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Short Communication

A note on the convergence rate of Kumar–Singh–Srivastava methods for solving nonlinear equations

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ABSTRACT

In the present article, it is shown that both the methods presented in Kumar et al. (2013) do not possess the order of convergence as claimed. One of the two methods, derivative involved method possesses the convergence rate of eighth order whereas the other derivative free method possesses sixth order convergence. The theoretical convergence rate is also validated by computational order of convergence.

1. Introduction

In the article [1], Kumar et al. presented ninth (with derivative) and seventh-order (derivative free) methods using two-step optimal fourth order methods followed by a step of modified Newton's method. Here, we denote these methods by KSSWD9 and KSSDF7, respectively. The derivative free scheme was obtained by approximating involved derivatives in the previous scheme by the expressions of divided differences. Here, we show that both the methods do not possess the order convergence as mentioned in the same article by the authors. The methods are given below: **I. KSSWD9:** The method is based on optimal fourth order method followed by a Newton-like step and is given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \left[1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2\right]\frac{f(y_n)}{f'(y_n)}, \\ x_{k+1} &= z_n - \left[1 + 2\left(\frac{f(y_n)}{f(x_n)}\right)^2 - 4\frac{f(z_n)}{f(y_n)}\right]\frac{f(z_n)}{f'(y_n)}, \end{aligned} \tag{1.1}$$

II. KSSDF7: Derivative free scheme was obtained by approximating the derivatives by means of divided differences. The method is given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[w_n, x_n]}, \\ z_n &= y_n - \left[1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2\right]\frac{f(y_n)f[w_n, x_n]}{f[y_n, x_n]f[y_n, w_n]}, \\ x_{k+1} &= z_n - \left[1 + 2\left(\frac{f(y_n)}{f(x_n)}\right)^2 - 4\frac{f(z_n)}{f(y_n)}\right]\frac{f(z_n)f[w_n, x_n]}{f[y_n, x_n]f[y_n, w_n]}, \end{aligned} \tag{1.2}$$

where $w = x_n + f(x_n)$.

2. Convergence rate

Here we prove that the convergence rate in the above two schemes is not nine and seven. In fact it is eight and six, respectively.

Theorem 2.1. *Let the function f(x) be sufficiently differentiable in a neighborhood of its root ξ. If the initial approximation x_0 is sufficiently close to ξ, then the rate of convergence of KSSWD9 and KSSDF7 is eight and six, respectively.*

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Proof. Convergence rate of KSSWD9: Let $e_n = x_n - \xi$, be the error in the nth iterate and $c_h = \frac{f^{(h)}(\xi)}{f'(\xi)h!}$, $h = 1, 2, 3\ldots$. We provide the Taylor's series expansion of each term involved in (1.1). By Taylor expansion around the simple root in the n^{th} iterate, we have

$$f(x_n) = f'(\xi)[e_n + c_2e_n^2 + \ldots + O(e_n^{10})], \tag{2.1}$$

and

$$f'(x_n) = f'(\xi)[1 + 2c_2e_n + \ldots + O(e_n^9)]. \tag{2.2}$$

Using these expressions in the first sub-step of (1.1), we obtain

$$y_n = \xi + c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + \ldots + O(e_n^9). \tag{2.3}$$

At this time, we should expand $f(y_n)$ and $f'(y_n)$ around the exact root ξ by taking into consideration (2.3). Accordingly, we have

$$f(y_n) = f'(\xi)(c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + \ldots + O(e_n^9)). \tag{2.4}$$

and

$$f'(y_n) = f'(\xi)(1 + 2c_2^2e_n^2 + (-4c_2^3 + 4c_2c_3)e_n^3 + \ldots + O(e_n^9)). \tag{2.5}$$

Using the Eqs. (2.1) and (2.3)–(2.5) in the second sub-step of (1.1), we can find

$$z_n = \xi + (4c_2^4 - 2c_2^2c_3)e_n^5 - c_2(30c_2^4 - 39c_2^2c_3 + 8c_3^2 + 3c_2c_4)e_n^6 + \ldots + O(e_n^9). \tag{2.6}$$

By virtue of the above equation, we have

$$f(z_n) = f'(\xi)((4c_2^4 - 2c_2^2c_3)e_n^5 - c_2(30c_2^4 - 39c_2^2c_3 + 8c_3^2 + 3c_2c_4)e_n^6 + \ldots + O(e_n^9)).$$

Finally, using (2.1) and (2.4)–(2.7), in the last sub-step of considered method (1.1), we get the final error expression which is given by

$$e_{n+1} = 24c_2^3(-2c_2^2 + c_3)e_n^8 + O(e_n^9). \tag{2.7}$$

This shows that the convergence rate of KSSWD9 is at least eight.

Convergence rate of KSSDF7: By virtue of the expression (2.1), one can write

$$w_n = (1 + f'(\xi))e_n + f'(\xi)[c_2e_n^2 + c_3e_n^3 + \ldots + O(e_n^7)], \tag{2.8}$$

and then

$$f(w_n) = f'(\xi)[(1 + f'(\xi))e_n + (f'(\xi)c_2 + (1 + f'(\xi))^2c_2)e_n^2 + \ldots + O(e_n^7)]. \tag{2.9}$$

With the help of Eqs. (2.9), (2.1) and (2.8), we obtain the Taylor's series expansion of $f[w_n, x_n] = \frac{f(w_n) - f(x_n)}{w_n - x_n}$ as follows:

$$f[w_n, x_n] = f'(\xi)[(2c_2 + f'(\xi)c_2)e_n + (f'(\xi)c_2^2 + 3c_3 + 3f'(\xi)c_3 + f'(\xi)^2c_3)e_n^2 + \ldots + O(e_n^9)]. \tag{2.10}$$

By putting the values of Eqs. (2.1) and (2.10) in the first sub-step of Eq. (1.2), we attain

$$y_n = \xi + (c_2 + f'(\xi)c_2)e_n^2 + (-2c_2^2 - 2f'(\xi)c_2^2 - f'(\xi)^2c_2^2 + 2c_3 + 3f'(\xi)c_3 + f'(\xi)^2c_3)e_n^3 + \ldots + O(e_n^9). \tag{2.11}$$

On the other hand, we find

$$f(y_n) = f'(\xi)[(c_2 + f'(\xi)c_2)e_n^2 + (-2c_2^2 - 2f'(\xi)c_2^2 - f'(\xi)^2c_2^2 + 2c_3 + 3f'(\xi)c_3 + f'(\xi)^2c_3)e_n^3 + \ldots + O(e_n^9)]. \tag{2.12}$$

Furthermore, we can also obtain

$$f[y_n, x_n] = f'(\xi)[1 + c_2e_n + (c_2^2 + f'(\xi)c_2^2 + c_3)e_n^2 + \ldots + O(e_n^9)], \tag{2.13}$$

Table 1
Computational order of convergence.

Method	*n*	*COC*
KSSWD9	3	8.0000
KSSDF7	4	6.0000

and

$$f[y_n, w_n] = f'(\xi)[1 + (1 + f'(\xi))c_2e_n + (c_2^2 + 2f'(\xi)c_2^2 + c_3 + 2f'(\xi)c_3 + f'(\xi)^2c_3)e_n^2 + \ldots + O(e_n^9)]. \tag{2.14}$$

By using the Eqs. (2.1), (2.12), (2.10), (2.13) and (2.14) in the second sub-step of method (1.2), we attain

$$z_n = \xi + -(1 + f'(\xi))^2c_2((-1 + f'(\xi))c_2^2 + c_3)e_n^4 + \ldots + O(e_n^9). \tag{2.15}$$

By virtue of the above equation, we acquire

$$f(z_n) = -f'(\xi)[(1 + f'(\xi))^2c_2((-1 + f'(\xi))c_2^2 + c_3)e_n^4 + \ldots + O(e_n^9)]. \tag{2.16}$$

Lastly making using of the required relations which are already obtained above and then simplifying, the final error expression is given by

$$e_{n+1} = [(1 + f'(\xi))^3c_2((5 - 11f'(\xi) + 6f'(\xi)^2)c_2^4 + (-10 + 11f'(\xi))c_2^2c_3 + 5c_3^2)]e_n^6 + O(e_n^7),$$

which confirms the correct convergence rate of KSSDF7. □

3. Numerical testing

In order to verify the theoretical rate of convergence of KSSWD9 and KSSDF7, we calculate the computational order of convergence (COC) using the formula [2]

$$COC = \frac{log|f(x_n)/f(x_{n-1})|}{log|f(x_{n-1})/f(x_{n-2})|}. \tag{3.1}$$

We consider the test function with corresponding zero and its initial approximation is given as

$$f(x) = e^{(-x^2+x+2)} - cosx + x^3 + 1, \quad \xi = -1.000000000000000, \quad x_0 = -0.5.$$

The computations are performed in the programming package Mathematica [3] using multiple-precision arithmetic. For each scheme, we calculate the number of iterations (n) needed to converge to the solution such that

$$|x_n - x_{n-1}| \leq 10^{-15}. \tag{3.2}$$

The nonlinear equation, initial guess and error of tolerance are taken same as considered in the paper [1]. The necessary iterations n required to converge to the zero of considered function and the computational order of convergence (COC) are displayed in Table 1. It is clear that the calculated values of computational order of convergence (COC) verify the theoretical rate of convergence as proved in theoretical study.

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