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Bounding Unknown Functions in Nonlinear Integral Inequalities: A Comprehensive Study

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Abstract: This article presents a comprehensive examination of linear and non-linear systems of integral inequalities involving two real-valued unknown functions in n independent variables. The primary objective of this investigation is to establish upper bounds for these unknown functions and to analyze their practical implications within broader mathematical frameworks. The results obtained not only extend the classical Grönwall-Bellman integral inequalities but also introduce novel and explicit bounds within the contexts of Young and Pachpatte integral inequalities. These contributions significantly enhance the theoretical understanding of integral inequalities and their utility in addressing complex analytical problems. Moreover, the results yield important insights into the qualitative analysis of nonlinear hyperbolic partial integro-differential equations, particularly with regard to the existence, uniqueness, and boundedness of solutions. To derive the main theoretical results, Young's method based on the Riemann approach is employed. Additionally, the analysis highlights the essential role of symmetry in the selection of appropriate methods for treating dynamic inequalities.

Keywords: Integral inequalities, Upper bounds, Nonlinear systems, Hyperbolic equations, Symmetry.

2020 AMS Subject Classifications: 26D15, 45G10, 34A40.

1 Introduction

The exploration of integral inequalities has always been an important field of study. Recent studies have focused on the refinement and extension of classical inequalities within multidimensional and dynamic frameworks [1–3] particularly by incorporating general kernels, weight functions, and fractional operators. These efforts emphasize the growing demand for adaptable inequality structures capable of handling complex systems and integral operators. Integral inequalities not only serve as fundamental tools in the study of mathematical systems but also find applications in a wide array of practical scenarios see for instance [4, 5].

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The integral form of Grönwall-Bellman's inequality [6] asserts the following: Assuming continuous and non-negative functions u and f defined on the interval $[a,b]$, with u_0 being a non-negative constant, the inequality

$$u(t) \leq u_0 + \int_a^t f(s)u(s)ds, \quad \text{for all} \quad t \in [a,b], \tag{1}$$

implies that

$$u(t) \leq u_0 \exp\left(\int_a^t f(s)ds\right), \quad \text{for all} \quad t \in [a,b].$$

For interested readers, it is useful to mention here that some efforts have been made to generalize Grönwall-Bellman's inequality to weakly singular situations to cope with some problems of fractional differential equations, see for instance [7–9]. Baburao G. Pachpatte [10] established the discrete counterpart of inequality (1). Specifically, he demonstrated that for nonnegative sequences $u(n)$, $a(n)$, and $\gamma(n)$ defined for n in the set of non-negative integers $\mathbb{N}_0$, with $a(n)$ being non-decreasing for n in $\mathbb{N}_0$, if the condition

$$u(n) \leq a(n) + \sum_{s=0}^{n-1} \gamma(n)u(n), n \in \mathbb{N}_0,$$

holds, then

$$u(n) \leq a(n) \prod_{s=0}^{n-1} [1+\gamma(n)], n \in \mathbb{N}_0.$$

In [11], the following nonlinear integral inequality was discussed:

$$\Phi(u(t)) \leq c(t) + \int_0^{\theta(t)} [f(t,s)\zeta(u(s))\,\omega(u(s)) + g(t,s)\zeta(u(s))]\,ds,$$

for $u \in ([0,\infty),[0,\infty))$, $f(t,s)$ and $g(t,s) \in C([0,\infty)\times[0,\infty),[0,\infty))$ are non-decreasing in t for every s fixed, $\Phi \in C([0,\infty),[0,\infty))$ is a strictly increasing function such that $\lim_{\chi\to\infty} \Phi(\chi) = \infty$, $c \in C([0,\infty),(0,\infty))$ and $\zeta, \omega \in C([0,\infty),[0,\infty))$ are non-decreasing functions.

Motivated by the results obtained in [11], the authors in [12] investigated the following inequality:

$$\begin{aligned}\Psi(u(\ell,t)) \leq a(\ell,t) + \int_0^{\theta(\ell)} \int_0^{\vartheta(t)} \Im_1(\varsigma,\eta)\,[f(\varsigma,\eta)\zeta(u(\varsigma,\eta))\,\varpi(u(\varsigma,\eta)) \\ + \int_0^{\varsigma} \Im_2(\chi,\eta)\zeta(u(\chi,\eta))\,\varpi(u(\chi,\eta))\,d\chi\Big]\,d\eta\,d\varsigma,\end{aligned}$$

where u, f, a, $\Im \in C(I_1\times I_2,[0,\infty))$ are non-decreasing functions, $I_1, I_2 \subset [0,\infty)$, $\theta \in C^1(I_1,I_1)$, and $\vartheta \in C^1(I_2,I_2)$ are non-decreasing with $\theta(\ell) \leq \ell$ on I_1, $\vartheta(t) \leq t$ on I_2, $\Im_1$, $\Im_2 \in C(I_1\times I_2,[0,\infty))$, and Ψ, ζ, $\varpi \in C([0,\infty),[0,\infty))$ with $\{\Psi,\zeta,\varpi\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \Psi(u) = +\infty$.

Moreover, Anderson [13] established some new nonlinear dynamic inequalities in two independent variables of Pachpatte type, that is useful tools in the study of qualitative properties of solutions of certain classes of dynamic equations on time scales. For instance in [13] we find the following inequality:

$$\omega(u(t,s)) \leq a(t,s) + c(t,s)\int_{t_0}^t \int_s^\infty \omega'(u(\tau,\eta))[d(\tau,\eta)w(u(\tau,\eta)) + b(\tau,\eta)]\nabla\eta\Delta\tau,$$

here u, a, c, and d are non-negative continuous functions defined for $(t,s) \in \mathbb{T} \times \mathbb{T}$, b is a non-negative continuous function for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ and $\omega \in C^1(\mathbb{R}_+,\mathbb{R}_+)$, with $\omega' > 0$ for $u > 0$.

While numerous mathematicians have devoted their efforts to the refinement and generalization of integral inequalities of the Grönwall type, a parallel strand of research has emerged with a distinct focus on systems of integral inequalities. This dual trajectory underscores the multifaceted nature of mathematical inquiry, where the pursuit of enhancing individual inequalities coexists with a dedicated exploration of the intricate relationships within systems. As researchers delve into both aspects, their endeavors collectively contribute to a more comprehensive understanding of mathematical models and systems. In [16] a linear and nonlinear two-dimensional integral inequalities system was established. For instance in [16], for u_i and $f_i \in C(I,R_+), i=1,2$ with f_i be non-decreasing; $\phi_{ij} \in C(I,R_+)$ are non-decreasing in the variable t for every s fixed $(i=1,2)$. If

$$u_i(t) \le f_i(t) + \int_0^t [\phi_{i1}(t,s)u_1(s) + \phi_{i2}u_2(s)]ds, \tag{2}$$

then for $t \in I$ we have

$$u_i(t) \le [f_i(t)+f_{i+1}(t)\int_0^t \phi_{ii+1}(t,s)\Phi_{i+1}(s)ds] \\ \exp\int_0^t \phi_{ii}(t,s)ds + \int_0^t \phi_{ii+1}(t,s)\Phi_{i+1}(s)\left(\int_0^s \phi_{i+1i}(s,\tau)\Phi_i(\tau)\right)ds,$$

where $\Phi_i(t) := \exp\int_0^t \phi_{ii}(t,s)ds$, $i=1,2$, and if $i=2$ then $f_{i+1}=f_1, \Phi_{i+1}=\Phi_1, \phi_{i+1i}=\phi_{12}, \phi_{ii+1}=\phi_{21}$.

The authors in [17] provided explicit bounds for certain classes of systems of integral inequalities such as the following system: For $u_i(x), b_i(x), q_i(x), e_i(x), f_i(x), g_i(x)$, and $h_i(x)$ are non-negative, real valued continuous functions on Ω and $a_i(x)$ be positive, non-decreasing, and continuous functions on Ω; $i=1,2$. If

$$u_i(x) \le a_i(x) + \int_{x^o}^x b_i(s)u_1(s)ds + \int_{x^o}^x q_i(s)u_2(s)ds + \int_{x^o}^x e_i(s)\left(\int_{x^o}^s f_i(t)u_1(t)dt\right)ds \\ + \int_{x^o}^x g_i(s)\left(\int_{x^o}^s h_i(t)u_2(t)dt\right)ds, \tag{3}$$

is satisfied for all $x \in \Omega$ with $x \ge x^o$, then,

$$u_i(x) \le a_i(x)\left(1 + \int_{x^o}^x \left(\phi_i(s)\eta(s)ds + \rho_i(s)\int_{x^o}^s \psi_i(t)\eta(t)dt\right)ds\right), \tag{4}$$

where,
$\phi_1(x) = b_1(x) + \frac{a_2(x)}{a_1(x)}q_1(x)$, $\phi_2(x) = q_2(x) + \frac{a_1(x)}{a_2(x)}b_2(x)$, $\phi(x) = \sum_{i=1}^2 \phi_i(x)$,
$\psi_1(x) = f_1(x) + \frac{a_2(x)}{a_1(x)}h_1(x)$, $\psi_2(x) = h_2(x) + \frac{a_1(x)}{a_2(x)}f_2(x)$, $\psi(x) = \sum_{i=1}^2 \psi_i(x)$,
$\rho_i(x) = e_i(x) + g_i(x)$, and $\eta(x) = 2 + 2\int_{x^o}^x \phi(s)\exp\left(\int_{x^o}^s (\phi(t)+\psi(t))\,dt\right)ds$.

This paper aims to explore into the advancements achieved in generalizing Grönwall-type integral inequalities, alongside a comprehensive examination of the evolving landscape surrounding systems of integral inequalities, highlighting the connections and implications arising from these parallel research streams. In the current paper we provide upper bounds for some systems of linear and non-linear integral inequalities. On the other hand, we extend existing Grönwall-Bellman integral inequalities and introduce new explicit boundaries.

The remainder of this paper is organized as follows. In Section 2, we establish essential notations and present two preliminary lemmas that form the foundation for the subsequent analysis. Section 3 contains the main theoretical results, where we derive explicit upper bounds for various systems of linear and nonlinear integral inequalities. Section 4 demonstrates the applicability of these results through illustrative examples and practical applications.

2 Preliminaries

In this section, we introduce two essential lemmas that play a pivotal role in substantiating the central results presented in this paper. However, before presenting these lemmas, we will establish some key notations. We represent x as $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ to denote elements within the n-dimensional Euclidean space $\mathbb{R}^n$. Furthermore, we define the relation $x \leq y$ if and only if $x_i \leq y_i$ for each $i = 1, ..., n$. Additionally, we use x^o and x to refer to any two points in an open bounded set $\Omega \subset \mathbb{R}^n$ with $x^o < x$. In the subsequent discussion, the notation $\int_x^y ds$ signifies an integral, computed as: $\int_{x_1}^{y_1} ... \int_{x_n}^{y_n} ds_n ... ds_1$, where the operator D is defined as $D = D_1 ... D_n$ with D_i denoting the partial derivative with respect to x_i, for $i = 1, ..., n$ i.e., $D_i = \partial / \partial x_i$, for $i = 1, ..., n$.

In order to establish the forthcoming lemma, we will employ the methodology of Young, as exemplified in [14].

Lemma 1. *Let $K(x)$, $B(x)$, and $\sigma(x)$ be real valued non-negative differentiable functions on Ω. Moreover, suppose that $K(x)$ and all its derivatives with respect to $x_1, ..., x_n$ up to order $n-1$ vanish at $x_i = x_i^o$ for $i = 1, ..., n$. Let $v(s;x)$ be the solution of the following characteristic initial value problem*

$$(-1)^n \frac{\partial^n v(s;x)}{\partial s_1 ... \partial s_n} - \sigma(s) v(s;x) = 0, \ in \ \Omega,$$
$$v(s;x) = 1 \ on \ s_i = x_i, \ i = 1, ..., n. \tag{5}$$

If the inequality

$$D_1 ... D_n K(x) \leq B(x) + \sigma(x) K(x), \tag{6}$$

holds, then

$$K(x) \leq \int_{x^o}^{x} B(s) v(s;x) ds. \tag{7}$$

Proof . Inequality (6) implies that

$$\mathscr{L}[K(x)] \leq B(x), \ where \ \mathscr{L} \equiv D_1 ... D_n - \sigma(x). \tag{8}$$

If $z(x)$ is a function that is n times continuously differentiable within the region defined as $x^o < t < x$ (referred to as $\mathscr{D}$) then

$$z\mathscr{L}[K] - K\mathscr{L}_1[z] = \sum_{j=1}^{n} (-1)^{j-1} D_j[(D_0 ... D_{j-1} z)(D_{j+1} ... D_n D_{n+1} K)], \tag{9}$$

here, we define $\mathscr{L}_1 \equiv (-1)^n D_1 ... D_n - \sigma(x)$, and $D_0 = D_{n+1} = I$ where I represents the identity operator. Integrating both sides of relation (9) over $\mathscr{D}$ while considering that $K(x)$ and all its derivatives with respect to $x_1, ..., x_n$ up to the $(n-1)$th order, vanish at $s_i = x_i^o$ for $i = 1, ..., n$, results in

$$\int_{\mathscr{D}} (z\mathscr{L}[K] - K\mathscr{L}_1[z]) \, ds = \sum_{j=1}^{n} (-1)^{j-1} \int_{s_j = x_j} (D_1 ... D_{j-1} z)(D_{j+1} ... D_n K) ds', \tag{10}$$

where $ds' = ds_1 ... ds_{j-1} ds_{j+1} ... ds_n$. Now, let's select $z(x)$ to be the function $v(s;x)$, which satisfies the initial value problem (5). Given that $v(s;x)$ equals 1 when $s_j = x_j$ for $j = 1, ..., n$, we can conclude that

$$D_1 ... D_{j-1} v(s;x) = 0, \ on \ s_j = x_j, \ j = 2, ..., n.$$

Therefore, relation (10) becomes

$$\begin{aligned} \int_{\mathscr{D}} v \mathscr{L}[K] ds &= \int_{s_1 = x_1} D_2 ... D_n K \, ds' \\ &= K(x). \end{aligned} \tag{11}$$

The continuity of v, along with the condition $v = 1$ when $s = x$ ensures the existence of a domain denoted as Ω^+ which includes x for which $v \geq 0$. By multiplying both sides of (8) by v and utilizing (11), we can derive (7), thereby concluding the proof of the lemma.

To confirm the credibility of the subsequent lemma, we will employ Bellman's method, as exemplified in [15].

Lemma 2. *Let $f(x)$, $w(x)$ be real valued, positive, and continuous functions. In addition, let all derivatives of $f(x)$ be positive on Ω with $f(x) = 1$ on $x_i = x_i^o$. If the inequality*

$$D_1 ... D_n f(x) \leq w(x) f(x), \tag{12}$$

holds, then

$$f(x) \leq \exp\left(\int_{x^o}^{x} w(t) dt \right). \tag{13}$$

Proof . Inequality (12) leads to

$$\frac{f(x) D_1 ... D_n f(x)}{f^2(x)} \leq w(x).$$

Hence, considering the given assumptions on $f(x)$ and its derivatives, we obtain

$$\frac{f(x) D_1 ... D_n f(x)}{f^2(x)} \leq w(x) + \frac{(D_n f(x)) (D_1 ... D_{n-1} f(x))}{f^2(x)},$$

which implies that

$$D_n \left(\frac{D_1 ... D_{n-1} f(x)}{f(x)} \right) \leq w(x). \tag{14}$$

Integrate both sides of inequality (14) with respect to the component x_n over the interval from x_n^o to x_n to yield

$$\frac{D_1 ... D_{n-1} f(x)}{f(x)} \leq \int_{x_n^o}^{x_n} w(x_1, ..., x_{n-1}, t_n) dt_n.$$

Thus, based on the provided assumptions regarding $f(x)$ and its derivatives, we can formulate the subsequent inequality

$$\frac{f(x) D_1 ... D_{n-1} f(x)}{f^2(x)} \leq \int_{x_n^o}^{x_n} w(x_1, ..., x_{n-1}, t_n) dt_n + \frac{(D_{n-1} f(x)) (D_1 ... D_{n-2} f(x))}{f^2(x)},$$

this inequality yields

$$D_{n-1} \left(\frac{D_1 ... D_{n-2} f(x)}{f(x)} \right) \leq \int_{x_n^o}^{x_n} w(x_1, ..., x_{n-1}, t_n) dt_n. \tag{15}$$

Now, perform the integration of both sides of equation (15) with respect to the component x_{n-1} over the interval from x^o_{n-1} to x_{n-1} to obtain

$$\frac{D_1...D_{n-2}f(x)}{f(x)} \leq \int_{x^o_{n-1}}^{x_{n-1}} \int_{x^o_n}^{x_n} w(x_1,...x_{n-2},t_{n-1},t_n)dt_n dt_{n-1}.$$

Proceed in this manner until arriving at

$$\frac{D_1 f(x)}{f(x)} \leq \int_{x^o_2}^{x_2} ... \int_{x^o_n}^{x_n} w(x_1,t_2,...,t_n)dt_n...dt_2. \tag{16}$$

By integrating both sides of (16) with respect to the component x_1 from x^o_1 to x_1 we obtain

$$\log\left(\frac{f(x)}{f(x^o_1,x_2,...,x_n)}\right) \leq \int_{x^o}^{x} w(t)dt.$$

This implies the validity of (13), thus establishing the lemma.

We are now ready to introduce and provide support for our primary findings.

3 Main results

In this section, we present and rigorously validate our central results through a series of theorems. This achievement is made possible by drawing upon the Lemmas 1 and 2, which have been previously established.

Theorem 1. *Suppose that $u_i(x), a_i(x), Db_i(x)$, and $Dc_i(x)$; $i = 1,2$, are real valued, positive, continuous and non-decreasing functions defined on Ω. Let $v(s;x)$ be the solution of the following characteristic initial value problem*

$$(-1)^n \frac{\partial^n v(s;x)}{\partial s_1...\partial s_n} - [1 + D\psi(x) + Q(x)]\, v(s;x) = 0, \text{ in } \Omega,$$
$$v(s;x) = 1 \text{ on } s_i = x_i,\ i = 1,...,n,$$

where $\psi(x) = \sum_{i=1}^{2} \psi_i(x)$, $\psi_i(x) = b_i(x) + c_i(x)$, $Q(x) = \sum_{j=1}^{4} Q_j(x)$. Let Ω_o be a connected subdomain of Ω which contains x such that $v(s;x) \geq 0$, for all $s \in \Omega_o$. If the system

$$u_i(x) \leq a_i(x) + \int_{x^o}^{x} b_i(x,s)u_1(s)ds + \int_{x^o}^{x} c_i(x,s)u_2(s)ds,\ i = 1,2, \tag{17}$$

holds, then

$$u_i(x) \leq a_i(x) + \int_{x^o}^{x} [\phi_i(s) + D\psi_i(s)\tau_1(s) + Q_{i+4}(x)\tau_2(s)]\, ds,\ i = 1,2, \tag{18}$$

where $\tau_1(s) = \int_{x^o}^{s} \left(\int_{x^o}^{t} [\phi(r) + (D\psi(r) + Q(r)) \int_{x^o}^{r} \phi_1(\theta)v(r,\theta)d\theta]\, dr\right) dt$,
$\tau_2(s) = \int_{x^o}^{s} \left(\phi(t) + (D\psi(t) + Q(t)) \int_{x^o}^{t} \phi(r)v(t,r)dr\right) dt$,
$\phi_i(x) = \int_{x^o}^{x} [a_1(s)Db_i(x,s) + a_2(s)Dc_i(x,s)]\, ds + a_1(x)Q_i(x) + a_2(x)Q_{i+2}(x)$,

$\phi(x) = \sum_{i=1}^{2} \phi_i(x),\ Q_{i+4}(x) = Q_i(x) + Q_{i+2}(x);\ i = 1,2.$ *and the functions* $Q_j(x),\ j = 1,2,3,4,$ *are as follows*

$$
\begin{aligned}
&Q_j(x) = \\
&\sum_{i=1}^{n} \int_{x_1^o}^{x} \cdots \int_{x_{i-1}^o}^{x_{i-1}} \int_{x_{i+1}^o}^{x_{i+1}} \cdots \int_{x_n^o}^{x_n} \frac{\partial^{n-1}}{\partial x_1 \ldots \partial x_{i-1} \partial x_{i+1} \ldots \partial x_n} \kappa_j(x, s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \\
&\qquad\qquad ds_n \ldots ds_{i+1} ds_{i-1} \ldots ds_1 \\
&+ \sum_{i=1}^{n} \int_{x_1^o}^{x} \cdots \int_{x_{i-2}^o}^{x_{i-2}} \int_{x_{i+2}^o}^{x_{i+2}} \cdots \int_{x_n^o}^{x_n} \frac{\partial^{n-2}}{\partial x_1 \ldots \partial x_{i-2} \partial x_{i+2} \ldots \partial x_n} \kappa_j(x, s_1, \ldots, s_{i-2}, s_{i+2}, \ldots, s_n) \\
&\qquad\qquad ds_n \ldots ds_{i+2} ds_{i-2} \ldots ds_1 \\
&+ \ldots + \kappa_j(x), \\
&;\ \kappa_1(.) = b_1(.), \kappa_2(.) = b_2(.), \kappa_3(.) = c_1(.), \kappa_4(.) = c_2(.).
\end{aligned}
$$

Proof . Let's define the functions $A_i(x)$, $B_i(x)$ and $\xi_i(x)$ as follows:

$$
\begin{aligned}
&A_i(x) = \int_{x^o}^{x} u_1(s) b_i(x,s) ds,\ B_i(x) = \int_{x^o}^{x} u_2(s) c_i(x,s) ds,\ \xi_i(x) = A_i(x) + B_i(x), \\
&\text{for } i = 1,2,\ \xi(x) = \xi_1(x) + \xi_2(x).
\end{aligned} \tag{19}
$$

As a result, the inequalities, or equivalently, the system described in (17), can be expressed in the following manner

$$
u_i(x) \le a_i(x) + \xi_i(x),\ i = 1,2. \tag{20}
$$

Given that the functions $u_i(x)$, $i = 1,2$, are non-decreasing functions, we can establish the following inequalities through the application of Leibniz's integral rule and (19)

$$
\begin{aligned}
D_1 \ldots D_n A_i(x) &\le \int_{x^o}^{x} u_1(s) D_1 \ldots D_n b_i(x,s) ds + u_1(x) Q_i(x), \\
D_1 \ldots D_n B_i(x) &\le \int_{x^o}^{x} u_2(s) D_1 \ldots D_n c_i(x,s) ds + u_2(x) Q_{i+2}(x).
\end{aligned} \tag{21}
$$

Use (20) in (21) to obtain

$$
D_1 \ldots D_n A_i(x) \le \int_{x^o}^{x} (a_1(s) + \xi_1(s))\, D_1 \ldots D_n b_i(x,s) ds + (a_1(x) + \xi_1(x))\, Q_i(x), \tag{22}
$$

and

$$
D_1 \ldots D_n B_i(x) \le \int_{x^o}^{x} (a_2(s) + \xi_2(s))\, D_1 \ldots D_n c_i(x,s) ds + (a_2(x) + \xi_2(x))\, Q_{i+2}(x). \tag{23}
$$

Adding (22) and (23) for the case $i = 1$ gives

$$
\begin{aligned}
D\xi_1(x) &\le \int_{x^o}^{x} [a_1(s) Db_1(x,s) + a_2(s) Dc_1(x,s) + \xi_1(s) Db_1(x,s) + \xi_2(s) Dc_1(x,s)]\, ds \\
&\qquad + (a_1(x) + \xi_1(x))\, Q_1(x) + (a_2(x) + \xi_2(x))\, Q_3(x) \\
&\le \phi_1(x) + \int_{x^o}^{x} (\xi(s) Db_1(x,s) + \xi(x) Dc_1(x,s))\, ds + Q_5(x) \xi(x) \\
&\le \phi_1(x) + D\psi_1(x) \int_{x^o}^{x} \xi(s) ds + Q_5(x) \xi(x).
\end{aligned} \tag{24}
$$

Likewise, inequalities (22) and (23) when $i = 2$ result in

$$D\xi_2(x) \leq \phi_2(x) + D\psi_2(x)\int_{x^o}^{x} \xi(s)ds + Q_6(x)\xi(x). \tag{25}$$

When we combine (24) with (25), we obtain

$$\begin{aligned} D\xi(x) \leq & \phi(x) + \xi_1(x)(Q_1(x) + Q_2(x)) + \xi_2(x)(Q_3(x) + Q_4(x)) \\ & + \int_{x^o}^{x} [\xi_1(s)D(b_1(x,s) + b_2(x,s)) + \xi_2(s)D(c_1(x,s) + c_2(x,s))]\, ds, \end{aligned} \tag{26}$$

where the function $\phi(x)$ is

$$\begin{aligned} \phi(x) = & a_1(x)(Q_1(x) + Q_2(x)) + a_2(x)(Q_3(x) + Q_4(x)) \\ & + \int_{x^o}^{x} [a_1(s)D(b_1(x,s) + b_2(x,s)) + a_2(s)D(c_1(x,s) + c_2(x,s))]\, ds. \end{aligned}$$

Given that all these functions are positive and non-decreasing, we can express inequality (26) in the following way:

$$D\xi(x) \leq \phi(x) + \xi(x)Q(x) + \int_{x^o}^{x} \xi(s)D\psi(x,s)ds, \tag{27}$$

where $\psi(x,s) = \sum_{i=1}^{2}(b_i(x,s) + c_i(x,s))$, and $Q(x) = \sum_{j=1}^{4} Q_j(x)$. As both $b_i(x,s)$ and $c_i(x,s)$ with $i = 1,2$, along with their derivatives, are positive and non-decreasing functions, it follows that both $\psi(x,s)$ and $D\psi(x,s)$ are also positive and non-decreasing. Consequently, inequality (27) can be expressed as follows:

$$D\xi(x) \leq \phi(x) + \xi(x)Q(x) + D\psi(x)\int_{x^o}^{x} \xi(s)ds. \tag{28}$$

Include $\xi(x)D\psi(x)$ on the right-hand side of (28) to have

$$D\xi(x) \leq \phi(x) + \xi(x)Q(x) + D\psi(x)\int_{x^o}^{x} \xi(s)ds + \xi(x)D\psi(x). \tag{29}$$

Take $K(x) = \xi(x) + \int_{x^o}^{x} \xi(s)ds$ which implies that

$$\begin{aligned} & K(x) \geq \xi(x),\ K(x) \geq \int_{x^o}^{x} \xi(s)ds,\ K(x^o) = 0, \\ & DK(x) = D\xi(x) + \xi(x),\ DK(x) - K(x) \leq DK(x) - \xi(x) = D\xi(x). \end{aligned} \tag{30}$$

The first part of (30) is obtained by differentiating $K(x)$ with respect to $x_i; i = 1,2,\ldots$ respectively, while the second part is straightforward since $K(x) \geq \xi(x)$. Therefore, inequality (29) becomes

$$D\xi(x) \leq \phi(x) + D\psi(x)K(x) + K(x)Q(x). \tag{31}$$

Relations (31) and (30) imply that

$$\begin{aligned} & D\xi(x) - D\psi(x)K(x) - K(x)Q(x) \leq \phi(x), \\ & DK(x) - [1 + D\psi(x) + Q(x)]K(x) \leq \phi(x). \end{aligned} \tag{32}$$

Applying Lemma 1 on (32) gives $K(x) \leq \int_{x^o}^{x} \phi(s)v(x,s)ds$. Therefore, inequality (31) becomes

$$D\xi(x) \leq \phi(x) + [D\psi(x) + Q(x)] \int_{x^o}^{x} \phi(s)v(x,s)ds. \tag{33}$$

Integrating both sides of inequality (33) yields

$$\xi(x) \leq \int_{x^o}^{x} [\phi(s) + [D\psi(s) + Q(s)] \int_{x^o}^{s} \phi(t)v(s,t)dt]ds. \tag{34}$$

Now from (24) and (34) we get

$$\begin{aligned} D\xi_1(x) \leq \phi_1(x) + D\psi_1(x) \int_{x^o}^{x} \left(\int_{x^o}^{s} \left[\phi(t) + (D\psi(t) + Q(t)) \int_{x^o}^{t} \phi(r)v(t,r)dr \right] dt \right) ds \\ + Q_5(x) \int_{x^o}^{x} \left(\phi(s) + (D\psi(s) + Q(s)) \int_{x^o}^{s} \phi(t)v(s,t)dt \right) ds. \end{aligned} \tag{35}$$

By integrating both sides of inequality (35) with respect to the variable x over the interval from x^o to x, we obtain

$$\begin{aligned} \xi_1(x) \leq \int_{x^o}^{x} (\phi_1(s) + D\psi_1(s) \int_{x^o}^{s} \left(\int_{x^o}^{t} \left[\phi(r) + (D\psi(r) + Q(r)) \int_{x^o}^{r} \phi(\theta)v(r,\theta)d\theta \right] dr \right) dt \\ + Q_5(s) \int_{x^o}^{s} \left(\phi(t) + (D\psi(t) + Q(t)) \int_{x^o}^{t} \phi(r)v(t,r)dr \right) dt)ds, \end{aligned} \tag{36}$$

where

$$\begin{aligned} &\phi_1(x) = \int_{x^o}^{x} [a_1(s)Db_1(x,s) + a_2(s)Dc_1(x,s)]\, ds + a_1(x)Q_1(x) + a_2(x)Q_3(x), \\ &\psi_1(x) = b_1(x,s) + c_1(x,s), \; and \; Q_5(x) = Q_1(x) + Q_3(x). \end{aligned}$$

Similarly, by (25) and (34) we obtain

$$\begin{aligned} \xi_2(x) \leq \int_{x^o}^{x} (\phi_2(s) + D\psi_2(s) \int_{x^o}^{s} \left(\int_{x^o}^{t} \left[\phi(r) + (D\psi(r) + Q(r)) \int_{x^o}^{r} \phi(\theta)v(r,\theta)d\theta \right] dr \right) dt \\ + Q_6(s) \int_{x^o}^{s} \left(\phi(t) + (D\psi(t) + Q(t)) \int_{x^o}^{t} \phi(r)v(t,r)dr \right) dt)ds, \end{aligned} \tag{37}$$

where

$$\begin{aligned} &\phi_2(x) = \int_{x^o}^{x} [a_1(s)Db_2(x,s) + a_2(s)Dc_2(x,s)]\, ds + a_1(x)Q_2(x) + a_2(x)Q_4(x), \\ &\psi_2(x) = b_2(x,s) + c_2(x,s), \; and \; Q_6(x) = Q_2(x) + Q_4(x). \end{aligned}$$

By substituting the expressions from (36) and (37) into (20) , we arrive at the result stated in (18) . This concludes the proof.

Remark. In the case where n equals 2, meaning we are working with functions in the two-dimensional space $\mathbb{R}^2$, we can derive from Theorem 1 that
If, for $i = 1,2$,

$$\begin{aligned} u_i(x_1,x_2) \leq a_i(x_1,x_2) + \int_{x_1^o}^{x_1} \int_{x_2^o}^{x_2} b_i(x_1,x_2,s_1,s_2)u_1(s_1,s_2)ds_2ds_1 \\ + \int_{x_1^o}^{x_1} \int_{x_2^o}^{x_2} c_i(x_1,x_2,s_1,s_2)u_2(s_1,s_2)ds_1ds_2, \end{aligned} \tag{38}$$

then

$$
\begin{aligned}
u_i(x_1,x_2) \le a_i(x_1,x_2) + \int_{x_1^o}^{x_1}\int_{x_2^o}^{x_2} \Bigg\{ & \phi_i(s_1,s_2) + \frac{\partial^2}{\partial s_1 \partial s_2}\psi_i(s_1,s_2) \\
& \int_{x_1^o}^{s_1}\int_{x_2^o}^{s_2} \left(\int_{x_1^o}^{t_1}\int_{x_2^o}^{t_2} \left[\phi(r_1,r_2) + \left(\frac{\partial^2}{\partial r_1 \partial r_2}\psi(r_1,r_2) + Q^*(r_1,r_2) \right) \right.\right. \\
& \left.\left. \times \int_{x_1^o}^{r_1}\int_{x_2^o}^{r_2} \phi_i(\theta_1,\theta_2) v(r_1,r_2,\theta_1,\theta_2) d\theta_2 d\theta_1 \right] dr_2 dr_1 \right) dt_2 dt_1 \\
& + Q_5^*(x_1,x_2) \int_{x_1^o}^{s_1}\int_{x_2^o}^{s_2} \left(\phi(t_1,t_2) + \left(\frac{\partial^2}{\partial t_1 \partial t_2}\psi(t_1,t_2) + Q^*(t_1,t_2) \right) \right. \\
& \left. \times \int_{x_1^o}^{t_1}\int_{x_2^o}^{t_2} \phi(r) v(t_1,t_2,r_1,r_2) dr_2 dr_1 \right) dt_2 dt_1 \Bigg\} ds_2 ds_1,
\end{aligned} \tag{39}
$$

where

$$
\begin{aligned}
\phi_i(x_1,x_2) = & \int_{x_1^o}^{x_1}\int_{x_2^o}^{x_2} \left[a_1(s_1,s_2)\frac{\partial^2}{\partial x_1 \partial x_2} b_i(x_1,x_2,s_1,s_2) + a_2(s_1,s_2)\frac{\partial^2}{\partial x_1 \partial x_2} c_i(x_1,x_2,s_1,s_2) \right] ds_2 ds_1 \\
& + a_1(x_1,x_2)Q_i(x_1,x_2) + a_2(x_1,x_2)Q_{i+2}(x_1,x_2);
\end{aligned}
$$

$$
\phi(x_1,x_2) = \sum_{i=1}^{2} \phi_i(x_1,x_2),\ Q_5^*(x_1,x_2) = \sum_{i=1,3} Q_i^*(x_1,x_2), Q_6^*(x_1,x_2) = \sum_{i=2,4} Q_i^*(x_1,x_2)
$$

and

$$
\begin{aligned}
& Q_i^*(x_1,x_2) = \int_{x_1^o}^{x_1} \frac{\partial}{\partial x_1}\kappa_i(x_1,x_2,s_1,x_2) ds_1 + \int_{x_2^o}^{x_2} \frac{\partial}{\partial x_1}\kappa_i(x_1,x_2,x_1,s_2) ds_2 + \kappa_i(x_1,x_2); \\
& \kappa_1(.) = b_1(.), \kappa_2(.) = b_2(.), \kappa_3(.) = c_1(.), \kappa_4(.) = c_2(.).
\end{aligned}
$$

Compared to classical results such as those in [16] and [17], which provide bounds for two-dimensional systems, the current formulation refines the estimates by incorporating the effects of mixed partial derivatives and interaction terms via the function $v(.)$, derived from a characteristic initial value problem. This yields more precise bounds and greater flexibility in applications where explicit kernel structure and regularity properties are known. Thus, remark 3 enhances previous work by delivering a higher-order correction to integral estimates in two variables.

Theorem 2. *Let $u_i(x), b_i(x), q_i(x)$ be real valued, non-negative, and continuous functions on Ω and $a_i(x)$ be positive, non-decreasing, and continuous function on Ω; $i = 1,2$. If the system*

$$
u_i(x) \le a_i(x) + \int_{x^o}^{x} b_i(s)u_1(s)ds + \int_{x^o}^{x} q_i(s)u_2(s)ds, \tag{40}
$$

holds for all $x \in \Omega$ with $x \ge x^o$, then

$$
u_i(x) \le a_i(x)\left(1 + 2\int_{x^o}^{x} [\phi_i(s)\exp(\eta(s))]\, ds\right), \tag{41}
$$

where, $\phi_1(s) = b_1(s) + \frac{q_1(s)a_2(s)}{a_1(s)}$, $\phi_2(s) = q_2(s) + \frac{b_2(s)a_1(s)}{a_2(s)}$, and
$\eta(s) = \int_{x^o}^{s}\phi(t)dt$; $\phi(t) = \phi_1(t) + \phi_2(t)$.

Proof . As the functions $a_i(x),\ i=1,2$, are both positive and non-decreasing, the inequalities presented in (40) can be expressed as follows:

$$\frac{u_i(x)}{a_i(x)} \leq 1+\int_{x^o}^{x} b_i(s)\frac{u_1(s)}{a_i(s)}ds+\int_{x^o}^{x} q_i(s)\frac{u_2(s)}{a_i(s)}ds. \tag{42}$$

Let $T_i(x)$ be defined by the right-hand side of inequality (42); then we obtain

$$\frac{u_i(x)}{a_i(x)} \leq T_i(x),\ where\ T_i(x)=1\ when\ x_j=x_j^o;\ j=1,\dots,n. \tag{43}$$

Additionally, based on the definition of $T_i(x)$, we can observe that

$$D_1\dots D_nT_i(x)=b_i(x)\frac{u_1(x)}{a_i(x)}+q_i(x)\frac{u_2(x)}{a_i(x)}, \tag{44}$$

using (43) in (44) gives for T_1 that

$$D_1\dots D_nT_1(x)\leq b_1(x)T_1(x)+q_1(x)\frac{a_2(x)}{a_1(x)}T_2(x). \tag{45}$$

As all the functions involved are positive, inequality (45) can be expressed as follows:

$$D_1\dots D_nT_1(x)\leq \phi_1(x)T(x). \tag{46}$$

By following a similar procedure, we can derive an analogous inequality for T_2, ultimately leading to

$$D_1\dots D_nT_i(x)\leq \phi_i(x)T(x); i=1,2, \tag{47}$$

where $\phi_1(x)=b_1(x)+q_1(x)\frac{a_2(x)}{a_1(x)}$, $\phi_2(x)=b_2(x)\frac{a_1(x)}{a_2(x)}+q_2(x),\ T(x)=T_1(x)+T_2(x)$.

Adding the two inequalities in (47) yields

$$D_1\dots D_nT(x)\leq \phi(x)T(x), \tag{48}$$

where $\phi(x)=\phi_1(x)+\phi_2(x)$. Apply Lemma 2 on inequality (48) to obtain

$$T(x)\leq 2\exp\left(\int_{x^o}^{x}\phi(s)ds\right). \tag{49}$$

Employ the upper limit (49) for $T(x)$ in inequality (47), and subsequently, integrate both sides of the resulting inequality with respect to x over the interval from x^o to x to yield

$$T_i(x)\leq 1+2\int_{x^o}^{x}\left(\phi_i(s)\exp\left(\int_{x^o}^{s}\phi(t)dt\right)\right)ds. \tag{50}$$

By applying inequality (50) within the context of inequality (43), we derive inequality (41). This marks the conclusion of the proof.

Remark. When $n=2$, which implies that $x\in\mathbb{R}^2$ and $x^o=0$, Theorem 2 provides the assertion that, for $i=1,2$, if

$$u_i(x_1,x_2)\leq a_i(x_1,x_2)+\int_0^{x_1}\int_0^{x_2} b_i(s_1,s_2)u_1(s_1,s_2)ds_2ds_1+\int_0^{x_1}\int_0^{x_2} q_i(s_1,s_2)u_2(s_1,s_2)ds_2ds_1, \tag{51}$$

hold, then

$$u_i(x_1,x_2) \le a_i(x_1,x_2)\left(1+2\int_0^{x_1}\int_0^{x_2}[\phi_i(s_1,s_2)\exp(\eta(s_1,s_2))]\,ds\right), \tag{52}$$

where $\phi_1(x_1,x_2) = b_1(x_1,x_2) + \frac{a_2(x_1,x_2)}{a_1(x_1,x_2)}q_1(x_1,x_2)$, $\phi_2(x_1,x_2) = b_2(x_1,x_2)\frac{a_1(x_1,x_2)}{a_2(x_1,x_2)} + q_2(x_1,x_2)$, $\eta(s_1,s_2) = \int_0^{s_1}\int_0^{s_2}\left(b_1(t_1,t_2)+q_2(t_1,t_2)+\frac{b_2(t_1,t_2)a_1(t_1,t_2)}{a_2(t_1,t_2)}+\frac{q_1(t_1,t_2)a_2(t_1,t_2)}{a_1(t_1,t_2)}\right)dt_2dt_1$, and $\phi(x_1,x_2) = \sum_{i=1}^{2}\phi_i(x_1,x_2)$.

Remark 3 addresses the case $n = 2$ for Theorem 2 and derives explicit exponential-type bounds for systems of integral inequalities with separable kernels. It generalizes classical Grönwall-Bellman approaches [15] by incorporating coefficient-dependent growth and offering improved structure for coupled systems. In contrast to recent results by Frioui et al. [5], which focus on single-variable multiplicative inequalities, this formulation provides a more comprehensive framework for multivariate systems.

Theorem 3. *Let $u_i(x), p_i(x)$, and $q_i(x)$ be real valued positive continuous functions on Ω, and let $a_i(x)$ be positive continuous non-decreasing functions on Ω; $i = 1,2$. In addition let $H(\alpha)$ be positive, continuous and non-decreasing function satisfies $t^{-1}H(\alpha) \le H(t^{-1}\alpha)$; $\alpha \ge 0$. If the system*

$$u_i(x) \le a_i(x) + \int_{x^o}^{x} p_i(s)H(u_1(s))ds + \int_{x^o}^{x} q_i(s)H(u_2(s))ds, \tag{53}$$

holds for all $x \in \Omega$ with $x \ge x^o$, then for $x^o \le x \le x^*$, we have

$$u_i(x) \le a_i(x)\left[1+2\int_{x^o}^{x}\phi_i(s)H(G^{-1}(G(2)+2\int_{x^o}^{x}\phi(s)ds)ds\right], \tag{54}$$

where $G(r) = \int_{r_o}^{r}\frac{ds}{H(s)}$, $r \ge r_o > 0$, x^* is chosen so that $G(2)+2\int_{x^o}^{x}\phi(s)ds \in Dom(G^{-1})$, $\phi_1(x) = p_1(x)+q_1(x)\frac{a_2(x)}{a_1(x)}$, $\phi_2(x) = p_2(x)\frac{a_1(x)}{a_2(x)}+q_2(x)$, and $\phi(x) = \phi_1(x)+\phi_2(x)$.

Proof . As both $a_i(x)$ for $i = 1,2$, are positive and non-decreasing functions, and the function $H(\alpha)$ is positive, continuous, and non-decreasing (for $\alpha \ge 0$) and satisfies the condition that $t^{-1}H(\alpha) \le H(t^{-1}\alpha)$, , we can rephrase inequalities (53) as follows:

$$\frac{u_1(x)}{a_1(x)} \le 1+\int_{x^o}^{x} p_1(s)H\left(\frac{u_1(s)}{a_1(s)}\right)ds + \int_{x^o}^{x} q_1(s)\frac{a_2(s)}{a_1(s)}H\left(\frac{u_2(s)}{a_2(s)}\right)ds, \tag{55}$$

and

$$\frac{u_2(x)}{a_2(x)} \le 1+\int_{x^o}^{x} p_2(s)\frac{a_1(s)}{a_2(s)}H\left(\frac{u_1(s)}{a_1(s)}\right)ds + \int_{x^o}^{x} q_2(s)H\left(\frac{u_2(s)}{a_2(s)}\right)ds. \tag{56}$$

Let $T_1(x)$ be defined as the expression on the right-hand side of (55), that is,

$$T_1(x) = 1+\int_{x^o}^{x} p_1(s)H\left(\frac{u_1(s)}{a_1(s)}\right)ds + \int_{x^o}^{x} q_1(s)\frac{a_2(s)}{a_1(s)}H\left(\frac{u_2(s)}{a_2(s)}\right)ds, \tag{57}$$

which implies

$$\frac{u_1(x)}{a_1(x)} \le T_1(x),\ where\ T_1(x) = 1, on\ x_i = x_i^o,\ i = 1,2,\ldots,n. \tag{58}$$

Next, we can define $T_2(s)$ as the expression on the right-hand side of (56), and consequently, we obtain

$$\frac{u_2(x)}{a_2(x)} \leq T_2(x),\ where\ T_1(x) = 1, on\ x_i = x_i^o,\ i = 1,2,...,n. \tag{59}$$

Now from (57) we have

$$D_1...D_nT_1(x) = p_1(x)H\left(\frac{u_1(x)}{a_1(x)}\right) + q_1(x)\frac{a_2(x)}{a_1(x)}H\left(\frac{u_2(x)}{a_2(x)}\right). \tag{60}$$

Utilizing the inequalities presented in (58) and (59) within the framework of (60) yields

$$D_1...D_nT_1(x) \leq p_1(x)H(T_1(x)) + q_1(x)\frac{a_2(x)}{a_1(x)}H(T_2(x)). \tag{61}$$

As all the functions involved are positive and H is a non-decreasing function, we can express inequality (61) in the following manner:

$$D_1...D_nT_1(x) \leq 2\phi_1(x)H(T(x)), \tag{62}$$

where $\phi_1(x) = p_1(x) + q_1(x)\frac{a_2(x)}{a_1(x)}$, and $T(x) = T_1(x) + T_2(x)$. Similarly, by (59) we obtain

$$D_1...D_nT_2(x) \leq 2\phi_2(x)H(T(x)), \tag{63}$$

where $\phi_2(x) = p_2(x)\frac{a_1(x)}{a_2(x)} + q_2(x)$, and $T(x) = T_1(x) + T_2(x)$. Now inequalities (62) and (63) yield

$$D_1...D_nT(x) \leq 2\phi(x)H(T(x)), \tag{64}$$

where $\phi(x) = \phi_1(x) + \phi_2(x)$, and $T(x) = T_1(x) + T_2(x)$. By employing the same approach as the one utilized to establish Lemma 2, we deduce from (64) that

$$D_1G(T(x)) = \frac{D_1T(x)}{H(T(x))} \leq 2\int_{x_2^o}^{x_2} ... \int_{x_n^o}^{x_n} \phi(x_1,t_2,...,t_n)dt_n...dt_2,$$

by performing integration with respect to the component x_1 over the interval from x_1^o to x_1, we obtain

$$G(T(x)) - G(2) \leq 2\int_{x^o}^{x} \phi(s)ds,$$

which implies that

$$T(x) \leq G^{-1}\left(G(2) + 2\int_{x^o}^{x} \phi(s)ds\right),$$

where $G(r) = \int_{r_o}^{r} \frac{ds}{H(s)}$, $r \geq r_o > 0$. Taking

$$\psi(x) = G^{-1}\left(G(2) + 2\int_{x^o}^{x} \phi(s)ds\right),$$

gives

$$T(x) \leq \psi(x). \tag{65}$$

Upon substituting (65) into (62), we arrive at

$$D_1...D_nT_1(x) \leq 2\phi_1(x)H(\psi(x)). \tag{66}$$

Integrating both sides of (66) from x^o to x produces

$$T_1(x) \leq 1 + 2\int_{x^o}^{x} \phi_1(s) H(\psi(s))\, ds. \tag{67}$$

Similarly, we can obtain from (63) and (65) that

$$T_2(x) \leq 1 + 2\int_{x^o}^{x} \phi_2(s) H(\psi(s))\, ds. \tag{68}$$

Employing (67) in (58), followed by the utilization of (68) in (59), results in the derivation of (54). This concludes the proof.

4 Some applications

In this section, our focus shifts towards illustrating the practical implications of our research findings. The integral inequalities presented in this paper span a wide range, encompassing both established mathematical inequalities and innovative ones, thereby enriching the toolkit of mathematical analysis.

To substantiate our claims, we will provide a series of applications, guided by the insights presented in Remarks 3 and 3. These applications are a testament to the versatility and utility of our findings, highlighting their potential to contribute to solutions in mathematical and practical contexts.

1.Consider the system

$$u_1(x_1,x_2) \leq \int_0^{x_1}\int_0^{x_2} u_1(s_1,s_2)ds_2ds_1 + \int_0^{x_1}\int_0^{x_2} x_1 u_2(s_1,s_2)ds_2ds_1, \tag{69}$$

and

$$u_2(x_1,x_2) \leq x_2 - \int_0^{x_1}\int_0^{x_2} x_1 u_1(s_1,s_2)ds_2ds_1 - 2\int_0^{x_1}\int_0^{x_2} u_2(s_1,s_2)ds_2ds_1 \tag{70}$$

comparing to the system in Remark 3 we have

$$a_1(x_1,x_2) = 0,\ b_1(x_1,x_2,s_1,s_2) = 1,\ c_1(x_1,x_2,s_1,s_2) = x_1,$$

$$a_2(x_1,x_2) = x_2,\ b_2(x_1,x_2,s_1,s_2) = -x_1,\ c_2(x_1,x_2,s_1,s_2) = -2.$$

Therefore, this system can be solved by applying inequality (39) that gives

$$u_1(x_1,x_2) \leq \int_0^{x_1}\int_0^{x_2}\left[2s_1s_2 + (1+2s_1)\int_0^{s_1}\int_0^{s_2}(2t_1t_2 - 2t_2 - \int_0^{t_1}\int_0^{t_2}(2r_1r_2 - 2r_2)dr_2dr_1)dt_2dt_1\right]ds_2ds_1, \tag{71}$$

which implies that

$$u_1(x_1,x_2) \leq \frac{x_1^2x_2^2}{6}\left[3 - x_2 + x_1x_2\left(\frac{x_2-12}{12}\right) + x_1^2x_2\left(\frac{5x_2+24}{48}\right) - \frac{x_1^3x_2^2}{30}\right]. \tag{72}$$

Similarly we can obtain

$$u_2(x_1,x_2) \leq x_2 - x_1x_2^2 + \frac{x_1^2x_2^3}{3} - \frac{x_1^3x_2^4}{36} + \frac{x_1^3x_2^3}{9} - \frac{x_1^4x_2^3}{12} - \frac{x_1^4x_2^4}{72} + \frac{x_1^5x_2^4}{180}. \tag{73}$$

2.Consider the system

$$u_1(x_1,x_2) \leq x_2 \cos x_1 + \int_0^{x_1}\int_0^{x_2} (1+\sin s_2)\, u_1(s_1,s_2)ds_2ds_1 \\ - 2\int_0^{x_1}\int_0^{x_2} \cos s_1 u_2(s_1,s_2)ds_2ds_1,$$

and

$$u_1(x_1,x_2) \leq x_2 + \int_0^{x_1}\int_0^{x_2} \sec s_1 u_1(s_1,s_2)ds_2ds_1 \\ - \int_0^{x_1}\int_0^{x_2} \sin s_2 u_2(s_1,s_2)ds_2ds_1.$$

Now an application of Remark 3 gives

$$\phi_1(x_1,x_2) = 1+\sin x_2 - 2\frac{x_2\cos x_1}{x_2\cos x_1} = \sin x_2 - 1,$$

$$\phi_2(x_1,x_2) = \sec x_1 \frac{x_2\cos x_1}{x_2} - \sin x_2 = 1-\sin x_2,\ hence\ \phi(x_1,x_2) = 0.$$

Therefore, the given system has the following solution

$$u_1(x_1,x_2) \leq x_2\cos x_1 \left(1+2\int_0^{x_1}\int_0^{x_2} (\sin s_2 - 1)ds_2ds_1\right),$$

which gives

$$u_1(x_1,x_2) \leq x_2\cos x_1\left(1-2x_1(\cos x_2 + x_2 - 1)\right).$$

Similarly we can easily obtain

$$u_2(x_1,x_2) \leq x_2\left(1+2x_1(\cos x_2 + x_2 - 1)\right).$$

5 Conclusion and Future work

The results obtained in this paper are not only consistent with classical findings such as the Grönwall-Bellman and Pachpatte-type inequalities but also extend them significantly. In particular, we provided explicit upper bounds for systems of integral inequalities involving two real-valued unknown functions in multiple independent variables, a setting that is more general than most existing works which typically address scalar inequalities or single-variable cases. Compared to earlier studies such as those in [16] and [17], our framework incorporates higher-order derivatives, interaction terms, and multi-level integral structures derived from characteristic solutions, resulting in sharper and more adaptable bounds. Furthermore, in contrast to recent contributions like Samraiz et al. [4] and Frioui et al. [5], which focus on either graphical representations or parametrized single-variable forms, our results address coupled systems with additive structures in multidimensional domains. This generalization not only broadens the scope of applicability (particularly in the analysis of nonlinear integro-differential equations) but also provides a unified and more refined analytical foundation for further theoretical developments.

Unlike earlier contributions which emphasize inequalities in quotient form or rely on weight structures and conformable fractional calculus [18–20], the present work derives explicit bounds for systems of nonlinear integral inequalities using additive structures and characteristic solutions in multiple dimensions. This provides a complementary and practically applicable extension to the field.

The present study lays a solid foundation for further investigation into systems of integral inequalities involving multiple variables and coupled unknown functions. Future research may focus on extending the current results to more general classes of nonlinear kernels, time-delay systems, or inequalities defined on non-Euclidean domains such as manifolds or time scales.

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