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Constrained Probabilistic Inventory with Varying Rent and Discount of Expiration Cusp for Crisp and Fuzzy: Power Lomax Distribution

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Abstract: Two models of the multi-item probabilistic continuous review inventory model of expiration cusp with varying rent cost and all units discount are deduced. The restricted mathematical model is solved utilising the Lagrange multiplier approach. Here, the shortage is permissible, which is a combination of the backorder cost and lost sales cost. Also, the holding cost is a mixture of insurance cost and rent cost. The expected total cost is composed of the expected purchase cost, the expected ordering cost, the expected mixture holding cost, the expected mixture shortage cost, and the expected deterioration cost. The demand during lead time follows the Power Lomax distribution, analyzed for both unimodal and decreasing cases.

The first model assumes that the total cost components are crisp values, whereas the second model assumes that the costs are trapezoidal fuzzy numbers. For these proposed models, we found the optimal order quantity and the optimal reorder point which minimizes the expected total cost. To demonstrate the suggested two models in two cases a numerical example was added.

Keywords: Probabilistic inventory model; Expiration cusp; Power Lomax distribution; All units discount; Varying rent.

2020 AMS Subject Classifications: 62P20, 90B05.

1. Introduction

Over a hundred years ago, an analysis of the inventory system appeared. The significance of this field of study has drawn the interest of researchers. Numerous publications and literatures have examined and debated a wide variety of inventory models. The economic order quantity (EOQ) model is one of the simplest, as stated in [1]. The EOQ model has evolved over the last decade as

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a foundational model for the creation of further intricate inventory models. Still, very few of such models address inventory models that take into consideration perishables, deteriorated items, or expiration cusp. Food products with a short shelf life usually account for the majority of product sales because they are perishable. Nevertheless, taking advantage of supplier discounts will typically promote larger order quantities. Discounts reduce the purchase price and holding costs per unit of the product (holding costs per unit are typically proportionate to the product price) in the inventory system [2]. Research has been conducted on inventory systems that account for factors such as all-units discount and expiration dates, as shown in the publications [3], [4], and [5]. [6] provided economic order quantity model for expanding items with budget constraints, a storage facility that can accommodate them, and incremental quantity discounts. A probabilistic multi-item inventory model introduced by [7]. [8] introduced optimal simple algorithm for an EOQ multi-item situation that has deteriorated. [9] explained Two-storage facility inventory model with selling price, time-dependent demand, and variable holding costs. [10] examined optimum inventory rules and precise lead time demand modelling in continuous review systems. [11] introduced restricted probabilistic inventory model with variable holding cost and continuous distributions. [12] examined probabilistic multi-item inventory model with variable mixture shortage cost under limitations. [13] studied the power Lomax distribution with an application to bladder cancer data. [14] developed a probabilistic inventory model with multi-items, taking into account purchase bonus and perishable aspects. In recent years, there has been a growing interest in probabilistic inventory models due to their significance in enhancing inventory management efficiency. Recent research has shown notable developments in this field. [15] introduced a multi-item probabilistic inventory model that takes into account warehouse capacity constraints, the all-unit discount policy, and the expiration factor. [16] examined optimal replenishment policy for multi-item probabilistic inventory model with all-units discount. [17] explored inventory optimization in a green environment with two warehouses. [18] developed a multi-item inventory model that incorporates stock-dependent demand and all-units discount. [19] presented probabilistic inventory model with multiple items and variable lead times under all-units discount.

As with other parameters, such prices, marketing, production, and inventory, the cost parameter in real inventory problems is uncertain. Utilising fuzzy sets theory as a mathematical approach to address these concerns, numerous scholars have published multiple articles in recent years. For example, [20] discussed scheduling period inventory model with Weibull deteriorating for crisp and fuzzy. [21] introduced Economic design of a multi echelon inventory system in a fuzzy cost environment with a variable lead time and service level constraints. [22] studied an energy-efficient dual-channel inventory model with trapezoidal fuzzy demand. [23] developed a probabilistic fuzzy set and triangular fuzzy number-based multi-objective wholesaler-retailers inventory-distribution model with control-label lead-time. [24] studied Continuous review inventory model in a fuzzy random environment with variable lead time.

2. The Model for a Probabilistic Inventory with Expiration Cusp and All-Units Discount

Retailers commonly face two main inventory-related challenges: determining the optimal order quantity and the best timing for placing orders with suppliers. Addressing these challenges is crucial for minimizing issues like stock shortages and potential losses. A mathematical model can help retailers identify the optimal order quantity and reorder point.

This paper presented a multi-item probabilistic continuous review inventory model with discount offered by the supplier and expiration cusp for the items. In order to take advantage of the offer, the supplier may decide to order goods in bulk at the discounted rate of all units. This model is applicable when the lead time is constant, and the lead time demand is random variable that follows Power Lomax distribution. It also applies in situations where orders are placed but a shortage arises, and part of the orders are filled in the following cycle at the same price and request time (backorder), while the remaining portion is lost forever, and a penalty clause is paid. Also, the expected holding cost is divided into the Insurance part and varying rent part. There is a constrained on the expected mixture shortage cost. The constrained problem is solved by applying the Lagrange multiplier approach. Our objective is to minimize the expected total cost, where the order quantity and the reorder point are the policy variables for this model. We evaluate the optimal policy variables in two submodels: one considers cost components as crisp values, while the other uses trapezoidal fuzzy numbers, reflecting a more realistic environment. The results were derived using Mathematica version 12.3.

2.1 The Model Notations and Presumptions

The following notations and presumptions will be applied in this paper to develop the inventory model with expiration cusp and all-units discount:

1. The same source is used to order all types of items (MISS).
2. Inventory is continuously reviewed, and whenever the inventory level drops to the reorder point r, replenishments are made.
3. The good fraction value (θ_i) is for all types of items.
4. The percentage of good items indicates the number of deteriorated items

   which is (1-θ_i).
5. Items that have expired can still be sold to certain parties at S_i price (where $S_i < C_{pi}$). Assuming this, expired items can still be used (sold) at a discount, but they cannot be consumed (used as food).
6. Lead time demand follows continuous distributions.
7. Lead time is constant.
8. Shortage is a mixture of backorder and lost sales.

The average number of items is used to determine holding cost divided into an insurance part and another part is rent, which is a decreasing function.

$\overline{D}_i$	Average demand for the i^{th} item in one planning period
C_{pi}	The purchase price for the i^{th} item per unit before and after discount
Q_i	The order quantity for the i^{th} item (decision variable)
${Q_i}^*$	The optimal order quantity for the i^{th} item
Q_{di}	The quantity of deteriorated items for the i^{th} item
C_{oi}	The order cost for the i^{th} item per unit per planning period
$\grave{C}_{hi}$	The holding cost for the i^{th} item represents insurance per unit per planning period
$C_{hi}(Q_i)$	The decreasing holding cost for the i^{th} item representing rent per unit per planning Period, $= C_{hi}{Q_i}^{-\beta}$, β is a constant real number selected to provide the best fit of estimated expected total cost function
C_{bi}	The backorder cost for the i^{th} item per unit per planning period
C_{li}	The lost sales cost for the i^{th} item per unit per planning period
$\tilde{C}_{oi}$	The fuzzy order cost for the i^{th} item per unit per planning period
$\tilde{C}_{hi}$	The fuzzy holding cost for the i^{th} item representing rent per unit per planning Period
$\tilde{\grave{C}}_{hi}$	The fuzzy holding cost for the i^{th} item representing insurance per unit per planning period
$\tilde{C}_{bi}$	The fuzzy backorder cost for the i^{th} item per unit per planning period
$\tilde{C}_{li}$	The fuzzy lost sales cost for the i^{th} item per unit per planning period
$\tilde{C}_{pi}$	The fuzzy purchase price for the i^{th} item per unit before and after discount
$\tilde{\overline{D}}_i$	Average fuzzy demand for the i^{th} item in one planning period
r_i	The reorder point for the i^{th} item (decision variable)
${r_i}^*$	The optimal reorder point for the i^{th} item
S_i	Selling price for the i^{th} item of deteriorated items
$\overline{H}_i$	The average on hand inventory of the i^{th} item per period
θ_i	Fraction of good items ($0 < \theta_i < 1$)
$1-\theta_i$	Fraction of deteriorated items
U_{ji}	The minimum number of order quantity for the i^{th} item allowed by the supplier for the j^{th} price
γ_i	Fraction of the unsatisfied demand per replenishment cycle can be backordered, and the remaining fraction (1 - γ_i) is lost, $0 < \gamma_i < 1$
δ_i	Fraction of Insurance cost and the remaining fraction $(1-\delta_i)$ is rent cost, $0 < \delta_i < 1$
$\bar{S}(r_i)$	The amount of expected shortage
$E_i(PC)$	The expected purchase cost for the i^{th} item
$E_i(OC)$	The expected order cost for the i^{th} item
$E_i(HC_I)$	The expected insurance cost for the i^{th} item
$E_i(HC_R)$	The expected rent cost for the i^{th} item
$E_i(HC)$	The expected mixture holding cost = $E_i(HC_I) + E_i(HC_R)$ for the i^{th} item
$E_i(BC)$	The expected backorder cost for the i^{th} item
$E_i(LC)$	The expected lost sales cost for the i^{th} item
$E_i(SC)$	The expected mixture shortage cost = E_i(BC) + E_i(LC) for the i^{th} item
E_i(DC)	The expected deteriorated cost for the i^{th} item
$f(x)$	The density function for lead time demand where x is the lead time demand
Min E(TC_i)	The minimum expected total cost for the i^{th} item

2.2 Model I: The Model for Crisp Environmental

The expected total inventory cost in one year shall be the sum of the expected purchase cost, the expected ordering cost, the expected mixture holding cost, the expected mixture shortage cost, and the expected deterioration cost.

$$E(TC(Q_i, r_i))=\sum_{i=1}^{m}[E_i(PC)+E_i(OC)+E_i(HC)+E_i(SC)+E_i(\mathrm{DC})] \quad (1)$$

The expected purchase cost is the price paid to the supplier for the goods. The supplier offers discounts in this model, so the purchase cost per unit can be expressed as follows:

$$\boldsymbol{C_{pi}} = \begin{cases} \boldsymbol{C_{p0i}} & \boldsymbol{for\ U_{0i} \le Q_i < U_{1i}} \\ \boldsymbol{C_{p1i}} & \boldsymbol{for\ U_{1i} \le Q_i < U_{2i}} \\ & \vdots \\ \boldsymbol{C_{pji}} & \boldsymbol{for\ U_{ji} \le Q_i < U_{(j+1)i}} \end{cases}$$

Where $\boldsymbol{C_{pji}} > \boldsymbol{C_{p(j+1)i}}$, j = 0,1,2,3, … for all units of the i^{th} item. If the annual average demand for the i^{th} item is $\overline{D}_i$ unit, then the annual purchasing cost is:

$$E_i(PC) = C_{pi}\,\overline{D}_i \quad (2)$$

The expected ordering cost occurs in every item procurement from the supplier to the retailer. If the retailer is required to pay C_{oi} for every procurement, then the ordering cost per planning period is:

$$E_i(OC) = \frac{C_{oi}\,\overline{D}_i}{Q_i} \quad (3)$$

The expected mixture holding cost which will be interested in our study split into two parts which is called rent and insurance cost as follows:

I) Insurance part: If one can express the annual holding cost per unit as a fraction of the purchase cost per unit, which is $\delta_i C_{pi}\,\grave{C}_{hi}$ so this part's annual cost can be calculated as $E_i(HC_I) = \delta_i C_{pi}\,\grave{C}_{hi} n_i \overline{H}_i$.

Where, $\overline{H}_i = \frac{1}{n_i}\,[[\frac{Q_i}{2} + r_i - E(x) + (1-\gamma_i)\int_r^{\infty}(x-r)f(x)dx]$, x is a random variable for demand during lead time.

II) Rent part: which is decreasing function depending on the stored items and the annual cost of this part can be formulated as $E_i(HC_R) = (1-\delta_i)\ C_{hi}\,(Q_i)n_i\overline{H}_i$. Then the expected annual holding cost is follows:

$$E_i(HC) = E_i(HC_I) + E_i(HC_R)$$

$$= [\delta_i C_{pi}\ \grave{C}_{hi} + (1-\delta_i)C_{hi}{Q_i}^{-\beta}]\,[\tfrac{Q_i}{2} + r_i - \mu + (1-\curlyvee_i)\ \int_r^{\infty}(x-r)f(x)dx] \qquad (4)$$

The expected shortage cost is cost because when demand comes in, the retailer is empty. Only when lead time demand exceeds the amount of inventory in the retailer's warehouse does there exist this shortage. During a stockout period, demand is typically considered to be forever lost or fully backordered. Some consumers may be impatient and insist on getting their needs met right away from other sources (lost sales), while others may be willing to wait until the next shipment of stock (backorder instance). This paper involves a mixture shortage (backorders and lost sales) which is given as:

$$E_i(SC) = E_i(BC) + E_i(LC)$$

Where,

$$E_i(BC) = C_{bi}\,\curlyvee_i \frac{\overline{D}_i}{Q_i}\overline{S}(r_i) \quad \text{and} \quad E_i(LC) = C_{li}(1-\curlyvee_i)\ \frac{\overline{D}_i}{Q_i}\overline{S}(r_i) \qquad (5)$$

The expected deterioration cost is the cost that occurs when items are expiration cusp. The retailer will offer all deteriorating items at a lower price in this situation. The expected annual deteriorated cost is thus:

$$E_i(\mathrm{DC}) = \frac{Q_d\,(\mathrm{C_{pi}} - S_i)\,\overline{D}_i}{Q_i} = (1-\theta_i)(C_{pi} - S_i)\,\overline{D}_i \qquad (6)$$

Then from equations (2), (3), (4), (5) and (6), in equation (1) the expected total cost can be obtained by:

$$\mathrm{E(TC(Q_i, r_i))} =$$

$$\sum_{i=1}^{m}\left[\begin{array}{l} C_{pi}\,\overline{D}_i + \frac{C_{oi}\overline{D}_i}{Q_i} + [\delta_i C_{pi}\,\grave{C}_{hi} + (1-\delta_i)\,C_{hi}{Q_i}^{-\beta}]\,[\frac{Q_i}{2} + r_i - \mu + (1-\curlyvee_i)\ \int_r^{\infty}(x-r)f(x)dx] \\ +C_{bi}\frac{\curlyvee_i\overline{D}_i}{Q_i}\int_r^{\infty}(x-r)f(x)dx + C_{li}(1-\curlyvee_\mathrm{i})\frac{\overline{D}_i}{Q_i}\ \int_r^{\infty}(x-r)f(x)dx + (1-\theta_i)(\mathrm{C_{pi}} - S_i)\,\overline{D}_i \end{array}\right] \qquad (7)$$

where; $\int_r^{\infty}(x-r)f(x)dx = \bar{S}(r_i)$

The main objective is to minimize the optimal values ${Q_i}^*$ and ${r_i}^*$ that minimize the expected annual total cost E(TC(Q_i, r_i)) under the expected mixture shortage cost constraint. If certain regularity constraints are met, the Karush-Kuhn-Tucker (KKT) condition [25] is a first-order necessary condition for a nonlinear programming solution to be optimal. This constraint problem can be solved using the Lagrange multiplier method.

Consider a restriction on the expected mixture shortage cost, i.e.,

$$\sum_{i=1}^{m} E_i(SC) \ \leq k_i$$

$$\sum_{i=1}^{m} \frac{\overline{D}_i}{Q_i}[\, C_{bi}\, \gamma_i + C_{li}(1-\gamma_{\rm i})]\bar{S}(r_i) \ \leq k_i \tag{8}$$

Let's develop the previews equations in the following manner to solve the convex programming issue of the primal function:

$$\mathrm{E}(\mathrm{TC}(Q_i,r_i)) = \sum_{i=1}^{m}\begin{bmatrix} C_{pi}\,\overline{D}_i + \frac{C_{oi}\,\overline{D}_i}{Q_i} + \ [\delta_i C_{pi}\,\grave{C}_{hi} + (1-\delta_i)\,C_{hi}{Q_i}^{-\beta}]\,[\frac{Q_i}{2} + r_i - \mu + (1-\gamma_i)\,\bar{S}(r_i)] \\ +C_{bi}\frac{\gamma_i\overline{D}_i}{Q_i}\ \bar{S}(r_i) + C_{li}(1-\gamma_{\rm i})\frac{\overline{D}_i}{Q_i}\ \bar{S}(r_i) + (1-\ \theta_i\,)\big(\mathrm{C_{pi}} -\ S_i\big)\,\overline{D}_i \end{bmatrix} \tag{9}$$

Subject to:

$$\sum_{i=1}^{m} \frac{\overline{D}_i}{Q_i}[\, C_{bi}\, \gamma_i + C_{li}(1-\gamma_{\rm i})]\bar{S}(r_i) - k_i \ \leq\ 0 \tag{10}$$

To find the optimum values ${Q_i}^*$ and ${r_i}^*$ which minimize equation (9) under the constraint (10), the Lagrange multiplier function with the conditions [25] is used as follows:

$$\mathrm{L} = \sum_{i=1}^{m}[\,\mathrm{E}(TC_i) +\ \lambda_i\,\{E_i(SC) - k_i\}], \qquad \lambda_i\ > 0$$

$$= \sum_{i=1}^{m}\begin{bmatrix} C_{pi}\,\overline{D}_i + \frac{C_{oi}\,\overline{D}_i}{Q_i} + \ [\delta_i C_{pi}\,\grave{C}_{hi} + (1-\delta_i)C_{hi}{Q_i}^{-\beta}]\,[\frac{Q_i}{2} + r_i - \mu + (1-\gamma_i)\,\bar{S}(r_i)] \\ +C_{bi}\frac{\gamma_i\overline{D}_i}{Q_i}\ \bar{S}(r_i) + C_{li}(1-\gamma_i)\frac{\overline{D}_i}{Q_i}\ \bar{S}(r_i) + (1-\ \theta_i\,)\big(C_{pi} -\ S_i\big)\,\overline{D}_i \\ +\lambda_i[\frac{\overline{D}_i}{Q_i}\ [\, C_{bi}\,\gamma_i + C_{li}(1-\gamma_i)]\bar{S}(r_i) - k_i] \end{bmatrix} \tag{11}$$

By setting each of the first partial derivatives of equation (11) to zero, the optimal values ${Q_i}^*$ and ${r_i}^*$ may be computed. Subsequently, we acquire:

$$\frac{\partial L}{\partial Q_i} = -\frac{C_{oi}\overline{D}_i}{{Q_i}^2} + \frac{\delta_i C_{pi} \grave{C_{hi}}}{2} + \frac{(1-\delta_i)C_{hi}{Q_i}^{-\beta}}{2} - \beta\,(1-\delta_i)C_{hi}{Q_i}^{-1-\beta}\left[\frac{Q_i}{2} + r_i - \mu + (1-\curlyvee_i)\overline{S}(r_i)\right] - \frac{C_{bi}\curlyvee_i\overline{D}_i}{{Q_i}^2}\overline{S}(r_i) - \frac{C_{li}(1-\curlyvee_i)\overline{D}_i}{{Q_i}^2}\overline{S}(r_i) - \frac{\lambda_i}{{Q_i}^2}[C_{bi}\curlyvee_i\overline{D}_i + C_{li}(1-\curlyvee_i)\overline{D}_i]\overline{S}(r_i)$$

hence,

$$\delta_i C_{pi}\grave{C}_{hi}({Q_i}^*)^2 + (1-\delta_i)C_{hi}({Q_i}^*)^{(2-\beta)} - 2\beta(1-\delta_i)C_{hi}({Q_i}^*)^{(1-\beta)}\{\frac{Q_i}{2} + {r_i}^* - \mu + (1-\curlyvee_i)\overline{S}({r_i}^*)\} - 2\left(C_{oi}\overline{D}_i + (1+\lambda_i)\,[C_{bi}\curlyvee_i\overline{D}_i + C_{li}(1-\curlyvee_i)\overline{D}_i]\overline{S}({r_i}^*)\right) = 0$$

Where, $A = \delta_i C_{pi}\grave{C}_{hi}$, $B = (1-\delta_i)C_{hi}$, $W = C_{oi}\overline{D}_i$, $M = (1+\lambda_i)\,[C_{bi}\curlyvee_i\overline{D}_i + C_{li}(1-\curlyvee_i)\overline{D}_i]$

i.e.

$$A{Q_i}^{*2} + (1-\beta)\,B{Q_i}^{*(2-\beta)} - 2\beta B\,{Q_i}^{*(1-\beta)}\{{r_i}^* - \mu + (1-\curlyvee_i)\overline{S}({r_i}^*)\} - 2\left(W + M\overline{S}({r_i}^*)\right) = 0 \quad (12)$$

Also,

$$\frac{\partial L}{\partial r_i} = [\delta_i C_{pi}\grave{C_{hi}} + (1-\delta_i)C_{hi}{Q_i}^{-\beta}][1-(1-\curlyvee_i)R(r_i)] - \frac{C_{bi}\curlyvee_i\overline{D}_i}{Q_i}R(r_i)$$

$$-\frac{C_{li}(1-\curlyvee_i)\overline{D}_i}{Q_i}R\,(r_i) - \lambda_i[\frac{\overline{D}_i}{Q_i}\,[C_{bi}\curlyvee_i + C_{li}(1-\curlyvee_i)]R(r_i)$$

$$R({r_i}^*) = \frac{A{Q_i}^* + B{Q_i}^{*(1-\beta)}}{[M + A(1-\curlyvee_i){Q_i}^* + 1 - B(1-\curlyvee_i){Q_i}^{*(1-\beta)}]} \quad (13)$$

It is obvious that equations (12) and (13) cannot be solved in closed form to provide ${Q_i}^*$, ${r_i}^*$. Then, an iterative method must be used to determine ${Q_i}^*$, ${r_i}^*$ that are used to calculate the minimum expected total cost.

2.3 Model II: The Model for Fuzzy Environmental

The cost parameters and other characteristics in real inventory systems, such as price, marketing, and provider demand elasticity, are by their very nature uncertain and imprecise. Fuzziness notation was established as a result of this a misunderstanding. Since the suggested model operates in a fuzzy environment, it is necessary to make a fuzzy decision to satisfy the decision's requirements, and the outcome should also be fuzzy for the model to be taken into account in a fuzzy environment. It is difficult to specify every parameter accurately due to uncertainty.

Let

$\tilde{C}_{oi}$ = $(C_{oi} - a_{1i}, C_{oi} - a_{2i}, C_{oi} + a_{3i}, C_{oi} + a_{4i})$,

$\tilde{C}_{hi}$ = $(C_{hi} - a_{5i}, C_{hi} - a_{6i}, C_{hi} + a_{7i}, C_{hi} + a_{8i})$,

$\tilde{\grave{C}}_{hi}$ = $(\grave{C}_{hi} - a_{9i}, \grave{C}_{hi} - a_{10i}, \grave{C}_{hi} + a_{11i}, \grave{C}_{hi} + a_{12i})$,

$\tilde{C}_{bi}$ = $(C_{bi} - a_{13i}, C_{bi} - a_{14i}, C_{bi} + a_{15i}, C_{bi} + a_{16i})$,

$\tilde{C}_{li}$ = $(C_{li} - a_{17i}, C_{li} - a_{18i}, C_{li} + a_{19i}, C_{lr} + a_{20i})$,

$\tilde{C}_{pi}$ = $(C_{pi} - a_{21i}, C_{pi} - a_{22i}, C_{pi} + a_{23i}, C_{pi} + a_{24i})$,

and $\tilde{\overline{D}}_i$ = $(\overline{D}_i - a_{25i}, \overline{D}_i - a_{26i}, \overline{D}_i + a_{27i}, \overline{D}_i + a_{28i})$.

are trapezoidal fuzzy numbers, Where a_{ri}, r = 1,2, … , 28, i = 1,2,…,n, are arbitrary positive values satisfy the following limitations:

$C_{oi} > a_{1i} > a_{2i}$, $a_{3i} < a_{4i}$, $C_{hi} > a_{5i} > a_{6i}$, $a_{7i} < a_{8i}$,

$\grave{C}_{hi} > a_{9i} > a_{10i}$, $a_{11i} < a_{12i}$, $\tilde{C}_{bi} > a_{13i} > a_{14i}$, $a_{15i} < a_{16i}$,

$\tilde{C}_{li} > a_{17i} > a_{18i}$, $a_{19i} < a_{20i}$, $\tilde{C}_{pi} > a_{21i} > a_{22i}$, $a_{23i} < a_{24i}$,

and $\overline{D}_i > a_{25i} > a_{26i}$, $a_{27i} < a_{28i}$.

Thus, the left and right limits α cuts of C_{oi} , C_{hi}, $\grave{C}_{hi}$, C_{bi} , C_{li}, C_{pi} and $\overline{D}_i$ are given as follows:

$\tilde{C}_{oi_v}(\alpha) = C_{oi} - a_{1i} + (a_{1i} - a_{2i})\alpha$, $\tilde{C}_{oi_u}(\alpha) = C_{oi} + a_{4i} - (a_{4i} - a_{3i})\alpha$,

$\tilde{C}_{hi_v}(\alpha) = C_{hi} - a_{5i} + (a_{5i} - a_{6i})\alpha$, $\tilde{C}_{hi_u}(\alpha) = C_{hi} + a_{8i} - (a_{8i} - a_{7i})\alpha$,

$\tilde{\grave{C}}_{hiv}(\alpha) = \grave{C}_{hi} - a_{9i} + (a_{9i} - a_{10i})\alpha$, $\tilde{\grave{C}}_{hi_u}(\alpha) = \grave{C}_{hi} + a_{12i} - (a_{12i} - a_{11i})\alpha$,

$\tilde{C}_{bi_v}(\alpha) = C_{bi} - \delta_{13i} + (\delta_{13i} - \delta_{14i})\alpha$, $\tilde{C}_{bi_u}(\alpha) = C_{bi} + \delta_{16i} - (\delta_{16i} - \delta_{15i})\alpha$,

$\tilde{C}_{li_v}(\alpha) = C_{li} - a_{17i} + (a_{17i} - a_{18i})\alpha$, $\tilde{C}_{li_u}(\alpha) = C_{li} + a_{20i} - (a_{20i} - a_{19i})\alpha$,

$\tilde{C}_{pi_v}(\alpha) = C_{pi} - a_{21i} + (a_{21i} - a_{22i})\alpha$, $\tilde{C}_{pi_u}(\alpha) = C_{pi} + a_{24i} - (a_{24i} - a_{23i})\alpha$, and

$\tilde{\tilde{D}}_{iv}(\alpha) = \overline{D}_i - a_{25i} + (a_{25i} - a_{26i})\alpha$, $\tilde{\tilde{D}}_{iu}(\alpha) = \overline{D}_i + a_{28i} - (a_{28i} - a_{27i})\alpha.$

The defuzzified value of a fuzzy number can be obtained by applying the signed distance method, as follows:

$\tilde{C}_{oi} = \frac{1}{4}[4\, C_{oi} - a_{1i} - a_{2i} + a_{3i} + a_{4i}]$, $\tilde{C}_{hi} = \frac{1}{4}[4C_{hi} - \delta_{5i} - \delta_{6i} + \delta_{7i} + \delta_{8i}]$

$\tilde{\grave{C}}_{hi} = \frac{1}{4}[4\grave{C}_{hi} - \delta_{9i} - \delta_{10i} + \delta_{11i} + \delta_{12i}]$, $\tilde{C}_{bi} = \frac{1}{4}[4C_{bi} - a_{13i} - a_{14i} + a_{15i} + a_{16i}]$

$\tilde{C}_{li} = \frac{1}{4}[4C_{li} - a_{17i} - a_{18i} + a_{19i} + a_{20i}]$, $\tilde{C}_{pi} = \frac{1}{4}[4C_{pi} - a_{21i} - a_{22i} + a_{23i} + a_{24i}]$

and $\tilde{\tilde{D}}_i = \frac{1}{4}[4\overline{D}_i - \delta_{29i} - a_{30i} + a_{31i} + a_{32i}].$

Utilising approximated value of trapezoidal fuzzy numbers which is observed in Figure 1.

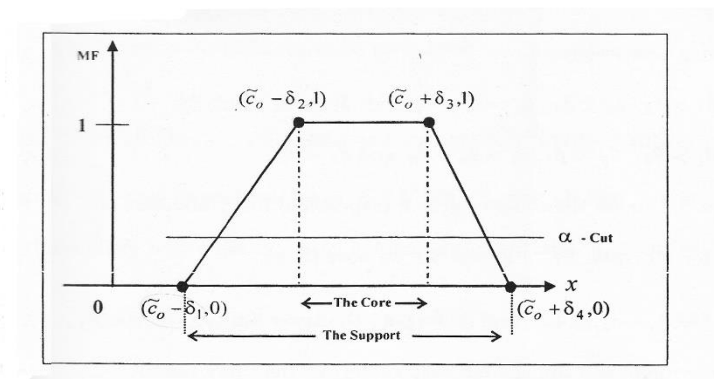

Fig. 1: Trapezoidal fuzzy number for order cost

The same steps can be used here as they were in model I, with the exception that the crisp values of C_{oi}, C_{hi}, $\grave{C}_{hi}$, C_{bi}, C_{li}, C_{pi} and $\overline{D}_i$ will be swapped out with the fuzzy values of $\tilde{C}_{oi}$, $\tilde{C}_{hi}$, $\tilde{\grave{C}}_{hi}$, $\tilde{C}_{bi}$, $\tilde{C}_{li}$, $\tilde{C}_{pi}$ and $\tilde{\tilde{D}}_i$. Then, using the same previous equations, optimal values for Q_i^* and r_i^* may be determined in order to minimize the excepted annually total cost E$(\widetilde{TC}(Q_i, r_i))$ for model II.

3. The model when the lead time demand follows Power Lomax distribution

When the lead time demand x follows the Power Lomax distribution (POLO) with parameters α, ω and Ψ, the PDF is:

$$f(x) = \alpha\, \omega\, \Psi^{\alpha}\, x^{\omega-1}(\Psi + x^{\omega})^{-\alpha-1}, \qquad x > 0, \alpha, \omega, \Psi > 0 \tag{14}$$

The reliability function of POLO distribution is given by,

$$\mathrm{R(r)} = \int_r^{\infty} f(x)dx = \Psi^{\alpha}\,(\Psi + x^{\omega})^{-\alpha}, \qquad x > 0, \alpha, \omega, \Psi > 0 \tag{15}$$

The expected shortage quantity will be as follows:

$$\bar{S}(\mathrm{r}) = r\Psi^{\alpha}(-(r^{\omega} + \Psi)^{-\alpha} + \frac{r^{-\alpha\omega}\alpha\omega\mathrm{Hypergeometric2F1}[1+\alpha,\alpha-\frac{1}{\omega},1+\alpha-\frac{1}{\omega},-r^{-\omega}\Psi]}{-1+\alpha\omega})] \tag{16}$$

And the mean μ of POLO distribution is given by:

$$\mu = \frac{\alpha\, \Psi^{\frac{1}{\omega}}\Gamma(\alpha-\frac{1}{\omega})\Gamma(\frac{1}{\omega})}{\omega\Gamma(1+\alpha)}$$

where Figure 2 represents the Power Lomax distribution's PDF, which is described by equation (14) as mentioned in [13] is:

a) Unimodal if $\alpha > 0$, $\omega > 1$, $\Psi > 0$.

b) Decreasing if $\alpha > 0$, $0 < \omega \leq 1$, $\Psi > 0$.

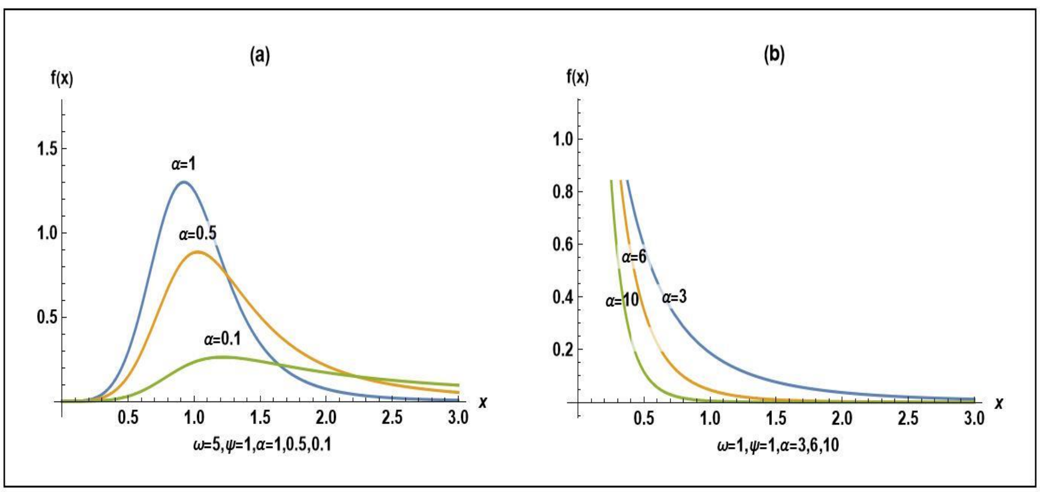

Fig. 2: The POLO density function plots for various values of ω, Ψ, and α.

Therefore, by inserting from (16) and (15) into (12) and (13), respectively, for every i^{th} item, it is mathematically feasible to minimize the expected total cost. The optimum values for ${Q_i}^*$ and ${r_i}^*$ are found to be as follows:

$$\mathrm{A}{Q_i}^{*2}+(1\text{-}\beta)\mathrm{B}{Q_i}^{*(2-\beta)}-2\beta\mathrm{B}\ {Q_i}^{*(1-\beta)}\{\ \mathrm{r_i}^*-\ \frac{\alpha\,\Psi^{\frac{1}{\omega}}\Gamma\left(\alpha-\frac{1}{\omega}\right)\Gamma\left(\frac{1}{\omega}\right)}{\omega\Gamma(1+\alpha)}+\ (1-\gamma_{\mathrm{i}})r\Psi^{\alpha}(-(r^{\omega}+\Psi)^{-\alpha}+\frac{r^{-\alpha\omega}\alpha\omega\mathrm{Hypergeometric2F1}\left[1+\alpha,\alpha-\frac{1}{\omega},1+\alpha-\frac{1}{\omega},-r^{-\omega}\Psi\right]}{-1+\alpha\omega})]\}-2\left(\mathrm{W}+\mathrm{M}\,r\Psi^{\alpha}(-(r^{\omega}+\Psi)^{-\alpha}+\frac{r^{-\alpha\omega}\alpha\omega\mathrm{Hypergeometric2F1}\left[1+\alpha,\alpha-\frac{1}{\omega},1+\alpha-\frac{1}{\omega},-r^{-\omega}\Psi\right]}{-1+\alpha\omega})]\right)=0 \quad (17)$$

hence,

$$\frac{A{Q_i}^*+B{Q_i}^{*(1-\beta)}}{[M+A(1-\gamma_i){Q_i}^*+1-B(1-\gamma_i){Q_i}^{*(1-\beta)}]}-\Psi^{\alpha}(\Psi+x^{\omega})^{-\alpha}=0 \quad (18)$$

Except for substituting fuzzy costs for crisp costs, the decision variables and minimum expected annual total cost for model II can be established in the same manner.

4. Numerical Example

The numerical data set from [15] was utilised to test the model with an additional varying rent cost and expiration cusp factor with assumption the good fraction factor value θ_i is 95 % and the remaining is the number of deteriorated items (1-θ_i) which is 5%. Also, the parameters of POLO distribution are α_i = Ψ_i = 5 and ω_i = (0.5, 0.7, 0.95) for decreasing case, α_i = Ψ_i = 5 and ω_i = (1.2, 1.5, 1.8) for unimodal case as in Figure 3. The data for three items can be seen in Table 1, Table 2 and Table 3 as follows:

Table 1. Product data (numerical set data)

Parameters	Item I	Item II	Item III
$\overline{D}_i$(unit/year)	550	400	800
δ_i	0.2	0.4	0.6
γ_i	0.3	0.5	0.7
k_i(decreasing case)	19	12	1.5
k_i(unimodal case)	1	2	0.4

Table 2. Crisp cost component data (numerical set data)

Costs description	Item I		Item II		Item III	
C_{pi} (decreasing case)	$Q < 112$	12	$Q < 49$	15	$Q < 124$	8
	$Q \geq 112$	10.5	$Q \geq 49$	13	$Q \geq 124$	7
C_{pi} (unimodal case)	$Q < 91$	12	$Q < 44$	15	$Q < 122$	8
	$Q \geq 91$	10.5	$Q \geq 44$	13	$Q \geq 122$	7
C_{oi}($)	9		9		9	
C_{hi}($/year)	0.12		0.225		0.08	
$\grave{C}_{hi}$($/year)	0.5		0.6		0.2	
C_{bi}($/unit)	1		2		1	
C_{li}($/unit)	2		2		1	
S_i($/unit)	10.2		12.75		6.8	

Table 3. Fuzzy cost component data (numerical set data)

Costs description	Item I	Item II	Item III
$\tilde{C}_{pi}$ (decreasing case)	$Q < 114$ (11,11.5,12.5,12.75)	$Q < 50$ (14, 14.5,15.5,15.75)	$Q < 124$ (7.5,7.75,8.5,8.75)
	$Q \geq 114$ (10,10.25,10.75,10.9)	$Q \geq 50$ (12,12.5,13.5,13.75)	$Q \geq 124$ (6.5,6.75,7.25,7.4)
$\tilde{C}_{pi}$ (unimodal case)	$Q < 94$ (11,11.5,12.5,12.75)	$Q < 46$ (14, 14.5,15.5,15.75)	$Q < 123$ (7.5,7.75,8.5,8.75)
	$Q \geq 94$ (10,10.25,10.75,10.9)	$Q \geq 46$ (12,12.5,13.5,13.75)	$Q \geq 123$ (6.5,6.75,7.25,7.4)
$\tilde{C}_{oi}$($)	(7, 8, 11, 12)	(7, 8, 11, 12)	(7, 8, 11, 12)
$\tilde{C}_{hi}$($/year)	(0.02, 0.04, 0.22, 0.32)	(0.125, 0.145, 0.325, 0.425)	(0.06, 0.07, 0.09, 0.11)
$\tilde{\grave{C}}_{hi}$($/year)	(0.3, 0.4, 0.6, 0.62)	(0.4, 0.5, 0.7, 0.72)	(0.09, 0.1, 0.3, 0.32)
$\tilde{C}_{bi}$($/unit)	(0.7, 0.8, 1.1, 1.2)	(1.7, 1.8, 2.1, 2.2)	(0.7, 0.8, 1.1, 1.2)
$\tilde{C}_{li}$($/unit)	(1.8, 1.9, 2.2, 2.3)	(1.8, 1.9, 2.2, 2.3)	(0.8, 0.9, 1.2, 1.3)
$\tilde{\tilde{D}}_i$(unit/year)	(520, 530, 560, 570)	(370, 380, 410, 420)	(770,780, 810,820)
S_i($/unit)	9.75	11.5	6

By using the Mathematica program V. 12.3, utilize the parameters of Tables 1, 2 and 3 in equations (17) and (18) for POLO distribution to obtain the optimal solutions $\lambda_i{}^*$, $Q_i{}^*$, $r_i{}^*$ and the minimum expected total cost for each item given by Table 4, Table 5 in decreasing and unimodal cases respectively for model I and model II at various values of β, the region where β fits this data the best is 0 < β < 1.

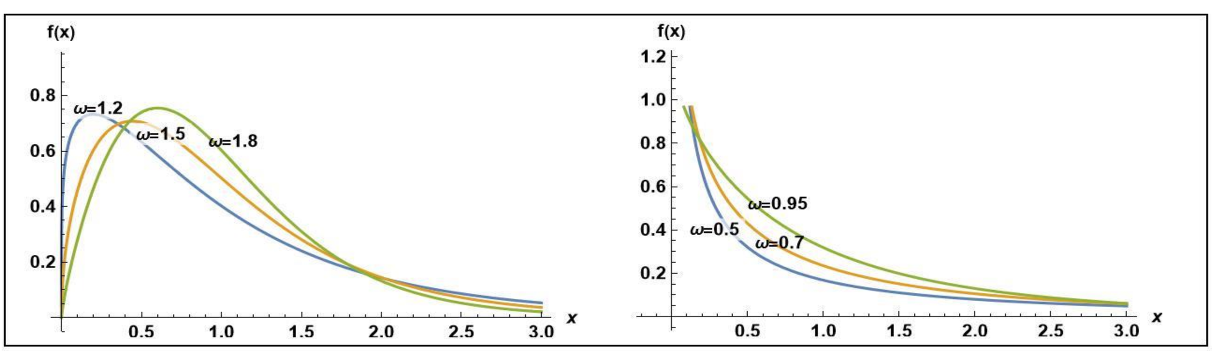

Fig. 3: The POLO density function plots for the unimodal and decreasing case

Table 4. The results of crisp and fuzzy for POLO distribution (decreasing case)

β	Item I						Item II						Item III					
	λ_1	Q_1	r_1	Q_{d1}	$E(HC_{R1})$	ETC_{1i}	λ_2	Q_2	r_2	Q_{d2}	$E(HC_{R2})$	ETC_{2i}	λ_3	Q_3	r_3	Q_{d3}	$E(HC_{R3})$	ETC_{3i}
0.1	0.3128	110.993	8.065	5.550	3.827	61.207	0.3952	49.006	3.303	2.450	2.497	109.744	0.4278	123.570	3.047	6.179	1.268	53.173
0.2	0.2771	111.652	7.939	5.583	2.395	60.833	0.3799	49.195	3.284	2.460	1.695	109.305	0.4149	124.071	3.038	6.204	0.786	46.122
0.3	0.2562	112.007	7.872	5.600	1.496	52.812	0.3699	49.310	3.273	2.465	1.150	109.04	0.4063	124.351	3.032	6.218	0.486	46.016
0.4	0.2439	112.190	7.837	5.609	0.933	52.721	0.3634	49.377	3.266	2.469	0.779	108.883	0.4023	124.506	3.030	6.225	0.300	45.957
0.5	0.2368	112.279	7.821	5.614	0.582	52.676	0.3591	49.416	3.262	2.471	0.527	108.792	0.3994	124.589	3.029	6.229	0.185	45.925
0.6	0.2326	112.318	7.814	5.616	0.363	52.656	0.3564	49.438	3.260	2.472	0.357	108.741	0.3977	124.634	3.028	6.232	0.114	45.908
0.7	0.2302	112.332	7.811	5.617	0.226	52.648	0.3545	49.449	3.259	2.472	0.242	108.714	0.3966	124.656	3.027	6.233	0.071	45.899
0.8	0.2288	112.335	7.811	5.617	0.141	52.646	0.3533	49.454	3.258	2.473	0.164	108.701	0.3960	124.667	3.027	6.233	0.044	45.895
0.9	**0.2280**	**112.335**	**7.811**	**5.617**	**0.088**	**52.645**	**0.3525**	**49.456**	**3.258**	**2.473**	**0.111**	**108.695**	**0.3956**	**124.673**	**3.027**	**6.234**	**0.027**	**45.893**
β	$\tilde{\lambda}_1$	$\tilde{Q}_1$	$\tilde{r}_1$	$\tilde{Q}_{d1}$	$E(\widetilde{HC}_{R1})$	$E\widetilde{TC}_{1i}$	$\tilde{\lambda}_2$	$\tilde{Q}_2$	$\tilde{r}_2$	$\tilde{Q}_{d2}$	$E(\widetilde{HC}_{R2})$	$E\widetilde{TC}_{2i}$	$\tilde{\lambda}_3$	$\tilde{Q}_3$	$\tilde{r}_3$	$\tilde{Q}_{d3}$	$E(\widetilde{HC}_{R3})$	$E\widetilde{TC}_{di}$
0.1	0.2291	113.674	7.616	5.684	4.842	59.021	0.3037	50.539	3.093	2.527	2.881	105.098	0.4511	124.753	2.967	6.238	1.317	45.707
0.2	0.1853	114.574	7.454	5.729	3.025	51.157	0.2869	50.770	3.072	2.539	1.951	104.599	0.4379	125.260	2.957	6.263	0.815	45.517
0.3	0.1597	115.065	7.367	5.753	1.885	50.929	0.2757	50.910	3.058	2.546	1.319	104.299	0.4299	125.544	2.952	6.277	0.504	45.412
0.4	0.1447	115.321	7.321	5.766	1.174	50.810	0.2685	50.993	3.051	2.550	0.891	104.121	0.4251	125.701	2.949	6.285	0.311	45.353
0.5	0.1359	115.446	7.299	5.772	0.730	50.751	0.2639	51.040	3.046	2.552	0.601	104.018	0.4221	125.785	2.948	6.289	0.192	45.322
0.6	0.1309	115.503	7.289	5.775	0.454	50.723	0.2608	51.067	3.044	2.553	0.406	103.960	0.4204	125.830	2.947	6.291	0.118	45.305
0.7	0.1279	115.525	7.285	5.776	0.282	50.712	0.2589	51.081	3.043	2.554	0.274	103.929	0.4193	125.853	2.947	6.293	0.073	45.297
0.8	0.1263	115.531	7.285	5.777	0.176	50.708	0.2575	51.087	3.042	2.554	0.185	103.913	0.4186	125.864	2.947	6.293	0.045	45.292
0.9	**0.1253**	**115.528**	**7.285**	**5.776**	**0.109**	**50.707**	**0.2567**	**51.090**	**3.042**	**2.555**	**0.125**	**103.906**	**0.4182**	**125.869**	**2.946**	**6.293**	**0.028**	**45.290**

Table 5. The results of crisp and fuzzy for POLO distribution (unimodal case)

	Item I						**Item II**						**Item III**					
β	λ_1	Q_1	r_1	Q_{d1}	$E(HC_{R1})$	ETC_{1i}	λ_2	Q_2	r_2	Q_{d2}	$E(HC_{R2})$	ETC_{2i}	λ_3	Q_3	r_3	Q_{d3}	$E(HC_{R3})$	ETC_{3i}
0.1	0.5827	90.094	2.801	4.505	2.888	75.087	0.3157	44.937	1.765	2.247	2.174	119.340	0.3036	121.890	1.759	6.095	1.227	53.886
0.2	0.5516	90.955	2.789	4.548	1.854	74.365	0.3045	45.136	1.762	2.257	1.491	118.799	0.2933	122.392	1.757	6.120	0.761	46.737
0.3	0.5320	91.461	2.782	4.573	1.185	64.397	0.2970	45.260	1.759	2.263	1.020	118.462	0.2807	122.675	1.756	6.134	0.472	46.627
0.4	0.5196	91.750	2.777	4.588	0.756	64.189	0.2918	45.336	1.759	2.267	0.697	118.256	0.2831	122.832	1.755	6.142	0.292	46.566
0.5	0.5119	91.913	2.775	4.596	0.481	64.072	0.2883	45.382	1.758	2.269	0.477	118.131	0.2807	122.918	1.755	6.146	0.180	46.532
0.6	0.5070	92.002	2.774	4.600	0.306	64.010	0.2860	45.409	1.757	2.270	0.325	118.057	0.2792	122.964	1.755	6.148	0.111	46.514
0.7	0.5039	92.049	2.773	4.602	0.195	63.974	0.2844	45.425	1.757	2.271	0.222	118.014	0.2783	122.988	1.754	6.149	0.069	46.505
0.8	0.5020	92.073	2.773	4.604	0.124	63.957	0.2836	45.433	1.757	2.272	0.152	117.989	0.2777	122.999	1.754	6.150	0.043	46.500
0.9	**0.5008**	**92.085**	**2.773**	**4.604**	**0.079**	**63.948**	**0.2826**	**45.438**	**1.757**	**2.272**	**0.104**	**117.976**	**0.2774**	**123.006**	**1.54**	**6.150**	**0.026**	**46.498**
β	$\tilde{\lambda}_1$	$\tilde{Q}_1$	$\tilde{r}_1$	$\tilde{Q}_{d1}$	$E(\widetilde{HC}_{R1})$	$E\widetilde{TC}_{1i}$	$\tilde{\lambda}_2$	$\tilde{Q}_2$	$\tilde{r}_2$	$\tilde{Q}_{d2}$	$E(\widetilde{HC}_{R2})$	$E\widetilde{TC}_{2i}$	$\tilde{\lambda}_3$	$\tilde{Q}_3$	$\tilde{r}_3$	$\tilde{Q}_{d3}$	$E(\widetilde{HC}_{R3})$	$E\widetilde{TC}_{3i}$
0.1	0.5219	93.608	2.754	4.680	3.727	71.379	0.2640	46.638	1.727	2.332	2.540	113.574	0.3387	123.099	1.739	6.155	1.276	46.304
0.2	0.4830	94.771	2.738	4.739	2.389	61.584	0.2514	46.881	1.723	2.344	1.736	112.968	0.3282	123.607	1.737	6.180	0.791	46.109
0.3	0.4588	95.456	2.729	4.773	1.522	61.133	0.2429	47.032	1..721	2.352	1.184	112.592	0.3217	123.894	1.736	6.194	0.489	45.999
0.4	0.4436	95.848	2.723	4.792	0.967	60.876	0.2371	47.124	1.719	2.356	0.806	112.363	0.3177	124.053	1.735	6.203	0.302	45.939
0.5	0.4340	96.068	2.720	4.803	0.613	60.733	0.2332	47.180	1.718	2.359	0.549	112.224	0.3152	124.139	1.735	6.207	0.187	45.906
0.6	0.4281	96.188	2.719	4.809	0.389	60.655	0.2306	47.213	1.718	2.361	0.373	112.142	0.3137	124.186	1.735	6.209	0.115	45.888
0.7	0.4244	96.251	2.718	4.813	0.246	60.614	0.2288	47.232	1.717	2.362	0.254	112.095	0.3127	124.210	1.734	6.210	0.071	45.879
0.8	0.4220	96.284	2.717	4.814	0.156	60.592	0.2276	47.242	1.717	2.362	0.173	112.068	0.3121	124.222	1.734	6.211	0.044	45.875
0.9	**0.4206**	**96.299**	**2.717**	**4.815**	**0.099**	**60.582**	**0.2268**	**47.248**	**1.717**	**2.362**	**0.117**	**112.054**	**0.3118**	**124.228**	**1.734**	**6.211**	**0.027**	**45.872**

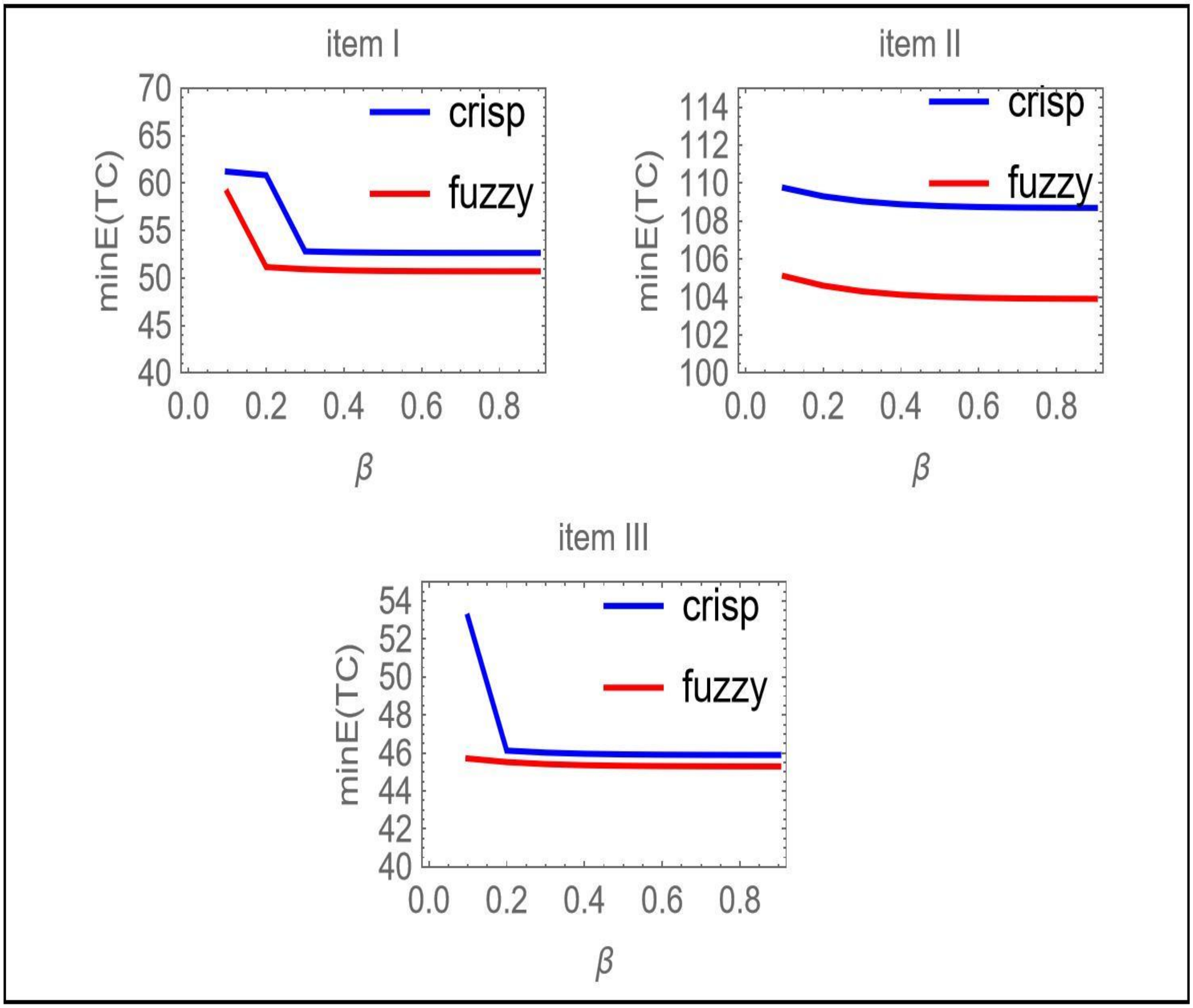

Fig. 4: The plots of crisp and fuzzy for three items (decreasing case)

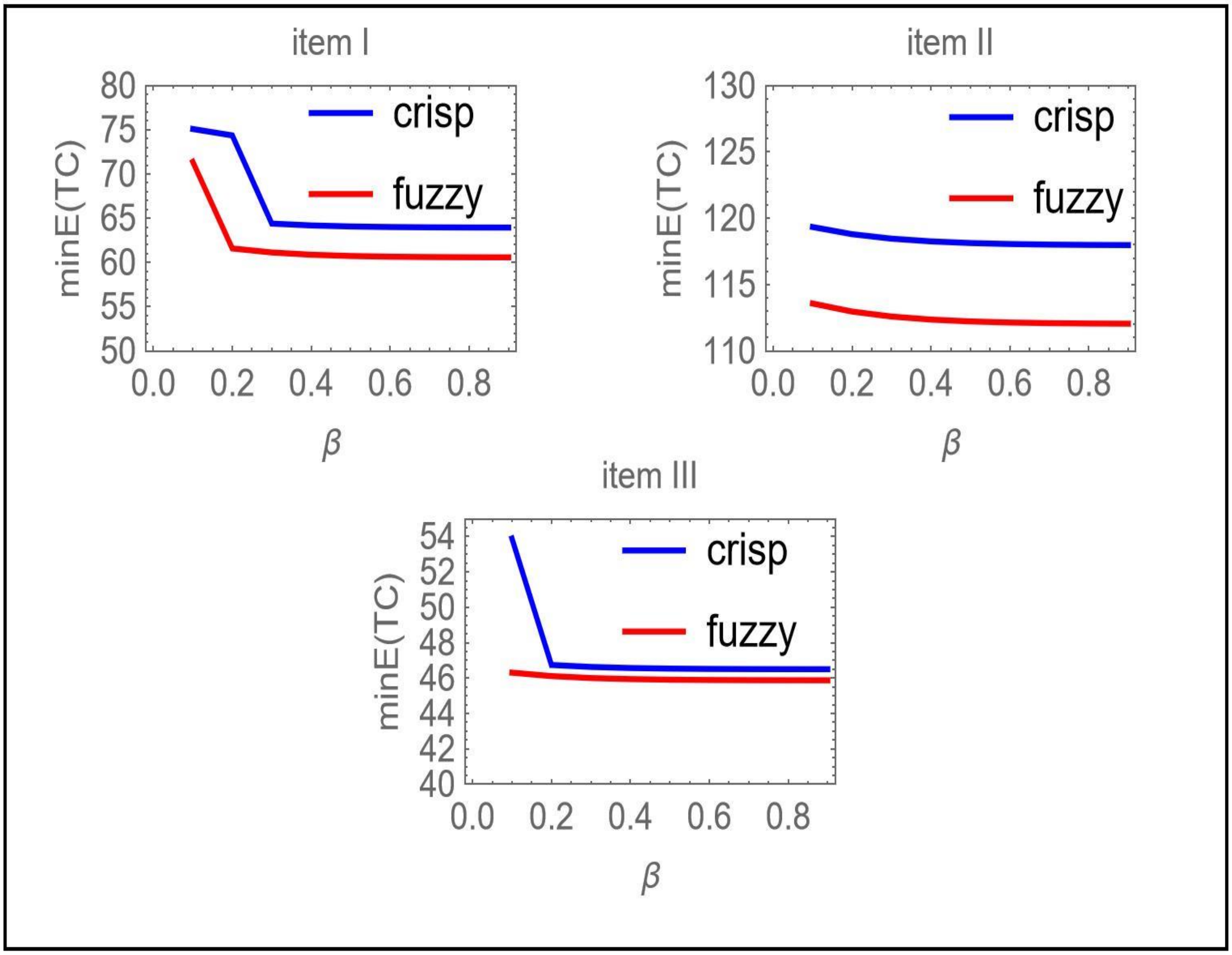

Fig. 5: The plots of crisp and fuzzy for three items (unimodal case)

Table 6. Comparison between the optimal policy variables of three items for unimodal and decreasing cases at the best value of β

Decreasing case												
	Crisp						Fuzzy					
Item	λ_i^*	Q_i^*	r_i^*	Q_{di}	$E(HC_{Ri})$	ETC_i	$\tilde{\lambda}_i^*$	$\tilde{Q}_i^*$	$\tilde{r}_i^*$	$\tilde{Q}_{di}$	$E(\widetilde{HC}_{Ri})$	$E\widetilde{TC}_i$
1	**0.2280**	**112.335**	**7.811**	**5.617**	**0.088**	**52.645**	**0.1253**	**115.528**	**7.285**	**5.776**	**0.109**	**50.707**
2	**0.3525**	**49.456**	**3.258**	**2.473**	**0.111**	**108.695**	**0.2567**	**51.090**	**3.042**	**2.555**	**0.125**	**103.906**
3	**0.3956**	**124.673**	**3.027**	**6.234**	**0.027**	**45.893**	**0.4182**	**125.869**	**2.946**	**6.293**	**0.028**	**45.290**
Unimodal case												
	Crisp						Fuzzy					
Item	λ_i^*	Q_i^*	r_i^*	Q_{di}	$E(HC_{Ri})$	ETC_i	$\tilde{\lambda}_i^*$	$\tilde{Q}_i^*$	$\tilde{r}_i^*$	$\tilde{Q}_{di}$	$E(\widetilde{HC}_{Ri})$	$E\widetilde{TC}_i$
1	**0.5008**	**92.085**	**2.773**	**4.604**	**0.079**	**63.948**	**0.4206**	**96.299**	**2.717**	**4.815**	**0.099**	**60.582**
2	**0.2826**	**45.438**	**1.757**	**2.272**	**0.104**	**117.976**	**0.2268**	**47.248**	**1.717**	**2.362**	**0.117**	**112.054**
3	**0.2774**	**123.006**	**1.54**	**6.150**	**0.026**	**46.498**	**0.3118**	**124.228**	**1.734**	**6.211**	**0.027**	**45.872**

5. Conclusion

In this study we introduced probabilistic multi-item continuous review inventory model with expiration cusp and all units discount under a varying rent cost of model I and model II for two cases of POLO distribution. We evaluated the approximated solutions of ${Q_i}^*$ and ${r_i}^*$ for each value of β and ${\lambda_i}^*$ under the expected mixture shortage cost constraint and then we obtain the numbers of deteriorated items, and the minimum expected total cost by utilising Lagrangian multiplier technique.

Based on the numerical example results, we found that, firstly, the best value for the minimum expected total cost is obtained at β = 0.9, as shown in Table 4 and Table 5. Secondly, the minimum expected total cost in model II is less than in model I as shown in Figure 4 and Figure 5, which suggests that fuzziness is fairly accurate to life as it really is. Thirdly, for the decreasing and unimodal cases of POLO distribution, we observed that the greater Q_i and the less E(TC(Q_i, r_i)) as shown in Table 4 and Table 5. Fourthly, ${Q_i}^*$ in the decreasing case is greater than ${Q_i}^*$ in the unimodal case. However, in the decreasing case, the minimum E(TC(Q_i, r_i)) is less than in the unimodal case as shown in Table 6. Therefore, it is preferable to apply the distribution in the decreasing case. Finally, when ${Q_i}^*$ increases, the expected varying rent cost $E(HC_{Ri})$ decreases and thus the E(TC(Q_i, r_i)) decreases.

Conflict of interest

The author declares that he has no competing interests.

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Authors' contributions

All authors read and agreed with the revised version of the paper.

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