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Complete paranormed Orlicz Lorentz sequence spaces over n-normed spaces

Renu Anand, Kuldip Raj*

Department of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K India

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ABSTRACT

In the present paper we introduce and study some Orlicz–Lorentz sequence space over real n-normed space as a base space. We make an effort to study some algebraic and topological properties of the space. Some inclusion relations are also establish in the paper. Finally, we show that the space is complete and paranormed.

1. Introduction and preliminaries

The theory of sequence spaces has several important applications in many branches of mathematical analysis. The classical sequence space l_2 is extended to l_p, $1 < p < \infty$ by Reisz [1] and further its generalizations to Lorentz sequence space $l_{p,\,q}$, for $0 < p, q < \infty$ due to Hardy and Littlewood [2]. In recent years, many mathematicians are interested to study the theory of sequence spaces generated by Cesàro mean, Orlicz function, Musielak–Orlicz function or using the combination of these. In [3] Lim and Lee introduced the Cesàro–Orlicz sequence space. Later on, Cui et al. [4] have studied and discussed its basic topological properties as well as geometric properties.

Let w be the set of all sequences of real numbers and let l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $||x||_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers. Let $\mathbb{R}$ and $\mathbb{N}$ stand for the set of real numbers and the set of all natural numbers, respectively.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $R > 0$ such that $M(2u) \le RM(u)$, $u \ge 0$.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach Space theory. Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \right.$$
$$\left. \text{for some } \rho > 0 \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

It is shown in [5] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$.

A sequence $\mathcal{M} = (M_i)$ of Orlicz functions is called a *Musielak-Orlicz function* (see [6,7]). For more details about sequence spaces (see [8–14]) and references therein.

Let $n \in \mathbb{N}$ and X be a linear space over the field of real numbers $\mathbb{R}$ of dimension d, where $d \ge n \ge 2$. A real valued function $|| \cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,

* Corresponding author.
E-mail addresses: renuanand71@gmail.com (R. Anand), kuldeepraj68@rediffmail.com, kuldipraj68@gmail.com (K. Raj).

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(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha|\ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$ and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$ is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{R}$. For more details on 2-normed spaces and n-normed spaces [15–19].

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by $||x_1, x_2, \cdots, x_{n-1}||_\infty =$

$$\max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if
$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0$ for every $z_1, \cdots, z_{n-1} \in X$. A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if $\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0$ for every $z_1, \cdots, z_{n-1} \in X$. If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let X be a linear space. A function $p : X \to \mathbb{R}$ is called *paranorm* if

(1) $p(x) \geq 0$ for each $x \in X$,
(2) $p(-x) = p(x)$ for any $x \in X$,
(3) $p(x + y) \leq p(x) + p(y)$ for any $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total paranorm* on X and the pair (X, p) is called a *total paranormed space*. It is well known that the metric of any linear metric space is given by some total paranorm (see [20] Theorem 10.4.2, pp. 183).

The Lorentz space was introduced by G. G. Lorentz in [21,22]. This space play an important role in the theory of Banach space.

Let $(E, ||.||)$ be a Banach space. The Lorentz sequence space $l(p, q, E)$ (or $l_{p,q}(E)$) for $1 \leq p, q \leq \infty$ is the collection of all sequences $\{a_i\} \in c_0(E)$ such that

$$||\{a_i\}||_{p,q} = \begin{cases} \left(\sum_{i=1}^{\infty} (i)^{\frac{q}{p}-1} ||a_{\phi(i)}||^q\right)^{\frac{1}{q}}, \\ 1 \leq p \leq \infty, 1 \leq q < \infty; \\ \sup_i\ (i)^{\frac{1}{p}} ||a_{\phi(i)}||, \\ 1 \leq p < \infty, q = \infty. \end{cases}$$

is finite, where $\{||a_{\phi(i)}||\}$ is non-increasing rearrangement of $\{||a_i||\}$ (it can be interpret that the decreasing rearrangement $\{||a_{\phi(i)}||\}$ is obtained by rearranging $\{||a_i||\}$ in decreasing order). This space was introduced by Miyazaki in [23] and examined by Kato in [24].

A weight sequence $\nu = \nu(i)$ is a positive decreasing sequence such that $\nu(1) = 1, \lim_{i\to\infty} \nu(i) = 0$ and $\lim_{i\to\infty} V(i) = \infty$, where $V(i) = \sum_{n=1}^{i}$ for every $i \in \mathbb{N}$. Popa [25] defined the generalized Lorentz sequence space $d(\nu, p)$ for $0 < p < \infty$ as follows

$$d(\nu, p) = \left\{x = \{x_i\} \in w : ||x_i||_{\nu,p} = \sup_\pi \left(\sum_{i=1}^{\infty} |x_{\pi(i)}|^p \nu(i)\right)^{\frac{1}{p}} < \infty\right\},$$

where π ranges over all permutations of the positive integers and $\nu = \nu(i)$ is a weight sequence . We know that $d(\nu, p) \subset c_0$ and hence for each $x \in d(\nu, p)$ there exists a non-increasing rearrangement $\{x^*\} = \{x_i^*\}$ of x and

$$||x_i||_{\nu,p} = \left(\sum_{n=1}^{\infty} |x_i^*|^p \nu(i)\right)^{\frac{1}{p}}$$

(See [25,26]).

Lemma 1.1 (*Hardy, Littlewood and Pólya* [27]). *Let $\{c_i^*\}$ and $\{^*c_i\}$ be the non-increasing and non-decreasing rearrangements of a finite sequence $\{c_i\}_{1 \leq i \leq n}$ of positive numbers, respectively. Then for two sequences $\{a_i\}_{1 \leq i \leq n}$ and $\{b_i\}_{1 \leq i \leq n}$ of positive numbers, we have*

$$\sum_i a_i^*.{}^*b_i \leq \sum_i a_i.b_i \leq \sum_i a_i^*.b_i^*.$$

Lemma 1.2 (*Kato* [24]). *Let $\{x_i^{(\mu)}\}$ be an X–valued double sequence such that $\lim_{i\to\infty} x_i^{(\mu)} = 0$ for each $\mu \in \mathbb{N}$ and let $\{x_i\}$ be an X–valued sequence such that $\lim_{\mu\to\infty} x_i^{(\mu)} = x_i$ (uniformly in i). Then $\lim_{i\to\infty} x_i = 0$ and for each $i \in \mathbb{N}$*

$$||x_{\phi(i)}|| \leq \lim_{\mu\to\infty} ||x^{(\mu)}_{\phi_\mu(i)}||,$$

where $\{||x_{\phi(i)}||\}$ and $\{||x^{(\mu)}_{\phi_\mu(i)}||\}_i$ are the non-increasing rearrangements of $\{||x_i||\}$ and $\{||x_i^{(\mu)}||\}_i$, respectively.

Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions, $p = (p_i)$ be a bounded sequence of positive real numbers, $\nu = (\nu_i)$ be a weight sequence and $u = (u_i)$ be a sequence of strictly positive real numbers. Also assume that $\{||x_{\phi(i)}||\}$ denotes the non-increasing rearrangements of the sequences $\{||x_i||\}$. Then we define the following space: $L(\mathcal{M}, u, \nu, p, \|., \ldots, .\|) = \Big\{x = x_i \in w :$ $\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big|\Big|\frac{x_{\phi(i)}\nu(i)}{\varrho}, z_1, \ldots, z_{n-1}\Big|\Big|\Big)\Big]^{p_i} < \infty,$ for some $\varrho > 0\Big\}$.

The following inequality will be used throughout the paper. If $0 \leq p_i \leq \sup p_i = H, K = \max(1, 2^{H-1})$ then

$$|a_i + b_i|^{p_i} \leq K\{|a_i|^{p_i} + |b_i|^{p_i}\} \tag{1}$$

for all i and $a_i, b_i \in \mathbb{R}$. Also $|a|^{p_i} \leq \max(1, |a|^H)$ for all $a \in \mathbb{R}$.

The main purpose of this paper is to introduce and study Lorentz sequence space defined by sequence of Orlicz functions over n-normed spaces. We study some topological and algebraic properties of the space and also show that the space is complete.

2. Main results

Theorem 2.1. *Suppose $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions, $p = (p_i)$ be a bounded sequence of positive real numbers, $\nu = (\nu_i)$ be a weight sequence and $u = (u_i)$ be a sequence of strictly positive real numbers. Then the sequence space $L(\mathcal{M}, u, \nu, p, \|., \ldots, .\|)$ is a linear over the real field $\mathbb{R}$.*

Proof. Suppose $x, y \in L(\mathcal{M}, u, \nu, p, \|., \ldots, .\|)$ and let $\{||x_{\phi(i)}||\}$, $\{||y_{\phi(i)}||\}$ be the non-increasing rearrangements of the sequences $\{||x_i||\}$, $\{||y_i||\}$ respectively and $\alpha, \beta \in \mathbb{R}$. Then there exist positive numbers ϱ_1, ϱ_2 such that

$$\left\{\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big|\Big|\frac{x_{\phi(i)}\nu(i)}{\varrho_1}, z_1, \ldots, z_{n-1}\Big|\Big|\Big)\Big]^{p_i} < \infty, \text{ for some } \varrho_1 > 0\right\} \text{ and}$$

$$\left\{\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i} < \infty, \text{ for some } \varrho_2>0\right\}.$$

Let $\varrho_3=\max(2|\alpha|\varrho_1, 2|\beta|\varrho_2)$. Since $\mathcal{M}=(M_i)$ is non-decreasing and convex so by using inequality (1.1), we have

$$\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{(\alpha x_{\phi(i)}+\beta y_{\phi(i)})\nu(i)}{\varrho_3},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}$$
$$\leq \frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(|\alpha|\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_3},z_1,\ldots,z_{n-1}\right|\right| + |\beta|\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_3},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}$$
$$\leq \frac{1}{2n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_1},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i} + \frac{1}{2n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}$$
$$\leq K\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_1},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i} + K\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}$$
$$< \infty.$$

Thus, $\alpha x+\beta y\in L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$. This proves that $L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$ is a linear space. □

Theorem 2.2. *Suppose $\mathcal{M}=(M_i)$ be a sequence of Orlicz functions, $p=(p_i)$ be a bounded sequence of positive real numbers, $\nu=(\nu_i)$ be a weight sequence and $u=(u_i)$ be a sequence of strictly positive real numbers. Also assume that $\{||x_{\phi(i)}||\}$ denotes the non-increasing rearrangements of the sequences $\{||x_i||\}$. Then $L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$ is a paranormed space with the paranorm*

$$g(x)=\inf\left\{\varrho^{\frac{p_r}{K}} : \left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}} \leq 1; r\in\mathbb{N}\right\}$$

where $0<p_i\leq \sup p_i=H<\infty$ and $K=\max(1,H)$.

Proof. Clearly $g(x)\geq 0$ for $x\in L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$. Since $M_i(0)=0$, we get $g(0)=0$. Also $g(x)=g(-x)$. Let $x,y\in L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$. Then there exists positive numbers ϱ_1 and ϱ_2 such that

$$\left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_1},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}\leq 1 \text{ and}$$
$$\left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}\leq 1$$

where $\{||x_{\phi(i)}||\}$, $\{||y_{\phi(i)}||\}$ are the non-increasing rearrangements of the sequences $\{||x_i||\}$, $\{||y_i||\}$ respectively. Let $\varrho_3=2^{\frac{K}{h}}(\varrho_1+\varrho_2)$ where $h=\inf p_i>0$. Since $\mathcal{M}$ is non-decreasing convex function and ν is decreasing so by Lemma (1.1), we have

$$\left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{(x_{\phi(i)}+y_{\phi(i)})\nu(i)}{\varrho_3},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}$$
$$\leq\left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{2^{\frac{K}{h}}(\varrho_1+\varrho_2)},z_1,\ldots,z_{n-1}\right|\right| + \left|\left|\frac{y_{\phi(i)}\nu(i)}{2^{\frac{K}{h}}(\varrho_1+\varrho_2)},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}$$
$$\leq\left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[\frac{\varrho_1}{2^{\frac{K}{h}}(\varrho_1+\varrho_2)}M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_1},z_1,\ldots,z_{n-1}\right|\right|\right) + \frac{\varrho_2}{2^{\frac{K}{h}}(\varrho_1+\varrho_2)}M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}$$
$$\leq\left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[\frac{1}{2^{\frac{K}{h}}}M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_1},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}} + \left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[\frac{1}{2^{\frac{K}{h}}}M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}$$
$$=\frac{1}{2n}\left(\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_1},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}} + \frac{1}{2n}\left(\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}\leq 1.$$

Since ϱ_1, ϱ_2 and ϱ_3 are positive real numbers, we get

$$g(x+y)$$
$$=\inf\left\{\varrho_3^{\frac{p_r}{K}}>0 : \left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{(x_{\phi(i)}+y_{\phi(i)})\nu(i)}{\varrho_3},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}\leq 1; r\in\mathbb{N}\right\}$$
$$\leq\inf\left\{\varrho_1^{\frac{p_r}{K}}>0 : \left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{x_{\phi(i)}\nu(i)}{\varrho_1},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}\leq 1; r\in\mathbb{N}\right\}$$
$$+\inf\left\{\varrho_2^{\frac{p_r}{K}}>0 : \left(\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{y_{\phi(i)}\nu(i)}{\varrho_2},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\right)^{\frac{1}{K}}\leq 1; r\in\mathbb{N}\right\}$$
$$=g(x)+g(y).$$

Let $(x^m)=(x_i^m)$ be any sequence in $L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$ such that $g(x^m-x)\to 0$ as $m\to\infty$ and (λ^m) is a sequence of reals with $\lambda^m\to\lambda$ as $m\to\infty$. Then since the inequality

$$g(x^m)\leq g(x)+g(x^m-x)$$

holds by subadditivity of the function g, $\{g(x^m)\}$ is bounded. Taking into account this fact, we therefore derive the inequality

$$g(\lambda^m x^m-\lambda x)\leq|\lambda^m-\lambda|g(x^m)+|\lambda|g(x^m-x)$$

which tends to zero as $m\to\infty$. Hence, the scalar multiplication is continuous follows from the above inequality and thus proving the theorem. □

Theorem 2.3. *Suppose $\mathcal{M}=(M_i)$ be a sequence of Orlicz functions, $p=(p_i)$ be a bounded sequence of positive real numbers, $\nu=(\nu_i)$ be a weight sequence and $u=(u_i)$ be a sequence of strictly positive real numbers. Also assume that $\{||x_{\phi(i)}||\}$ denotes the non-increasing rearrangements of the sequences $\{||x_i||\}$. Then $L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$ is complete with respect to its paranorm.*

Proof. Let $(x^s)=\{x_i^s\}$ be any Cauchy sequence in the space $L(\mathcal{M},u,\nu,p,\|.,\ldots,.\|)$. Since (x^s) is a Cauchy sequence, we have $g(x^s-x^t)\to 0$ as $s,t\to\infty$. Then, we have

$$\frac{1}{n}\sum_{i=1}^{n}u_i\left[M_i\left(\left|\left|\frac{(x_{\phi(i)}^s-x_{\phi(i)}^t)\nu(i)}{\varrho},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}\to 0 \text{ as } s,t\to\infty$$

for all $i\in\mathbb{N}$ where $\{||x_{\phi(i)}^s-x_{\phi(i)}^t||\}$ denotes non-increasing rearrangement of $\{||x_i^s-x_i^t||\}$. Therefore, we have $\{x_i^s\}$ is a Cauchy sequence in $\mathbb{R}$ for fixed $i\in\mathbb{N}$. Since $\mathbb{R}$ is complete, we have $x_i^s\to x_i$ as $s\to\infty$ for each i and $\mathcal{M}=(M_i)$ is continuous. For $\epsilon>0$, there exists a natural number N such that

$$\frac{1}{n}\sum_{i=1;s,t>N}^{n}u_i\left[M_i\left(\left|\left|\frac{(x_{\phi(i)}^s-x_{\phi(i)}^t)\nu(i)}{\varrho},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}<\epsilon.$$

Since for any fixed natural number D, we have

$$\frac{1}{n}\sum_{i\leq D;s,t>N}^{n}u_i\left[M_i\left(\left|\left|\frac{(x_{\phi(i)}^s-x_{\phi(i)}^t)\nu(i)}{\varrho},z_1,\ldots,z_{n-1}\right|\right|\right)\right]^{p_i}<\epsilon$$

by letting $t\to\infty$ in the above expression, we obtain

$$\frac{1}{n}\sum_{i\leq D; s,t>N}^{n} u_i\Big[M_i\Big(\Big\|\frac{(x^s_{\phi(i)}-x_{\phi(i)})v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i} < \epsilon.$$

Since D is arbitrary, by letting $D\to\infty$ we obtain

$$\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big\|\frac{(x^s_{\phi(i)}-x_{\phi(i)})v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i} < \epsilon.$$

Then, $g(x^s-x)\to 0$ as $t\to\infty$. Since $L(\mathcal{M},u,v,p,\|.,\ldots,.\|)$ is a linear space, we get $x=\{x(i)\}\in L(\mathcal{M},u,v,p,\|.,\ldots,.\|)$. This completes the proof. □

Theorem 2.4. *Let $\mathcal{M}=(M_i)$ and $\mathcal{S}=(S_i)$ be two sequences of Orlicz functions. Then*

$L(\mathcal{M},u,v,p,\|.,\ldots,.\|)\cap L(\mathcal{S},u,v,p,\|.,\ldots,.\|)\subseteq L(\mathcal{M}+\mathcal{S},u,v,p,\|.,\ldots,.\|)$.

Proof. Let $x\in L(\mathcal{M},u,v,p,\|.,\ldots,.\|)\cap L(\mathcal{S},u,v,p,\|.,\ldots,.\|)$. Then

$$\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho_1}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i} < \infty, \text{ for some } \varrho_1>0$$

and

$$\frac{1}{n}\sum_{i=1}^{n} u_i\Big[S_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho_2}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i} < \infty, \text{ for some } \varrho_2>0\,.$$

Let $\varrho=\max\{\varrho_1,\varrho_2\}$. The result follows from the inequality,

$$\frac{1}{n}\sum_{i=1}^{n} u_i\Big[(M_i+S_i)\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i}$$

$$=\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i} +\frac{1}{n}\sum_{i=1}^{n} u_i\Big[S_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i}$$

$$\leq K\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i} +K\frac{1}{n}\sum_{i=1}^{n} u_i\Big[S_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i}$$

$<\infty$, where $K=\max\{1,2^{H-1}\}$. Therefore, $x=(x_i)\in L(\mathcal{M}+\mathcal{S},u,v,p,\|.,\ldots,.\|)$. □

Theorem 2.5. *Suppose $\mathcal{M}=(M_i)$ be a sequence of Orlicz functions. Suppose that $\beta=\lim_{t\to\infty}\frac{M_i(t)}{t}<\infty$.*

Then $L(\mathcal{M},u,v,p,\|.,\ldots,.\|)\subset L(u,v,p,\|.,\ldots,.\|)$.

Proof. Let $\beta>0$. By definition of β, we have $M_i(t)\geq\beta t\ \forall\ t\geq 0$. Since $\beta>0$, we have $t\leq\frac{1}{\beta}M_i(t)\ \forall\ t\geq 0$. Let $x\in L(\mathcal{M},u,v,p,\|.,\ldots,.\|)$. Thus

$$\frac{1}{n}\sum_{i=1}^{n} u_i\Big[\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big]^{p_i}$$

$$\leq\frac{1}{\beta}\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i}$$

$<\infty$.

Thus, $x\in L(u,v,p,\|.,\ldots,.\|)$. This completes the proof. □

Theorem 2.6. *Let $\mathcal{M}=(M_i)$ and $\mathcal{S}=(S_i)$ be two sequences of Orlicz functions. Then*

$L(\mathcal{S},u,v,p,\|.,\ldots,.\|)\subset L(\mathcal{M},u,v,p,\|.,\ldots,.\|)$ if and only if $\sup_i(\frac{M_i(t)}{S_i(t)})<\infty$.

Proof. Let $\sup_i(\frac{M_i(t)}{S_i(t)})=N$. Then $M_i(t)\leq NS_i(t)$ for all $i\geq 0$. Let $x\in L(\mathcal{S},u,v,p,\|.,\ldots,.\|)$. Then we have

$$\frac{1}{n}\sum_{i=1}^{n} u_i\Big[M_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i}$$

$$\leq\frac{1}{n}\sum_{i=1}^{n} u_i\Big[NS_i\Big(\Big\|\frac{x_{\phi(i)}v(i)}{\varrho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_i}$$

$<\infty$.

Thus, $x\in L(\mathcal{M},u,v,p,\|.,\ldots,.\|)$. Hence the proof is complete. □

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