10.
[39] D. Matignon, Stability result on fractional differential equations with applications to control processing, Comput. Eng. Syst. Appl. 2 (1996) 963–968.
[40] E.J. Routh, A Treatise on the Stability of a Given State of Motion: Particularly Steady Motion, Macmillan, 1877.
[41] A. Hurwitz, Ueber die bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt Math. Ann. 46 (1895) 273–284.
[42] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002) 229–248.
[43] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, On some Routh–Hurwitz conditions for fractional-order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems, Phys. Lett. A 358 (2006) 1–4.
[44] K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. 29 (2002) 3–22.