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Original Article

Solutions of fractional order electrical circuits via Laplace transform and nonstandard finite difference method

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ABSTRACT

In this article, fractional linear electrical systems are investigated. Analytical solutions of the fractional models are derived using Laplace transform method. Also, numerical simulations using Grünwald–Letnikov definition are proposed. Comparisons between fractional and classical electrical systems are illustrated using Laplace transform and nonstandard finite difference method.

1. Introduction

In recent years, fractional calculus has an interest to mathematicians as it has many engineering applications [1–9,28–31]. It provides excellent instruments for the description of memory and properties of various materials and processes. Non-integer derivatives play an important role in modeling of electrical circuits that contains super capacitors and super inductors. Moreover, in such electrical circuits, singular linear systems were addressed in many papers and books [10–13]. The charging and discharging processes of different capacitors in R-C electrical circuits theoretically and experimentally are considered in [14]. Also the authors investigated the nonlocal behavior in these processes that arising from the time fractality via fractional calculus. The existence and uniqueness of the solution of an RLC circuit model were discussed and the solution of that model was obtained by Adomian Decomposition Method (ADM) and Laplace Transform method in [15]. The solution of a new class of singular fractional electrical circuits using Weierstrass regular pencil decomposition and Laplace transform are proposed in [16]. In this article, we will investigate analytical and numerical solutions for both R-L and R-C electrical circuit models using Laplace transform method and nonstandard finite difference methods (NSFDM).

This article is organized as: basic definitions and some properties of fractional calculus are given in Section 2. In Section 3 analytical solutions for different electrical circuits with fractional order derivatives and numerical solution using NSFDMs are derived, while some illustrative examples with their solutions are given in Section 4. Finally a brief conclusion is given in Section 5.

2. Preliminaries

In this section, we introduce some basic definitions and functions that have important rules in fractional calculus which are further used in this article, [17–25].

Definition 2.1. Caputo fractional derivative

The fractional derivative of a function $f(x)$ of non-integer order α is given as,

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$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-1)^{n-\alpha-1} \frac{d^n}{dt^n} f(t) dt, \; n-1 < \alpha < n. \tag{2.1}$$

Definition 2.2. Riemann–Liouville fractional derivative

The fractional derivative of the function $f(x)$ is given as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt, \; \alpha > 0. \tag{2.2}$$

Definition 2.3. Grünwald–Letnikov fractional derivative

In 1867, Grünwald–Letnikov defined the fractional derivative of a function $f(x)$ as:

$$D^{\alpha} f(x) = \lim_{N \to \infty} \frac{1}{h^{\alpha}} \sum_{j=0}^{N} C_j^{\alpha} f(x - jh), n-1 < \alpha < n, \; h = \frac{1}{N}, \tag{2.3}$$

where $C_0^{\alpha} = 1$ and $C_j^{\alpha} = (1 - \frac{1+\alpha}{j}) C_{j-1}^{\alpha}$.

The Caputo fractional derivative is not equivalent to the Riemann– Liouville fractional derivative and they are related by $D^{\alpha} f(t) = {}^{R}D^{\alpha}(f(t) - f(0))$ for $0 < \alpha < 1$. If the initial condition $f(0) = 0$, then we have that $D^{\alpha} f(t) = {}^{R}D^{\alpha} f(t)$ and the Grünwald–Letnikov fractional derivatives is equivalent to the Caputo fractional derivative, see [19,30].

Definition 2.4. Laplace Transform

If a function $f(t)$ is of exponential order α and is a piece-wise continuous on real line, then Laplace transform of $f(t)$ for $s > \alpha$ is defined by:

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt. \tag{2.4}$$

Laplace transform of Caputo derivative is:

$$L[D^{\alpha} f(t)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n = \lceil \alpha \rceil. \tag{2.5}$$

Laplace transform of the convolution of two functions:

The convolution of two functions $f(t)$ and $g(t)$ is defined by:

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau, \tag{2.6}$$

and the Laplace transform of the convolution of two functions $f(t), g(t)$ is defined by:

$$L\{f(t) * g(t)\} = L\left\{ \int_0^t f(t-\tau) g(\tau) d\tau \right\} = F(s) G(s), \tag{2.7}$$

where $F(s)$ and $G(s)$ are the Laplace transform of $f(t)$ and $g(t)$ respectively.

In this article, we will generalize some electrical circuits models to a fractional order system of order α in sense of caputo definition. We assume that the voltage across the inductor v_l and the capacitor current i_c are:

$$v_l = l \, \frac{d^{\alpha} i_l}{dt^{\alpha}}, \tag{2.8}$$

$$i_c = c \frac{d^{\alpha} v_c}{dt^{\alpha}}, \tag{2.9}$$

where $\frac{d^{\alpha}}{dt^{\alpha}} = D^{\alpha}$ is the fractional derivative operator in the sense of Caputo derivative. Also l is the inductance, c is the capacitance and i_l *and* v_c are the inductor current and the capacitor voltage respectively.

Definition 2.5. The function $E_t(\alpha, a)$ [18,19]

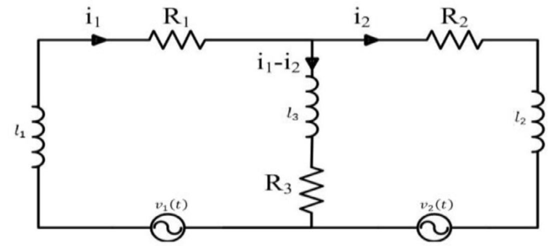

Fig. 1. Electrical circuit of application 1.

The function $E_t(\alpha, a)$ is a solution of the ordinary differential equation

$$(D - a)\; y = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \; Re(\alpha) > 0 \tag{2.10}$$

and it is defined by:

$$E_t(\alpha, a) = t^{\alpha} \; e^{a\,t} \; \gamma^*(\alpha, a\, t), \tag{2.11}$$

where $\gamma^*(\alpha, a\,t)$ is the incomplete gamma function defined in [19] as

$$\gamma^*(\alpha, z) = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha + k + 1)}. \tag{2.12}$$

If we replace a by ia in (2.11), then

$$E_t(\alpha, ia) = C_t(\alpha, a) + i\, S_t(\alpha, a), \tag{2.13}$$

where $C_t(\alpha, a) = t^{\upsilon} \; \sum_{k\; even}^{\infty} \frac{(-1)^{k/2} (a\; t)^k}{\Gamma(\alpha+k+1)}$, and $S_t(\alpha, a) = t^{\alpha} \; \sum_{k\; odd}^{\infty} \frac{(-1)^{(k-1)/2} \; (a\; t)^k}{\Gamma(\alpha+k+1)}$.

The function $S_t(\alpha, a)$ or $C_t(\alpha + 1, a)$ is a solution of the following ordinary differential equation

$$\left(D^2 + a^2\right) y = \frac{a\; t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0.$$

3. Fractional linear electrical systems

In this section, we will introduce some applications for some electrical circuits with fractional order $\alpha, 0 < \alpha \le 1$. Analytical solutions are briefly obtained for each application.

3.1. Analytical solutions of fractional systems

Firstly, we consider the following applications:

Application 1. Consider the electrical circuit shown in Fig. 1 with given resistances R_1, R_2, R_3, inductances l_1, l_2, l_3 and voltage sources $v_i(t), \; i = 1, 2$.

Using Kirchhoff's voltages and currents laws and considering (2.8) and (2.9) we get:

$$v_1(t) = i_1 R_1 + l_1 \frac{d^{\alpha} i_1}{dt^{\alpha}} + l_3 \frac{d^{\alpha}(i_1 - i_2)}{dt^{\alpha}} + \; R_3(i_1 - i_2), \tag{3.1}$$

$$v_2(t) + R_3(i_1 - i_2) + l_3 \frac{d^{\alpha}(i_1 - i_2)}{dt^{\alpha}} = i_2 R_2 + l_2 \frac{d^{\alpha} i_2}{dt^{\alpha}}. \tag{3.2}$$

We can write Eqs. (3.1) and (3.2) in the following form

$$\begin{bmatrix} l_1 + l_3 & -l_3 \\ l_3 & -(l_2 + l_3) \end{bmatrix} \frac{d^{\alpha}}{dt^{\alpha}} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} -(R_1 + R_3) & R_3 \\ -R_3 & (R_2 + R_3) \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ -v_2(t) \end{bmatrix}. \tag{3.3}$$

Let $q = [\begin{smallmatrix} l_1+l_3 & -l_3 \\ l_3 & -(l_2+l_3) \end{smallmatrix}]$, $I(t) = [\begin{smallmatrix} i_1(t) \\ i_2(t) \end{smallmatrix}]$, $R = [\begin{smallmatrix} -(R_1+R_3) & R_3 \\ -R_3 & (R_2+R_3) \end{smallmatrix}]$, $V(t) = [\begin{smallmatrix} v_1(t) \\ -v_2(t) \end{smallmatrix}]$, and $q^{-1} = \frac{-1}{l_1 l_2 + l_1 l_3 + l_2 l_3} [\begin{smallmatrix} -(l_2+l_3) & l_3 \\ -l_3 & l_1+l_3 \end{smallmatrix}]$ is the inverse of q.

Let

$$A = q^{-1}R = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{3.4}$$

where

$$A_{11} = \frac{-[l_2 R_1 + l_2 R_3 + l_3 R_1]}{l_1 l_2 + l_1 l_3 + l_2 l_3}, \quad A_{12} = \frac{-l_3 R_2 + l_2 R_3}{l_1 l_2 + l_1 l_3 + l_2 l_3},$$
$$A_{21} = \frac{l_1 R_3 - l_3 R_1}{l_1 l_2 + l_1 l_3 + l_2 l_3}, \quad A_{22} = \frac{-[l_1 R_2 + l_1 R_3 + l_3 R_2]}{l_1 l_2 + l_1 l_3 + l_2 l_3}, \tag{3.5}$$

then system (3.3) can be written in the form

$$\frac{d^\alpha}{dt^\alpha} I(t) = A\, I(t) + q^{-1} V(t). \tag{3.6}$$

From system (3.6) applying Laplace transform at zero initial conditions, we get

$$I_1(s) = \frac{A_{12}}{S^\alpha - A_{11}} I_2(s) - \frac{1}{(S^\alpha - A_{11})|q|}[V_1(s)(l_2 + l_3) + V_2(s) l_3], \tag{3.7}$$

$$s^\alpha I_2(s) = A_{21} I_1(s) + A_{22} I_2(s) - \frac{1}{|q|}[l_3 V_1(s) + (l_1 + l_3) V_2(s)]. \tag{3.8}$$

Substituting from (3.7) into (3.8) we get:

$$\begin{aligned} I_2(s) = & \frac{1}{|q|(s^\alpha + \omega_1)(s^\alpha + \omega_2)}[V_1(s)[l_3 A_{11} - A_{21}(l_2 + l_3)] \\ & + V_2(s)[A_{11}(l_1 + l_3) - A_{21} l_3] \\ & - l_3 s^\alpha V_1(s) - (l_1 + l_3) s^\alpha V_2(s)]. \end{aligned} \tag{3.9}$$

We observe that

$$\frac{1}{|q|(s^\alpha + \omega_1)(s^\alpha + \omega_2)} = \mathcal{M}\left[\frac{1}{s^\alpha + \omega_2} - \frac{1}{s^\alpha + \omega_1}\right] \text{where}$$
$$\mathcal{M} = \frac{1}{|q|(\omega_1 - \omega_2)}. \tag{3.10}$$

Applying Laplace Inverse for Eq. (3.10), then

$$L^{-1}\left\{\frac{1}{|q|(s^\alpha + \omega_1)(s^\alpha + \omega_2)}\right\} = \mathcal{M}\{e(t, -\omega_2) - e(t, -\omega_1)\}, \tag{3.11}$$

where

$$\omega_{1,2} = -0.5[tr(A) \pm \sqrt{(tr(A))^2 - 4|A|}] \tag{3.12}$$

and $e(t, \omega_i)$ is defined in [20] as

$$e(t, \omega_i) = \sum_{k=0}^{\rho-1} (\omega_i)^{\rho-k-1} E_t(-k\alpha, (\omega_i)^\rho), \quad i = 1, 2, \tag{3.13}$$

where $E_t(\alpha, a)$ is defined in (2.11) and ρ is defined in [20] as the least common multiple of the denominators of the equation $[D^{\frac{n}{\rho}} + b_1 D^{\frac{n-1}{\rho}} + \ldots + b_m]y = 0$.

Therefore applying inverse Laplace transform for Eq. (3.9) gives

$$\begin{aligned} i_2(t) = & \mathcal{M}\ \{e(t, -\omega_2) - e(t, -\omega_1)\} * \{\check{R}\ v_1(t) \\ & + \check{U}\ v_2(t) - l_3 D^\alpha v_1(t) - (l_1 + l_3) D^\alpha v_2(t)\}, \end{aligned} \tag{3.14}$$

where the symbol " $*$ " is the convolution rule defined in (2.6),

$$\check{R} = l_3 A_{11} - A_{21}(l_2 + l_3) \text{ and } \check{U} = A_{11}(l_1 + l_3) - A_{21} l_3. \tag{3.15}$$

Now applying inverse Laplace transform for Eq. (3.7) then

$$\begin{aligned} i_1(t) = & A_{12}\ i_2(t) * e(t, A_{11}\ ) \\ & - \frac{1}{|q|}\ e(t, A_{11}\ )\ * [(l_2 + l_3) v_1(t) + l_3 v_2(t)]. \end{aligned} \tag{3.16}$$

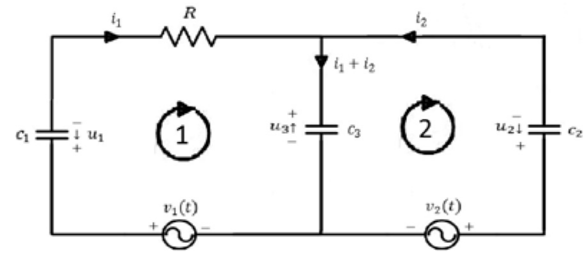

Fig. 2. Electrical circuit of application 2.

Application 2. Consider the electrical circuit shown in Fig. 2 with given resistance R, capacitance c_1, c_2, c_3 and voltage sources $v_i(t),\ i = 1, 2$.

From Eq. (2.7) the current in c_3 is given by

$$i_{c_3} = c_3 \frac{d^\alpha u_3}{dt^\alpha} = i_1 + i_2 = c_1 \frac{d^\alpha u_1}{dt^\alpha} + c_2 \frac{d^\alpha u_2}{dt^\alpha}. \tag{3.17}$$

From which

$$u_3 = \frac{1}{c_3} I^\alpha [c_1 D^\alpha u_1 + c_2 D^\alpha u_2] = \frac{1}{c_3}[c_1 u_1 + c_2 u_2], \tag{3.18}$$

where I^α, D^α are the Caputo fractional integral and derivative operators respectively . Using Kirchhoff's voltages and currents laws for the circuit shown in Fig. 2 and using (3.17) and (3.18) we get:

$$\frac{d^\alpha u_1(t)}{dt^\alpha} = \frac{v_1(t)}{Rc_1} - \frac{(c_1 + c_3) u_1(t)}{Rc_1 c_3} - \frac{c_2 u_2(t)}{Rc_1 c_3}, \tag{3.19}$$

$$v_2(t) = u_2(t) + \frac{1}{c_3}[c_1 u_1(t) + c_2 u_2(t)]. \tag{3.20}$$

Then

$$u_2(t) = \frac{c_3 v_2(t) - c_1 u_1(t)}{c_2 + c_3}. \tag{3.21}$$

Substituting from (3.21) into (3.19) we obtain:

$$\frac{d^\alpha u_1(t)}{dt^\alpha} = \frac{v_1(t)}{Rc_1} + B u_1(t) - \frac{c_2 v_2(t)}{Rc_1(c_2 + c_3)}, \tag{3.22}$$

where

$$B = -\frac{(c_1 + c_2 + c_3)}{Rc_1(c_2 + c_3)}. \tag{3.23}$$

Applying Laplace transform to Eq. (3.22) at zero initial conditions to obtain:

$$U_1(s) = \frac{1}{(s^\alpha - B)}\left\{\frac{V_1(s)}{Rc_1} - \frac{c_2 V_2(s)}{Rc_1(c_2 + c_3)}\right\}. \tag{3.24}$$

Applying inverse Laplace transform for Eq. (3.24) then,

$$u_1(t) = e(t, B) * \left\{\frac{v_1(t)}{Rc_1} - \frac{c_2 v_2(t)}{Rc_1(c_2 + c_3)}\right\}, \tag{3.25}$$

where $e(t, B)$ is defined in (3.13).

By Substituting from (3.25) into (3.21) we obtain the value of $u_2(t)$. Now the following analysis is to find the current expressions across the capacitor elements. Using Kirchhoff's voltages law for loop 1 in the circuit shown in Fig. 2, then

$$v_1(t) = \frac{1}{c_3}\ I^\alpha (i_1(t) + i_2(t)) + i_1(t) R + \frac{1}{c_1}\ I^\alpha i_1(t). \tag{3.26}$$

Operating both sides of Eq. (3.26) by D^α, then

$$D^\alpha i_1(t) = \frac{1}{R} D^\alpha v_1(t) - \frac{1}{R\ c_3} i_2(t) - \left(\frac{c_1 + c_3}{R\ c_1 c_3}\right) i_1(t). \tag{3.27}$$

Similarly applying Kirchhoff's voltages law for loop 2

$$v_2(t) = \frac{1}{c_2}\ I^\alpha i_2(t) + \frac{1}{c_3}\ I^\alpha (i_1(t) + i_2(t)). \tag{3.28}$$

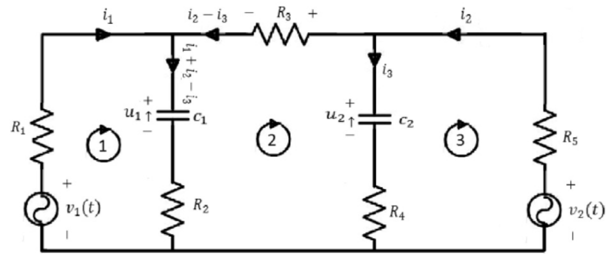

Fig. 3. Electrical circuit of application 3.

Operating both sides of Eq. (3.28) by D^α, then

$$i_2(t) = \frac{c_2 c_3}{c_2 + c_3}\; D^\alpha v_2(t) - \frac{c_2}{c_2 + c_3}\; i_1(t). \tag{3.29}$$

Substituting from Eq. (3.29) into Eq. (3.27), we get the following

$$D^\alpha i_1(t) = B\; i_1(t) + \frac{1}{R} D^\alpha v_1(t) + Q^*\; D^\alpha v_2(t), \tag{3.30}$$

where $Q^* = -\frac{c_2}{R\;(c_2+c_3)}$, B is defined in (3.23).

Applying Laplace transform to Eq. (3.30) at zero initial conditions to obtain:

$$I_1(s) = \frac{1}{S^\alpha - B}\left\{\frac{1}{R}\; s^\alpha\; V_1(s) + Q^*\; s^\alpha\; V_2(s)\right\}. \tag{3.31}$$

Applying inverse Laplace transform for Eq. (3.31) then,

$$i_1(t) = e(t, B) * \left\{\frac{1}{R} D^\alpha v_1(t) + Q^*\; D^\alpha v_2(t)\right\}. \tag{3.32}$$

Substituting from (3.32) into (3.29) to obtain the value of $i_2(t)$.

Application 3. Consider the electrical circuit shown in Fig. 3 with given resistances R_1, R_2, R_3, R_4, R_5, capacitance c_1, c_2 and voltage sources $v_i(t),\; i = 1, 2$

Applying Kirchhoff's voltages laws and using (2.8), (2.9) we get,

$$v_1(t) = i_1(R_1 + R_2) + \frac{1}{c_1} I^\alpha (i_1 + i_2 - i_3) + (i_2 - i_3)R_2, \tag{3.33}$$

$$(i_1 + i_2 - i_3)R_2 + \frac{1}{c_1} I^\alpha (i_1 + i_2 - i_3) + (i_2 - i_3)R_3 - \frac{1}{c_2} I^\alpha i_3 - i_3 R_4 = 0, \tag{3.34}$$

$$i_3 R_4 + \frac{1}{c_2} I^\alpha i_3 + i_2 R_5 = v_2(t). \tag{3.35}$$

We can write Eqs. (3.33)–(3.35) in the following system form:

$$D^\alpha I(t) = A^{-1}\{\psi\; I(t) - D^\alpha\; V(t)\}, \tag{3.36}$$

where

$$\psi = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -\lambda^* \\ 0 & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} -c_1(R_1 + R_2) & -c_1 R_2 & c_1 R_2 \\ -c_1 R_2 & -c_1(R_2 + R_3) & c_1(R_2 + R_4 + R_3) \\ 0 & -c_2 R_5 & -c_2 R_4 \end{bmatrix}, \tag{3.37}$$

$$I(t) = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad V(t) = \begin{bmatrix} c_1 v_1(t) \\ o \\ c_2 v_2(t) \end{bmatrix}, \quad \lambda^* = 1 + \frac{c_1}{c_2}. \tag{3.38}$$

where $A^{-1} = \frac{1}{|A|}\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$ is the inverse of A, and

$$J_{11} = c_1\, c_2\{R_4(R_2 + R_3) + R_5(R_4 - R_3)\}, \quad J_{12} = -c_1 c_2 R_2(R_4 + R_5),$$
$$J_{13} = c_1^2 R_2(R_2 - R_4 + 2R_3), \quad J_{21} = -c_1\, c_2\, R_2 R_4,$$
$$J_{22} = c_1 c_2 R_4(R_2 + R_1), \quad J_{23} = c_1^2\{(R_2 + R_1)(R_4 - R_3) - R_2^2\},$$
$$J_{31} = c_1 c_2 R_5 R_2, \quad J_{32} = -c_1 c_2 R_5(R_2 + R_1),$$
$$J_{33} = c_1^2\{(R_2 + R_1)(R_2 + R_3) - R_2^2\}. \tag{3.39}$$

From system (3.36) we can get:

$$D^\alpha i_1(t) = \phi_1 i_1(t) + \phi_1 i_2(t) + \phi_2 i_3(t) + q_1 D^\alpha v_1(t) + q_2 D^\alpha v_2(t), \tag{3.40}$$

$$D^\alpha i_2(t) = \phi_3 i_1(t) + \phi_3 i_2(t) + \phi_4 i_3(t) + q_3 D^\alpha v_1(t) + q_4 D^\alpha v_2(t), \tag{3.41}$$

$$D^\alpha i_3(t) = \phi_5 i_1(t) + \phi_5 i_2(t) + \phi_6 i_3(t) + q_5 D^\alpha v_1(t) + q_6 D^\alpha v_2(t), \tag{3.42}$$

where

$$\varphi_1 = \frac{J_{11} + J_{12}}{|A|}, \quad \varphi_2 = \frac{J_{13} - J_{11} - \lambda^* J_{12}}{|A|}, \quad \varphi_3 = \frac{J_{21} + J_{22}}{|A|},$$
$$\varphi_4 = \frac{J_{23} - J_{21} - \lambda^* J_{22}}{|A|}, \quad \varphi_5 = \frac{J_{31} + J_{32}}{|A|}, \quad \varphi_6 = \frac{-J_{31} - \lambda^* J_{32} + J_{33}}{|A|},$$
$$q_1 = \frac{-c_1 J_{11}}{|A|}, \quad q_2 = \frac{-c_2 J_{13}}{|A|}, \quad q_3 = \frac{-c_1 J_{21}}{|A|}, \quad q_4 = \frac{-c_2 J_{23}}{|A|},$$
$$q_5 = \frac{-c_1 J_{31}}{|A|}, \quad q_6 = \frac{-c_2 J_{33}}{|A|}. \tag{3.43}$$

Applying Laplace transform to Eqs. (3.40) and (3.41) at zero initial conditions we obtain:

$$I_1(s) = \frac{1}{s^\alpha - \phi_1}\{\phi_1 I_2(s) + \phi_2 I_3(s) + q_1\, s^\alpha\, V_1(s) + q_2\, s^\alpha\, V_2(s)\}, \tag{3.44}$$

$$I_2(s) = \frac{1}{s^\alpha - \phi_3}\{\phi_3 I_1(s) + \phi_4 I_3(s) + q_3 s^\alpha\, V_1(s) + q_4 s^\alpha\, V_2(s)\}. \tag{3.45}$$

Substituting from (3.45) into (3.44) to we obtain:

$$I_1(s) = \frac{1}{s^\alpha\,(s^\alpha - (\varphi_1 + \varphi_3))}\{[\varphi_1\, \varphi_4 + \varphi_2(s^\alpha - \varphi_3)]\, I_3(s) + \varphi_1(q_3\, s^\alpha\, V_1(s) + q_4\, s^\alpha\, V_2(s)) + (s^\alpha - \varphi_3)\,(q_1\, s^\alpha V_1(s) + q_2\, s^\alpha V_2(s))\,\}. \tag{3.46}$$

In Eq. (3.40) we observe that:

$$i_1(t) + i_2(t) = \frac{1}{\phi_1}\{D^\alpha i_1(t) - \phi_2 i_3 - (q_1 D^\alpha v_1(t) + q_2 D^\alpha v_2(t))\}. \tag{3.47}$$

Substituting from (3.47) into (3.42) and applying Laplace transform to the resultant form we get:

$$I_3(s) = \frac{\phi_5}{\phi_1\,(s^\alpha + P)}\left\{s^\alpha\, I_1(s) - (q_1\, s^\alpha\, V_1(s) + q_2\, s^\alpha\, V_2(s)) + \frac{\phi_1(q_5\, s^\alpha V_1(s) + q_6\, s^\alpha V_2(s))}{\phi_5}\right\}, \tag{3.48}$$

where $p = \frac{\phi_2 \phi_5}{\phi_1} - \phi_6$. Substituting from (3.48) into (3.46) then,

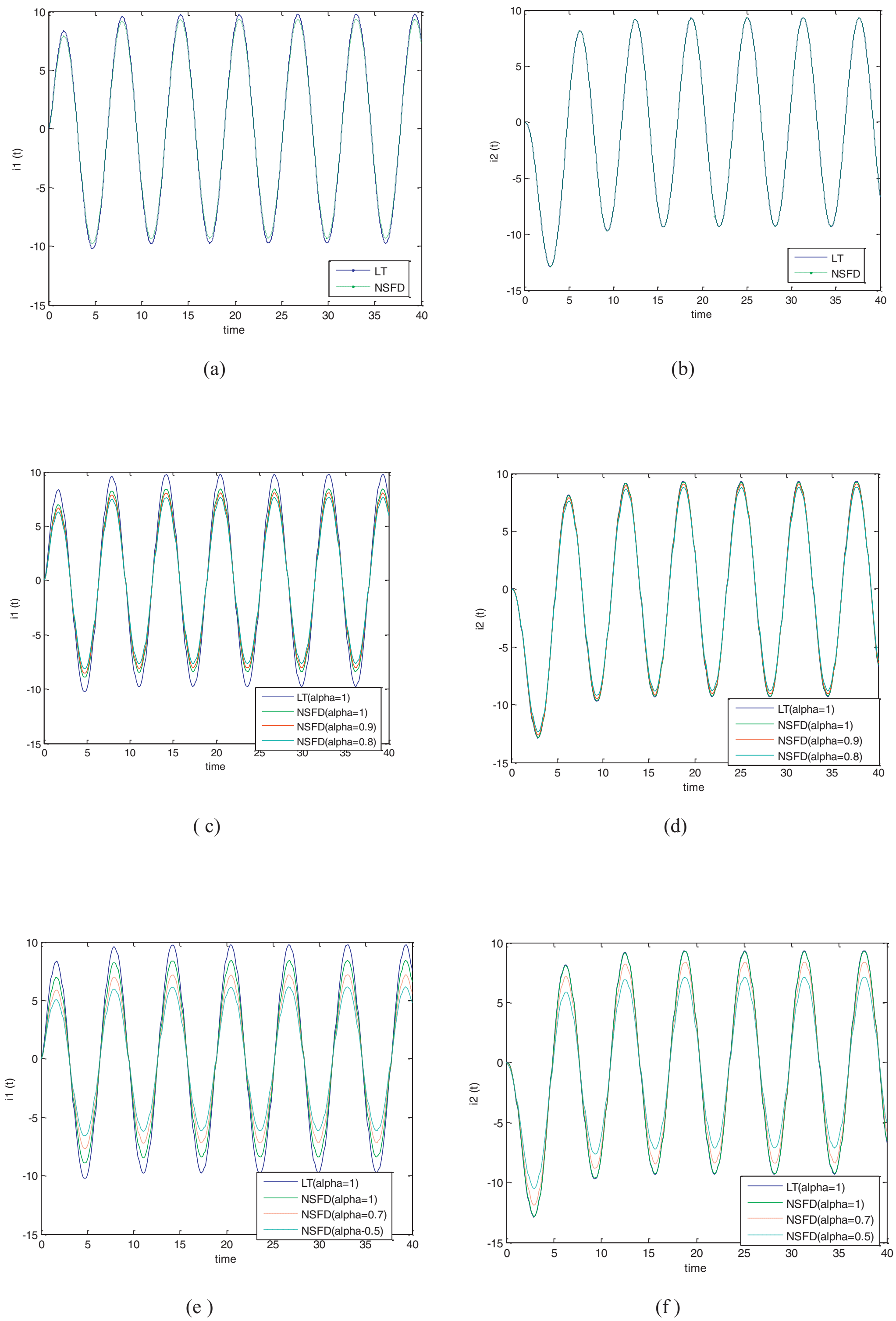

Fig. 4. Time response of the current $i_1(t), i_2(t)$ for Example 4.1.

$$I_1(s) = \frac{1}{(S^{2\alpha} - \psi_1 S^{\alpha} + \psi_2)}\{S^{2\alpha}\varpi_1(s) + S^{\alpha}[\varphi_1\varpi_3(s) + \sigma\varpi_1(s) + \varphi_2\varpi_5(s)] + [\zeta\varpi_5(s) + \delta\varpi_1(s)] - \varphi_1 p\varpi_3(s)\}, \tag{3.49}$$

where

$$\psi_1 = \varphi_1 + \varphi_3 + p + \frac{\varphi_2\varphi_5}{\varphi_1}, \quad \psi_2 = p(\varphi_1 + \varphi_3) + \frac{\varphi_2\varphi_3\varphi_5}{\varphi_1} - \varphi_4\varphi_5,\ \varpi_i(s) = q_i V_1(s) + q_{i+1}V_2(s),\ i = 1, 3, 5,$$

$$\sigma = \frac{\varphi_2\varphi_5}{\varphi_1} - \varphi_3 - p,\ \zeta = \varphi_1\varphi_4 - \varphi_2\varphi_3,$$

$$\delta = \varphi_3 p + \varphi_4\varphi_5 - \frac{\varphi_2\varphi_3\varphi_5}{\varphi_1}.$$

Applying inverse Laplace transform to Eq. (3.49) to get:

$$i_1(t) = L^{-1}\left\{\frac{1}{(S^{2\alpha} - \psi_1 S^{\alpha} + \psi_2)}\right\} * L^{-1}\{S^{2\alpha}\varpi_1(s)$$

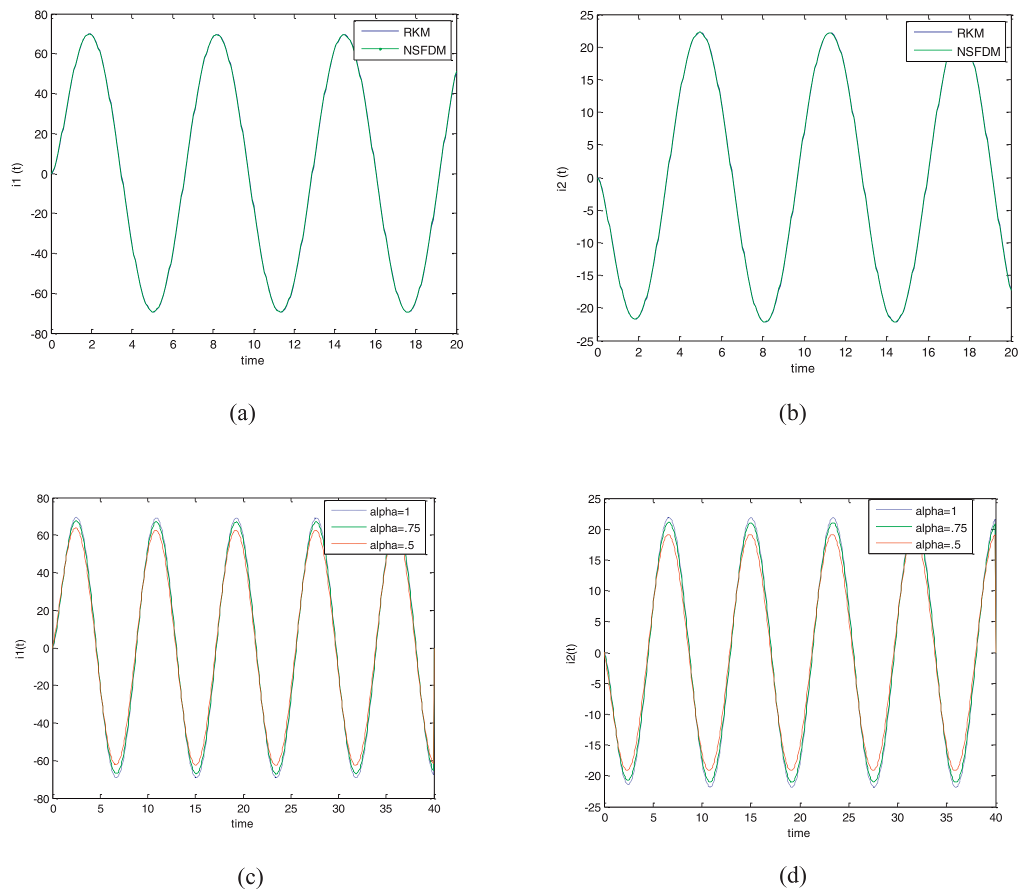

Fig. 5. Time response of the current $i_1(t), i_2(t)$ for Example 4.2.

$$+S^{\alpha}[\varphi_1\varpi_3(s)+\sigma\varpi_1(s)+\varphi_2\varpi_5(s)] + [\zeta\varpi_5(s)+\delta\varpi_1(s)]-\varphi_1 p\varpi_3(s)\}, \tag{3.50}$$

where

$$L^{-1}\left\{\frac{1}{(S^{2\alpha}-\psi_1 S^{\alpha}+\psi_2)}\right\} = L^{-1}\left\{\frac{1}{(\tau_1-\tau_2)}\left[\frac{1}{(S^{\alpha}+\tau_2)}-\frac{1}{(S^{\alpha}+\tau_1)}\right]\right\} = \frac{1}{(\tau_1-\tau_2)}[e(t,-\tau_2)-e(t,-\tau_1)], \tag{3.51}$$

and

$$\tau_{1,2}=-0.5\{\psi_1\pm\sqrt{{\psi_1}^2-4\psi_2}\}, \tag{3.52}$$

$e_i(t,\tau_i), i=1,2$ is defined by (3.13), also we get:

$$L^{-1}\{S^{2\alpha}\varpi_1(s)+S^{\alpha}[\varphi_1\varpi_3(s)+\sigma\varpi_1(s)+\varphi_2\varpi_5(s)] + \zeta\varpi_5(s)+\delta\varpi_1(s)-\varphi_1 p\varpi_3(s)\} = D^{2\alpha}\varpi_1(t)+D^{\alpha}[\varphi_1\varpi_3(t)+\sigma\varpi_1(t)+\varphi_2\varpi_5(t)] + \zeta\varpi_5(t)+\delta\varpi_1(t)-\varphi_1\varphi_6\varpi_3(t)\}, \tag{3.53}$$

where,

$$\varpi_i(t)=q_i\nu_1(t)+q_{i+1}\nu_2(t), i=1,3,5. \tag{3.54}$$

Similarly apply inverse Laplace transform to Eq. (3.48) then

$$i_3(t)=\frac{\varphi_5}{\varphi_1}e(t,-p)*\left\{D^{\alpha}i_1(t)-D^{\alpha}(q_1\nu_1(t)+q_2\nu_2(t)) + \frac{\varphi_1}{\varphi_5}D^{\alpha}(q_5\nu_1(t)+q_6\nu_2(t)\right\}, \tag{3.55}$$

where $e(t,-p)$ is defined by (3.13). Therefore substituting from (3.55) and (3.50) into (3.47), we get a relation of $i_2(t)$ as:

$$i_2(t)=\frac{1}{\phi_1}\{D^{\alpha}i_1(t)-D^{\alpha}(q_1\nu_1(t)+q_2\nu_2(t))\}-i_1(t). \tag{3.56}$$

3.2. Numerical simulation of fractional systems

Nonstandard finite difference method that introduced by Mickens [26] is used to solve the following system of differential equations:

$$i'_r=f_r(t,i_1,i_2,\dots,i_m), \quad r=1,2,..,m. \tag{3.57}$$

Using finite difference method, the discrete derivatives are

$$i'_1=\frac{i_{1,k+1}-i_{1,k}}{\varphi_1(\mathrm{h})},\quad i'_2=\frac{i_{2,k+1}-i_{2,k}}{\varphi_2(\mathrm{h})},\quad i'_m=\frac{i_{m,k+1}-i_{m,k}}{\varphi_{\mathrm{m}}(\mathrm{h})}, \tag{3.58}$$

where $\varphi_{\mathrm{r}}(\mathrm{h})=h+O(h^2)$, h is the step size, for details see [26,28,29].

Now we consider our fractional system as follows:

$$\frac{d^{\alpha}}{dt^{\alpha}}i_r(t)=f_r(t,i_1,i_2,\dots,i_m), \quad r=1,2,..,m. \tag{3.59}$$

We apply the above technique along with Grünwald–Letnikov definition for the fractional system (3.59), we have:

$$\sum_{j=0}^{k}C_j^{\alpha}\, i_r(t_{k-j})=f_r(t_k,i_{1,k},i_{2,k},\dots,i_{m,k}), \tag{3.60}$$

where $t_k=kh, k=0,1,2,\dots.$ and $C_0^{\alpha}=(\varphi(\mathrm{h}))^{\alpha}$.

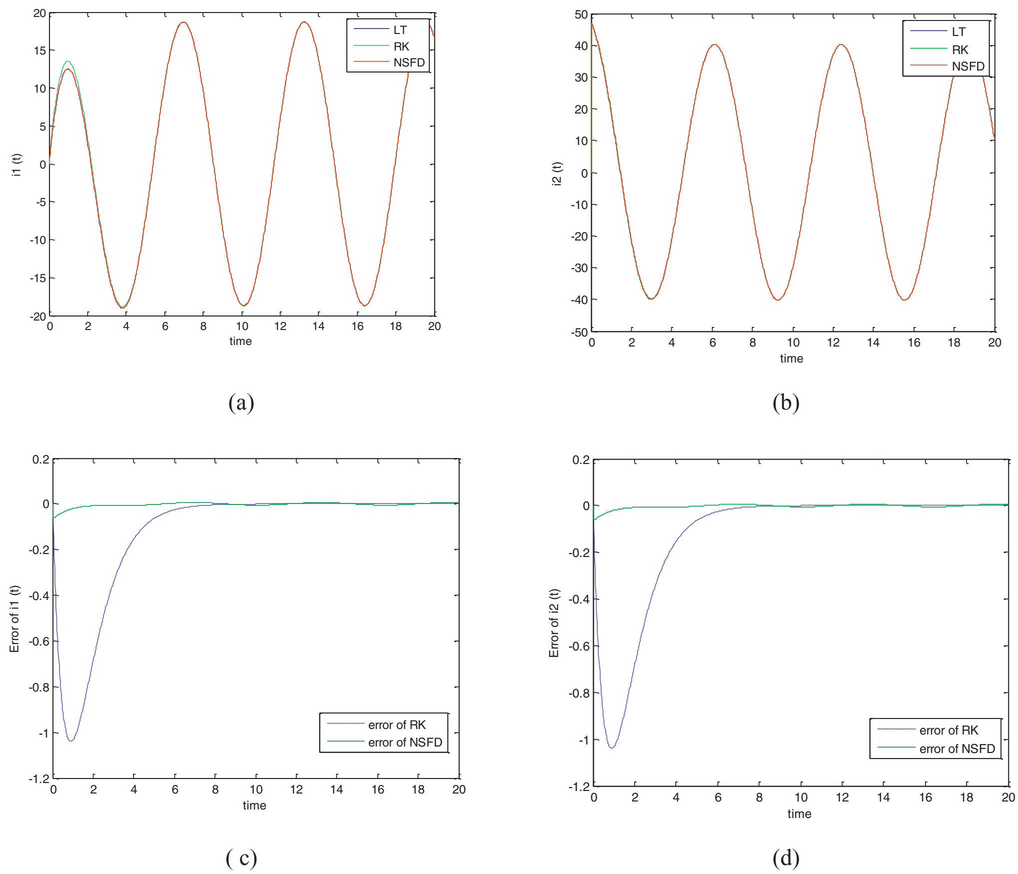

Fig. 6. Time response of the current $i_1(t), i_2(t)$ for Example 4.3.

4. Illustrative examples

In this section, we will give some remarks and illustrative examples for the previous applications with some numerical examples.

Example 4.1. Corresponding to the circuit of application 1 if $\alpha = 1$ and $v_i(t) = V_{mi}\sin(t), i = 1, 2$, then $Dv_i(t) = V_{mi}\, cos(t)$, then substituting in Eqs. (3.14) and (3.16), we get

$$i_2(t) = p_1 \sin(t) + p_2 \cos(t) + p_3\ e^{-\omega_1 t} + p_4\ e^{-\omega_2 t}, \tag{4.1}$$

$$i_1(t) = p_5 \sin(t) + p_6 \cos(t) + p_7\ e^{-\omega_1 t} + p_8\ e^{-\omega_2 t} + p_9\ e^{A_{11} t}, \tag{4.2}$$

Where $p_1 = \mathcal{M}\{a_1^*(\frac{\omega_2}{1+\omega_2^2} - \frac{\omega_1}{1+\omega_1^2}) + a_2^*(\frac{1}{1+\omega_2^2} - \frac{1}{1+\omega_1^2})\}$, $p_2 = \mathcal{M}\{a_1^*(\frac{1}{1+\omega_1^2} - \frac{1}{1+\omega_2^2}) + a_2^*(\frac{\omega_1}{1+\omega_1^2} - \frac{\omega_2}{1+\omega_2^2})\}$ $p_3 = \frac{\mathcal{M}(a_2^*-a_1^*)}{1+\omega_1^2}$, $p_4 = \frac{\mathcal{M}(a_1^*+\omega_2 a_2^*)}{1+\omega_2^2}$, $p_5 = \frac{1}{1+A_{11}^2}\{A_{12}(p_2 - A_{11}p_1) + \frac{A_{11}[(l_2+l_3)V_{m1}+l_3V_{m2}]}{|q|}\}$, $p_6 = \frac{1}{1+A_{11}^2}\{\frac{(l_2+l_3)V_{m1}+l_3V_{m2}}{|q|} - A_{12}(p_1 + A_{11}p_2)\}$, $p_7 = \frac{-A_{12}p_3}{A_{11}+\omega_1}$, $p_8 = \frac{-A_{12}p_4}{A_{11}+\omega_2}$, $p_9 = \frac{1}{1+A_{11}^2}\{A_{12}(p_1 + A_{11}p_2) - \frac{(l_2+l_3)V_{m1}+l_3V_{m2}}{|q|}\} - (p_7 + p_8)$, $a_1^* = \breve{R}\, V_{m1} + \breve{U}\, V_{m2}$, $a_2^* = l_3 V_{m1} + (l_1 + l_3)V_{m2}$, $\omega_{1,2}$ and Ř, Ŭ are defined in (3.12), (3.15) respectively.

We observe that $tr(A)\langle 0, |A|\rangle 0$ for all $l_i, R_i \in R^+, i = 1, 2, 3$. So $\omega_{1,2} > 0$ and the system of circuit 1 will be asymptotically stable. Assume that the parameters for application 1 take the values: $R_1 = 10\ \ \Omega, R_2 = 1\ \ \Omega, R_3 = 5\ \ \Omega, l_1 = 0.2\ \ H, l_2 = 10\ \ H, l_3 = 5\ \ H, V_{m1} = 100\ \ V, V_{m2} = 100\ \ V, \alpha = 1$. Hence $tr(A) = -3.8906$, $|A| = 1.2264$, *then* $\omega_1 = 0.3460, \omega_2 = 3.5446$ and the system of application 1 is stable. Fig. 4(a and b) represents the time response of the currents $i_1(t), i_2(t)$ for h = 0.05 and $\phi(h) = h$ using the exact solution by Laplace transform (LT) and NSFD method for $\alpha = 1$ which reveals that the NSFD method is in good agreement with the exact solution, while Fig. 4(c–f) is devoted to show the effect of different values of the fractional order α on the time response of the currents with $h = 0.1$. Also the amplitude of the time response of the currents attenuates as the fractional order α decreased.

Example 4.2. For application 1, using NSFD method [27,28,29] and assuming the following parameters: $R_1 = 1.4\ \ \Omega, R_2 = 0.2\ \ \Omega, R_3 = 1\ \ \Omega, l_1 = 0.8\ H, l_2 = 1\ H, l_3 = 0.1\ H, V_{m1} = 200\ V,\ V_{m2} = 100\ V$. The numerical results for this example are shown in Fig. 5(a and b) which represent the time response of the current $i_1(t), i_2(t)$ at $\alpha = 1$ using Runge–Kutta (RK) and NSFDM methods with h= 0.05 and $\phi(h) = 1 - e^{-h}$, while Fig. 5(c and d) gives the time response of the currents using NSFD method for different values of the fractional order $\alpha = 1, 0.75, 0.5$. And the amplitudes of the oscillations attenuate.

Example 4.3. Corresponding to the circuit of application 2, and considering $v_1(t) = V_{m1}\ \sin t\ ,\ v_2(t) = V_{m2}\ \sin t$, substituting in Eqs. (3.32) and (3.29) by $\alpha = 1$, then

$$i_1(t) = \left\{\frac{V_{m1}\ + R\ V_{m2}\ Q^*}{R\left(1 + B^2\right)}\right\}\left\{\sin(t) - B\cos(t) + B\ e^{Bt}\right\}, \tag{4.3}$$

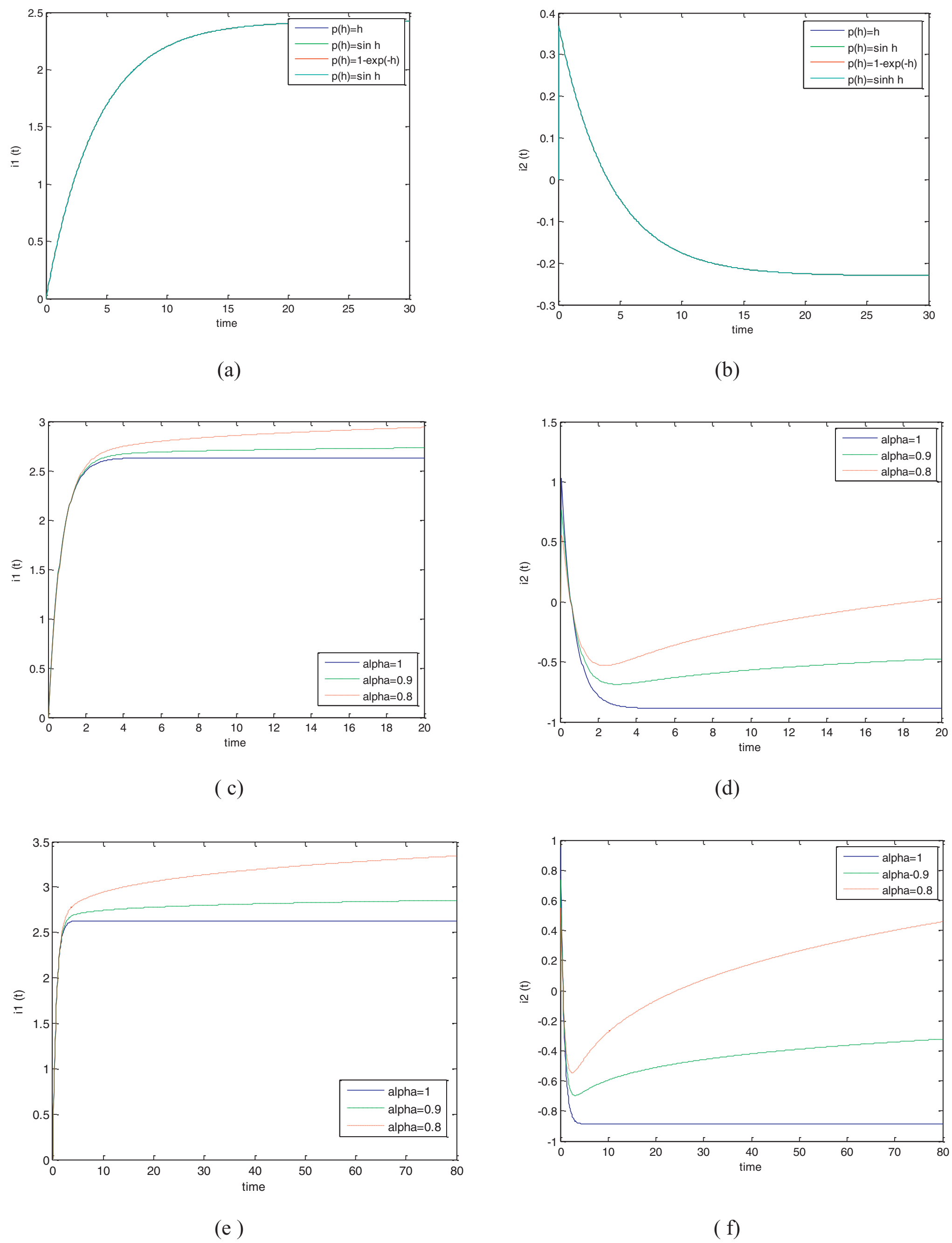

Fig. 7. Time response of the voltages v_i and the current $i_1(t), i_2(t)$ for Example 4.4.

and

$$i_2(t) = h_1^* \sin(t) + h_2^* \cos(t) + h_3^* \, e^{Bt}, \tag{4.4}$$

or simply

$$i_2(t) = \frac{c_2\, c_3}{c_2 + c_3}\left(\frac{d}{dt}\,(V_{m2}\,\sin\, t)\right) - \frac{c_2\, c_3}{c_2 + c_3}\,(i_1\,(t)), \tag{4.5}$$

where $h_1^* = \frac{-c_2\ (V_{m1}\ +R\ V_{m2}\ Q^*)}{R(c_2+c_3)(1+B^2)}, h_2^* = \frac{1}{(c_2+c_3)}\{c_2\ c_3 V_{m2} + \frac{B\ c_2\ (V_{m1}\ +R\ V_{m2}\ Q^*)}{R\ (1+B^2)}\}, h_3^* = B\ h_1^*$ and B is defined in (3.23). Then consider the following parameters for application 2:
$R = 5\ \Omega, c_1 = 0.2\ f, c_2 = 1\ f, c_3 = 0.9\ f, V_{m1} = 200\ V, V_{m2} = 100\ V$, $\alpha = 1$. We found that $B = -2.1333$ and the system is stable. Fig. 6(a and b) depicts the time response of the currents $i_1(t)$ and $i_2(t)$ using LT, RK and NSFD methods for $\phi(h) = h$ and $h = 0.025$ which declares that the NSFD method is more efficient than the RK method. This is clearly observed in Fig. 6(c and d) where the error of the numerical solutions of the currents are plotted using RK and NSFD methods.

Example 4.4. For application 2, assume that $R = 15\ \ \Omega, c_1 = 0.3\ f, c_2 = 0.1\ f, c_3 = 0.3\ f, v_1 = 10t, v_2 = 5t$. Fig. 7(a, b) shows the time response of the current $i_1(t)$ *and* $i_2(t)$ at $\alpha = 1$ using NSFD method for different values of $\phi(h)$ when $h = 0.05$. Also in Fig. 7(c and d) NSFD method is used with different values of the fraction order α, $h = 0.05$ and $\phi(h) = \sinh h$. Again in Fig. 7(e and f) we use NSFD method with different vales of the fraction order α, $h = 0.08$ and $\phi(h) = \sin h$ It is clear that the current tends to a constant value with increasing the time as they has negative exponential

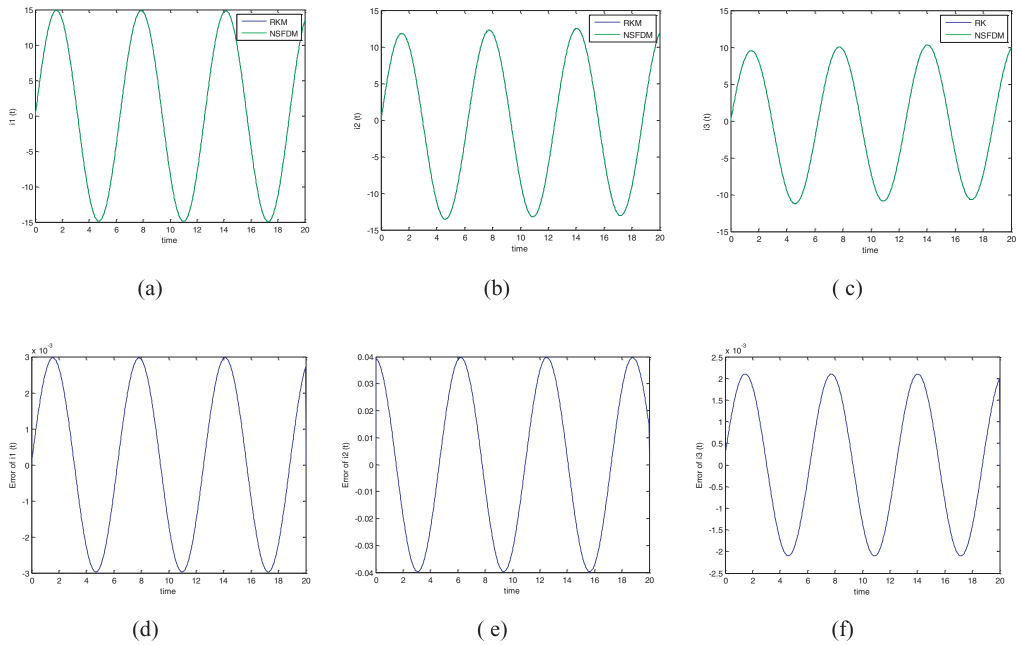

Fig. 8. Time response of the current $i_1(t), i_2(t), i_3(t)$ for Example 4.5.

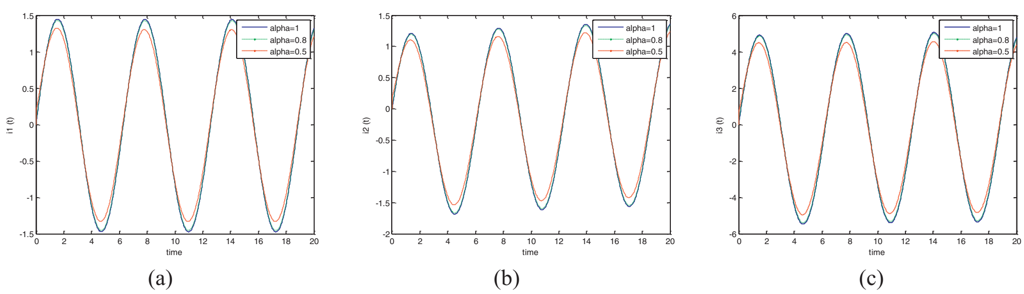

Fig. 9. Time response of the current $i_1(t), i_2(t), i_3(t)$ for Example 4.6.

forms when $\alpha = 1$ while by decreasing the value of the fractional order α the currents increasing by increasing the voltage applied to the circuit.

Example 4.5. For the circuit of application 3, substituting in Eqs. (3.50), (3.55) and (3.56) by $\alpha = 1$ and simplifying to get:

$$i_1(t) = x_1^* \sin(t) + x_2^* \cos(t) + x_3^*\ e^{-\tau_1 t} + x_4^*\ e^{-\tau_2 t}, \tag{4.6}$$

$$i_3(t) = x_5^* \sin(t) + x_6^* \cos(t) + x_7^*\ e^{-p\ t} + x_8^*\ e^{-\tau_1 t} + x_9^*\ e^{-\tau_2 t}, \tag{4.7}$$

$$i_2(t) = x_{10}^* \sin(t) + x_{11}^* \cos(t) + x_{12}^*\ e^{-\tau_1 t} + x_{13}^*\ e^{-\tau_2 t} + x_{14}^*\ e^{-p\ t}, \tag{4.8}$$

where

$$x_1^* = \frac{f_1^*}{\tau_1 - \tau_2}\{(\delta - 1)(q_1 V_{m1} + q_2 V_{m2}) + \zeta(q_5 V_{m1} + q_6 V_{m2}) - \phi_1 \phi_6 (q_3 V_{m1} + q_4 V_{m2})\} + \frac{f_2^* k_1}{\tau_1 - \tau_2},$$

$$x_2^* = \frac{f_2^*}{\tau_2 - \tau_1}\{(\delta - 1)(q_1 V_{m1} + q_2 V_{m2}) + \zeta(q_5 V_{m1} + q_6 V_{m2}) - \phi_1 \phi_6 (q_3 V_{m1} + q_4 V_{m2})\} + \frac{f_1^* k_1}{\tau_1 - \tau_2},$$

$$x_3^* = \frac{1}{(\tau_1 - \tau_2)(1 + \tau_1^2)}\{(1 - \delta)(q_1 V_{m1} + q_2 V_{m2}) - \zeta(q_5 V_{m1} + q_6 V_{m2}) + \phi_1 \phi_6 (q_3 V_{m1} + q_4 V_{m2}) + \tau_1 k_1\},$$

$$x_4^* = \frac{1}{(\tau_1 - \tau_2)(1 + \tau_1^2)}\{(\delta - 1)(q_1 V_{m1} + q_2 V_{m2}) + \zeta(q_5 V_{m1} + q_6 V_{m2}) - \phi_1 \phi_6 (q_3 V_{m1} + q_4 V_{m2}) + \tau_2 k_1\},$$

$$x_5^* = \frac{1}{1 + p^2}\left\{f_3^* - \frac{p\, x_2^* \phi_5}{\phi_1}\right\},\ x_6^* = \frac{1}{1 + p^2}\left\{\frac{x_2^* \phi_5}{\phi_1} + p\, f_3^*\right\},$$

$$x_7^* = \frac{\phi_5 \tau_1}{\phi_1}\left\{\frac{x_3^*}{p - \tau_1} + \frac{x_4^*}{p - \tau_1}\right\} - x_6^*, \quad x_8^* = \frac{\phi_5 \tau_1 x_3^*}{\phi_1(\tau_1 - p)},$$

$$x_9^* = \frac{\phi_5 \tau_1 x_4^*}{\phi_1(\tau_2 - p)},\ x_{10}^* = \frac{-1}{\phi_1}(x_2^* + \phi_2 x_6^*) - x_1^*,$$

$$x_{11}^* = \frac{-1}{\phi_1}\{\phi_2 x_7^* + q_1 V_{m1} + q_2 V_{m2} - x_1^*\} - x_2^*,$$

$$x_{12}^* = \frac{-1}{\phi_1}(\tau_1 x_3^* + \phi_2 x_9^*) - x_3^*,$$

$$x_{13}^* = \frac{-1}{\phi_1}(\tau_2 x_4^* + \phi_2 x_{10}^*) - x_4^*, \quad x_{14}^* = \frac{-\phi_2 x_8^*}{\phi_1},$$

$$f_1^* = \frac{\tau_2}{1 + \tau_2^2} - \frac{\tau_1}{1 + \tau_1^2},\ f_2^* = \frac{1}{1 + \tau_2^2} - \frac{1}{1 + \tau_1^2}$$

$$f_3^* = \frac{\phi_5}{\phi_1}\{x_1^* - (q_1 V_{m1} + q_2 V_{m2})\} + q_5 V_{m1} + q_6 V_{m2}$$

and $k_1 = \phi_1(q_3 V_{m1} + q_4 V_{m2}) + \sigma(q_1 V_{m1} + q_2 V_{m2}) + \phi_2(q_5 V_{m1} + q_6 V_{m2})$, $\varphi_i, q_i, i = 1{:}6$ are defined in (3.43), and $\tau_{1,2}$ are defined in (3.52). We considered the following parameters for application

3: $R_1 = 10\ \ \Omega, R_2 = 3\ \ \Omega, R_3 = 10\ \ \Omega, R_4 = 7\ \ \Omega,\ R_5 = 2\ \ \Omega,\ c_1 = 10\ f, c_2 = 1\ f, V_{m1} = 200\ V, V_{m2} = 100\ V, \alpha = 1$. Fig. 8(a–c) shows the time response of the currents $i_1(t), i_2(t)$ *and* $i_3(t)$ for h = 0.02 and $\phi(h) = \sin h$. We can conclude that NSFD method is superior to RK method as shown in Fig. 8(d–f).

Example 4.6. For the circuit of application 3, using the NSFD method and assume the following parameter $R_1 = 100\,\Omega,\ R_2 = 30\ \Omega,\ R_3 = 100\ \Omega,\ R_4 = 17\ \Omega,\ R_5 = 2\ \Omega,\ c_1 = 10\ f,\ c_2 = 1\ f, V_{m1} = 200V,\ V_{m2} = 100V$. Fig. 9 shows the time response of the currents $i_1(t),\ i_2(t)$ and $i_3(t)$ for h = 0.02, $\phi(h) = \sinh h$ and different values of the fractional order α

5. Conclusions

The fractional calculus is used to generalize RL and RC circuits. The fractional modeling introduces new parameters which provide more accurate representations of the real capacitor and real inductor. Solutions of the mathematical models of these circuits are derived using Laplace transform, RK, and NSFD methods. We observed that the amplitudes of the oscillations attenuate with decreasing the fractional order α. Also, the numerical results given by NSFD method is in a good agreement with LT method and more efficient than RK method in the case of integer order α. We conclude that the methodology applied in this paper can be extended for other electrical systems.

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