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# Alternating Repeated Games via $\pi$-Pre-Separation Axioms and Functions

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**Abstract:** This paper aims to link topology and game theory by using definitions of $\pi$-pre-separation axioms on $\pi$-pre-topological spaces. In this paper, we introduce and investigate infinitely long games using the concept of separation axioms on $\pi$-pre-topological spaces, specifically $\pi$-pre-$T_0$, $\pi$-pre-$T_1$, and $\pi$-pre-$T_2$. Winning and losing strategies for both players are studied with some examples. In addition, the effects of pre-open, surjective, injective, and pre-continuous functions on both players' strategies in different kinds of games are also studied.

**Keywords:** Pre-topology; $\pi$-pre-topology; separation axioms; game theory.

**2020 AMS Subject Classifications:** 54A05, 54C10, 54D10, 54D35, 91A05.

## 1 Introduction and Preliminaries

Pre-topology is a generalization of topology introduced by Brissaud [1,2], which depended on previous work of the Čech closure operator [3] and Kuratowski closure axioms [4]. A pre-topology on a non-empty set $\mathsf{X}$ is a pseudo closure operator $\tau$, from $P(\mathsf{X})$ to $P(\mathsf{X})$, where $P(\mathsf{X})$ is the power set of $\mathsf{X}$, which satisfies the following conditions: $\tau(\phi) = \phi$ and $\forall M \in P(\mathsf{X})$, $M \subseteq \tau(M)$. A pre-topology is a mathematical tool for modeling complex systems. Pre-topology has played a great role in data analysis [6,7], the generalization of graph theory [8,9] and game theory [10].

A function $\mathtt{f} : (\mathsf{X}, \tau) \to (\mathsf{Y}, \sigma)$ is said to be a pre-continuous function if, for all $H \in O(\sigma), \mathtt{f}^{-1}(H) \in O(\tau)$, where $O(\tau)$ and $O(\sigma)$ are collections of all $\tau$-pre-open and $\sigma$-pre-open respectively [11,12].

The concept of $\pi$-pre-topological space $(\mathsf{X}, \psi, \omega)$ has been defined in [13] which is a generalization of pre-topological space depending on two pre-topologies on an arbitrary universal set. New types of separation axioms on $\pi$-pre-topological spaces have been introduced in [13]. A $\pi$-pre-topological space $(\mathsf{X}, \psi, \omega)$ is called a $\pi$-pre-$T_0$ if it satisfies the following condition, $\forall \kappa, \rho \in \mathsf{X}$ with $\kappa \neq \rho, \exists \mathsf{H} \in \mathsf{O}(\psi) \cup \mathsf{O}(\omega)$ such that $\kappa \in \mathsf{H}, \rho \notin \mathsf{H}$ or $\kappa \notin \mathsf{H}, \rho \in \mathsf{H}$ (i.e, $\mathsf{H}$ contains only one of them), a $\pi$-pre-topological space $(\mathsf{X}, \psi, \omega)$ is said to be a $\pi$-pre-$T_1$ if it

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satisfies the following condition, $\forall \kappa, \rho \in \mathsf{X}$ with $\kappa \neq \rho$, then $\exists$ $\mathsf{Q} \in \mathsf{O}(\psi)$ and $\mathsf{R} \in \mathsf{O}(\omega)$ such that $\kappa \in \mathsf{Q}, \rho \notin \mathsf{Q}$ and $\rho \in \mathsf{R}, \kappa \notin \mathsf{R}$, or equivalently, $\kappa \in \mathsf{Q} - \mathsf{R}$ and $\rho \in \mathsf{R} - \mathsf{Q}$ and a $\pi$-pre-topological space $(\mathsf{X}, \psi, \omega)$ is called a $\pi$-pre-$T_2$ space if it satisfies the following condition, $\forall \kappa, \rho \in \mathsf{X}$ with $\kappa \neq \rho$, then $\exists$ $\mathsf{S} \in \mathsf{O}(\psi)$ and $\mathsf{T} \in \mathsf{O}(\omega)$ such that $\kappa \in \mathsf{S}$, $\rho \in \mathsf{T}$ and $S \cap T = \phi$, Where $\mathsf{O}(\psi)$ and $\mathsf{O}(\omega)$ are the collection of all pre-open sets with respect to $\psi$ and $\omega$ respectively. A mapping $\varphi : (\mathsf{X}, \psi, \Gamma) \to (\mathsf{Y}, \alpha, \beta)$ is called a $\pi$-pre-continuous function if the two functions $\varphi : (\mathsf{X}, \psi) \to (\mathsf{Y}, \alpha)$ and $\varphi : (\mathsf{X}, \Gamma) \to (\mathsf{Y}, \beta)$ are pre-continuous functions [13].

In 1957, Berge [14] introduced the term topological game with perfect information. A different meaning of topological games has been proposed by Telgarsky [15, 16]. The players in a topological game opt for some elements attached to the topological structures such as covers, open subsets, closed subsets, points, etc., and the condition on the game to be a winning strategy for a player must satisfy topological structures like compactness, convergence, closure, etc. These games always consider two players; Player **A** and Player **B**. If Player **A** begins the game (this means that he makes the first pace). A repeated game is one of the games that has more than one period, this game's number of repetitions may be finite or infinite. There are two kinds of games simultaneous and alternating games. The game is simultaneous if both players choose their paces at the same time, but none of them knows the choice of the other [17–19]. The game is alternating if one of the players (Player **A**) opts for one of the paces, then, the other player (Player **B**) opts for one of the other paces, after that Player **B** knows the pace of the first player and must note the player who begins the game.

A game G is said to be determined if either Player **A** or Player **B** has a winning strategy in G. A stationary strategy is a strategy that depends on the opponent's last pace only, and Markov strategy is a strategy that depends only on the ordinal number of the pace and opponent's last pace. Glavin and Scheeperst [20] studied infinite games. There is a lot of research talking about topological games such as [21–25]. There is a lot of research that linked topology and applications in various fields, such as medicine and economics [13, 26–28].

In Section 2, we define the infinitely long games $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ and $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$. We present two examples, one of which represents a winning strategy for Player **A** and a losing strategy for Player **B**, and the other example represents a winning strategy for Player **B** and a losing strategy for Player **A**. Also, we study winning and losing strategies for both Players with respect to each game. We prove that, if Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_i)$, $i = 1, 2$, then, Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_{i-1})$. Also, we prove that, if Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_i)$, $i = 0, 1$, then, Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_{i+1})$. We present infinite examples on $\mathbb{R}$ to clarify the previous two results. In addition, we introduce a diagram to explain the links between the three kinds of games. finally, we explain the relationship between game theory and pretopology in real-life situations.

In Section 3, we apply some properties of functions like pre-open, surjective, injective and pre-continuous to study the effect of these properties with games $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ and $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ on both players' strategies with respect to games $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$, $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_1)$ and $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_2)$. We prove that, if $\varphi : (\mathsf{X}, \Gamma, \lambda) \to (\mathsf{Y}, \alpha, \beta)$ is a pre-open function, surjective and Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_0)$, then, Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$. Also, we prove that, if $\varphi : (\mathsf{X}, \Gamma, \lambda) \to (\mathsf{Y}, \alpha, \beta)$ is a pre-open function, bijective and Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_{\mathbf{i}})$, then, Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_{\mathbf{i}})$, $\mathbf{i} = 1, 2$. In addition, we prove that, if $\varphi : (\mathsf{X}, \Gamma, \lambda) \to (\mathsf{Y}, \alpha, \beta)$ is a pre-continuous, injective function and Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_{\mathbf{i}})$, then, Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_{\mathbf{i}})$, $\mathbf{i} = 0, 1, 2$. Finally, a conclusion in Section 4 is given.

## 2 Games on $\pi$-pre-topological spaces

In this section, the definitions of $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ and $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ are presented.
In addition, winning and losing strategies for both players are studied.

**Definition 1.** *Let $(\mathsf{X},\psi,\omega)$ be a $\pi$-pre-topological space, we define a game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$ the following Player* **A** *and Player* **B** *are playing period for each natural number in the* $\mathbf{j}-th$ *period:*
*In the first pace, Player* **A** *opts for $\kappa_{\mathbf{j}} \neq \rho_{\mathbf{j}}$ where $\kappa_{\mathbf{j}},\rho_{\mathbf{j}} \in \mathsf{X}$. In the second pace, Player* **B** *opts for a pre-open subset $\mathsf{H}_{\mathbf{j}} \in \mathsf{O}(\psi)\cup\mathsf{O}(\omega)$ such that $\mathsf{H}_{\mathbf{j}}$ contains only one of the two points $\kappa_{\mathbf{j}},\rho_{\mathbf{j}}$.*

*Player* **B** *has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$ if $\mathbb{H} = \{\mathsf{H}_1,\mathsf{H}_2,\mathsf{H}_3,...,\mathsf{H}_{\mathbf{j}},...\}$ is a collection of pre-open subsets of $\mathsf{X}$ such that $\forall \kappa_{\mathbf{j}} \neq \rho_{\mathbf{j}}, \kappa_{\mathbf{j}},\rho_{\mathbf{j}} \in \mathsf{X}, \exists \mathsf{H}_{\mathbf{j}} \in \mathbb{H}$ contains only one of two points $\kappa_{\mathbf{j}},\rho_{\mathbf{j}}$. Otherwise, Player* **A** *wins. We represent this algorithm for this game in Figure 1.*

*Example 1.* Consider the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, where $\mathsf{X} = \{\kappa,\rho,\nu,\lambda\}$. From Table 1,
$\mathsf{O}(\psi) = \{\{\rho\},\{\lambda\},\{\rho,\nu\},\{\kappa,\nu\},\{\kappa,\nu,\lambda\},\{\kappa,\rho,\nu\},\mathsf{X},\emptyset\}$ and $\mathsf{O}(\omega) = \{\{\nu\},\{\kappa,\rho\},\{\nu,\lambda\},\{\rho,\nu,\lambda\},\mathsf{X},\emptyset\}$.

**Table 1:** $\psi(\mathsf{A})$ and $\omega(\mathsf{A})$

| A | $\psi$(A) | $\omega$(A) |
|---|---|---|
| $\{\kappa\}$ | $\{\kappa,\rho\}$ | $\{\kappa\}$ |
| $\{\rho\}$ | $\{\rho\}$ | $\{\kappa,\rho\}$ |
| $\{\nu\}$ | X | X |
| $\{\lambda\}$ | $\{\lambda\}$ | $\{\nu,\lambda\}$ |
| $\{\kappa,\rho\}$ | X | $\{\kappa,\rho\}$ |
| $\{\kappa,\nu\}$ | $\{\kappa,\rho,\nu\}$ | X |
| $\{\kappa,\lambda\}$ | $\{\kappa,\lambda\}$ | $\{\kappa,\rho,\lambda\}$ |

| A | $\psi$(A) | $\omega$(A) |
|---|---|---|
| $\{\rho,\nu\}$ | $\{\rho,\nu,\lambda\}$ | $\{\rho,\nu,\lambda\}$ |
| $\{\rho,\lambda\}$ | $\{\rho,\lambda\}$ | X |
| $\{\nu,\lambda\}$ | X | $\{\nu,\lambda\}$ |
| $\{\kappa,\rho,\nu\}$ | $\{\kappa,\rho,\nu\}$ | X |
| $\{\kappa,\rho,\lambda\}$ | X | $\{\kappa,\rho,\lambda\}$ |
| $\{\rho,\nu,\lambda\}$ | X | X |
| $\{\kappa,\nu,\lambda\}$ | $\{\kappa,\nu,\lambda\}$ | X |

Player **A** and Player **B** are playing for six periods. Then, the first period is the following: Player **A** opts for $\kappa \neq \rho$ where $\kappa,\rho \in \mathsf{X}$. Player **B** opts for $\mathsf{H}_1 = \{\rho\} \in \mathsf{O}(\psi)$ such that $\rho \in \mathsf{H}_1$ and $\kappa \notin \mathsf{H}_1$.

Then, the next period (the second period) is the following: Player **A** opts for $\kappa \neq \nu$ where $\kappa,\nu \in \mathsf{X}$.
Player **B** opts for $\mathsf{H}_2 = \{\nu\} \in \mathsf{O}(\omega)$ such that $\nu \in \mathsf{H}_2$ and $\kappa \notin \mathsf{H}_2$.

Then, the next period (the third period) is the following: Player **A** opts for $\kappa \neq \lambda$ where $\kappa,\lambda \in \mathsf{X}$.
Player **B** opts for $\mathsf{H}_3 = \{\lambda\} \in \mathsf{O}(\phi)$ such that $\lambda \in \mathsf{H}_3$ and $\kappa \notin \mathsf{H}_3$.

Then, the next period (the fourth period) is the following: Player **A** opts for $\rho \neq \nu$ where $\rho,\nu \in \mathsf{X}$.
Player **B** opts for $\mathsf{H}_4 = \{\rho\} \in \mathsf{O}(\phi)$ such that $\rho \in \mathsf{H}_4$ and $\nu \notin \mathsf{H}_1$.

Then, the next period (the fifth period) is the following: Player **A** opts for $\rho \neq \lambda$ where $\rho, \lambda \in X$. Player **B** opts for $H_5 = \{\lambda\} \in O(\phi)$ such that $\lambda \in H_5$ and $\rho \notin H_3$.

Then, the next period (the sixth period) is the following: Player **A** opts for $\nu \neq \lambda$ where $\nu, \lambda \in X$. Player **B** opts for $H_6 = \{\nu\} \in O(\omega)$ such that $\nu \in H_6$ and $\lambda \notin H_2$.

Then, $\mathbb{H} = \{H_1, H_2, H_3, H_4, H_5, H_6\}$ is the winning strategy for Player **B** in $G^X_{(\psi,\omega)}(T_0)$. Hence, Player **B** wins.

*Example 2.* Consider the game $G^X_{(\psi,\omega)}(T_0)$, where $X = \{\kappa, \rho, \nu\}$. From Table 2, $O(\psi) = \{X, \emptyset\}$ and $O(\omega) = \{\{\rho\}, X, \emptyset\}$.

**Table 2:** $\psi(A)$ and $\omega(A)$

| A | $\{\kappa\}$ | $\{\rho\}$ | $\{\nu\}$ | $\{\kappa,\rho\}$ | $\{\kappa,\nu\}$ | $\{\rho,\nu\}$ |
|---|---|---|---|---|---|---|
| $\psi(A)$ | $\{\kappa,\nu\}$ | $\{\kappa,\rho\}$ | X | X | X | X |
| $\omega(A)$ | X | $\{\rho,\nu\}$ | $\{\rho,\nu\}$ | X | $\{\kappa,\nu\}$ | X |

In the first period: The first pace, Player **A** opts for $\kappa \neq \rho$ where $\kappa, \rho \in X$. In the second pace, Player **B** opts for $H_1 = \{\rho\} \in O(\omega)$ which is a pre-open set containing $\rho$ and not containing $\kappa$.

In the second period: The first pace, Player **A** opts for $\kappa \neq \nu$ where $\kappa, \nu \in X$. In the second pace, Player **B** cannot find $H_2$ which is a pre-open set containing only one of the two points $\kappa, \nu$. Hence, Player **A** wins the game $G^X_{(\psi,\omega)}(T_0)$.

**Proposition 1.** *Player* **B** *has a winning strategy in the game* $G^X_{(\psi,\omega)}(T_0)$ *if and only if* $(X, \psi, \omega)$ *is* $\pi - pre - T_0$.

*Proof* . "$\Rightarrow$": If Player **B** has a winning strategy in $G^X_{(\psi,\omega)}(T_0)$, then, for any two distinct points $x_i, x_j \in X$ will be chosen by Player **A**, Player **B** can find a pre-open set $\mathscr{N} \in O(\psi) \cup O(\omega)$ which contains only one of the two points $x_i$, $x_j$. Hence, $(X, \psi, \omega)$ is $\pi - pre - T_0$ space.

"$\Leftarrow$": Follows immediately from the definition of $\pi - pre - T_0$ for a $\pi$-pre-topological space $(X, \psi, \omega)$.

**Corollary 1.** *Player* **B** *has a winning strategy in* $G^X_{(\psi,\omega)}(T_0)$ *if and only if for any two distinct points in* $X$*, a pre-closed set contained in* $C(\psi) \cup C(\omega)$ *exists, which contains only one of them. Where* $C(\psi)$ *and* $C(\omega)$ *are the collections of all pre-closed sets with respect to* $\psi$ *and* $\omega$ *respectively.*

*Proof* . Follows immediately from Proposition (2.4) and the concept complement pre-open sets.

**Corollary 2.** *Player* **A** *has a losing strategy in* $G^X_{(\psi,\omega)}(T_0)$ *if and only if a* $\pi$*-pre-topological space* $(X, \psi, \omega)$ *is* $\pi - pre - T_0$.

*Proof* . "$\Rightarrow$": Let Player **A** have a losing strategy in $G^X_{(\psi,\omega)}(T_0)$. If $(X, \psi, \omega)$ is not $\pi - pre - T_0$, then, there are two distinct points $x_i$, $x_j \in X$ such that Player **B** cannot find a pre-open set $\mathscr{U} \in O(\psi) \cup O(\omega)$ containing only one of them. Thus, Player **A** has a winning strategy. This gives a contradiction with the hypothesis. Hence, $(X, \psi, \omega)$ is $\pi - pre - T_0$.

"$\Leftarrow$": Let $(\mathsf{X},\psi,\omega)$ be $\pi - pre - T_0$. If Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, then, in $\mathbf{k}-th$ period, Player **A** opts for $\kappa_{\mathbf{k}} \neq \rho_{\mathbf{k}} \in \mathsf{X}$ and Player **B** cannot find a pre-open set $\mathscr{V} \in \mathsf{O}(\psi)\cup\mathsf{O}(\omega)$ which contains only one of them. This gives a contradiction with the hypothesis. Thus, Player **A** has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$.

**Proposition 2.** *Player* **A** *has a winning strategy in* $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$ *if and only if* $(\mathsf{X},\psi,\omega)$ *is not* $\pi - pre - T_0$.

*Proof* . "$\Rightarrow$": If Player **A** a winning strategy, this means that there is $\mathbf{r}-th$ period in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, Player **A** opts for two distinct points $\kappa_{\mathbf{r}},\rho_{\mathbf{r}} \in \mathsf{X}$ such that Player **B** cannot choose a pre-open set $\mathsf{H}_{\mathbf{r}}$ contained in $\mathsf{O}(\psi)\cup\mathsf{O}(\omega)$ containing only one of them. Hence, $(\mathsf{X},\psi,\omega)$ is not $\pi - pre - T_0$.

"$\Leftarrow$": Suppose that $(\mathsf{X},\psi,\omega)$ is not $\pi - pre - T_0$, then, there are two distinct points $\mathtt{x}_{\mathbf{i}}, \mathtt{x}_{\mathbf{j}} \in \mathsf{X}$ such that there is no existence of a pre-open set $\mathscr{S} \in \mathsf{O}(\psi)\cup\mathsf{O}(\omega)$ which contains only one of $\mathtt{x}_{\mathbf{i}}$, $\mathtt{x}_{\mathbf{j}}$. Then, $\mathtt{x}_{\mathbf{i}}$ and $\mathtt{x}_{\mathbf{j}}$ will be the choice of Player **A**. In this period, Player **B** cannot find a pre-open set $\mathscr{S} \in \mathsf{O}(\psi)\cup\mathsf{O}(\omega)$ which contains only one of $\mathtt{x}_{\mathbf{i}}$, $\mathtt{x}_{\mathbf{j}}$. So, Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$.

**Corollary 3.** *Player* **B** *has a losing strategy in* $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$ *if and only if* $(\mathsf{X},\psi,\omega)$ *is not* $\pi - pre - T_0$.

*Proof* . "$\Rightarrow$": Suppose that Player **B** has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$. If $(\mathsf{X},\psi,\omega)$ is $\pi - pre - T_0$, then, for any two distinct points $\mathtt{x}_{\mathbf{i}},\mathtt{x}_{\mathbf{j}} \in \mathsf{X}$, a pre-open set $\mathscr{R} \in \mathsf{O}(\psi)\cup\mathsf{O}(\omega)$ exists, which contains only one of $\mathtt{x}_{\mathbf{i}}$, $\mathtt{x}_{\mathbf{j}}$. Hence, Player **B** will win $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$. This gives a contradiction with the hypothesis. So, $(\mathsf{X},\psi,\omega)$ is not $\pi - pre - T_0$.

"$\Leftarrow$": Let $(\mathsf{X},\psi,\omega)$ is not $\pi - pre - T_0$. If Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, then, for any two distinct points $\mathtt{x}_{\mathbf{i}},\mathtt{x}_{\mathbf{j}} \in \mathsf{X}$ will be chosen by Player **A**, Player **B** can find a pre-open set $\mathscr{T} \in \mathsf{O}(\psi)\cup\mathsf{O}(\omega)$ which contains only one of them. This gives a contradiction with the hypothesis. So, Player **B** has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$.

**Definition 2.** *Let* $(\mathsf{X},\psi,\omega)$ *be a* $\pi$*-pre-topological space, we define a game* $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ *the following; Player* **A** *and Player* **B** *are playing period with each natural number in this game in the* $\mathbf{j}-th$ *period:*
*In the first pace, Player* **A** *opts for two distinct points* $\kappa_{\mathbf{j}},\rho_{\mathbf{j}} \in \mathsf{X}$. *In the second pace, Player* **B** *opts for two subsets* $\mathsf{S}_{\mathbf{j}} \in \mathsf{O}(\psi)$ *and* $\mathsf{R}_{\mathbf{j}} \in \mathsf{O}(\omega)$ *such that* $\kappa_{\mathbf{j}} \in \mathsf{S}_{\mathbf{j}} - \mathsf{R}_{\mathbf{j}}$ *and* $\rho_{\mathbf{j}} \in \mathsf{R}_{\mathbf{j}} - \mathsf{S}_{\mathbf{j}}$.
*Player* **B** *has a winning strategy in* $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ *if* $\mathbb{C} = \{(\mathsf{S}_1,\mathsf{R}_1),(\mathsf{S}_2,\mathsf{R}_2),...,(\mathsf{S}_{\mathbf{j}},\mathsf{R}_{\mathbf{j}}),...\}$ *is a collection of pre-open sets in* $\mathsf{X}$ *such that* $\forall \kappa_{\mathbf{j}} \neq \rho_{\mathbf{j}}; \kappa_{\mathbf{j}},\rho_{\mathbf{j}} \in \mathsf{X}, \exists (\mathsf{S}_{\mathbf{j}},\mathsf{R}_{\mathbf{j}}) \in \mathbb{C}$ *such that* $\kappa_{\mathbf{j}} \in \mathsf{S}_{\mathbf{j}} - \mathsf{R}_{\mathbf{j}}$ *and* $\rho_{\mathbf{j}} \in \mathsf{R}_{\mathbf{j}} - \mathsf{S}_{\mathbf{j}}$. *Otherwise, Player* **A** *wins. We represent this algorithm for this game in Figure 2.*

*Example 3.* Consider the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$, where $\mathsf{X} = \{\kappa,\rho,\nu,\lambda\}$. As follows, from Table 1, $\mathsf{O}(\psi) = \{\{\rho\},\{\lambda\},\{\rho,\nu\},\{\kappa,\nu\},\{\kappa,\nu,\lambda\},\{\kappa,\rho,\nu\},\mathsf{X},\emptyset\}$ and $\mathsf{O}(\omega) = \{\{\nu\},\{\kappa,\rho\},\{\nu,\lambda\},\{\rho,\nu,\lambda\},\mathsf{X},\emptyset\}$.

Player **A** and Player **B** are playing for six periods in this game. Then, the first period is the following: Player **A** opts for $\kappa \neq \rho$, where $\kappa,\rho \in \mathsf{X}$.
Player **B** opts for $S_1 = \{\kappa,\nu\} \in O(\psi)$ and $R_1 = \{\rho,\nu,\lambda\} \in O(W)$ such that $\kappa \in S_1 - R_1$ and $\rho \in R_1 - S_1$.
Then, the next period (the second period) is the following: Player **A** opts for $\kappa \neq \nu$, where $\kappa,\nu \in \mathsf{X}$.
Player **B** opts for $S_2 = \{\rho,\nu\} \in O(\psi)$ and $R_2 = \{\kappa,\rho\} \in O(W)$ such that $\nu \in S_2 - R_2$ and $\kappa \in R_2 - S_2$.
Then, the next period (the third period) is the following: Player **A** opts for $\kappa \neq \lambda$, where $\kappa,\lambda \in \mathsf{X}$.
Player **B** opts for $S_3 = \{\lambda\} \in O(\psi)$ and $R_3 = \{\kappa,\rho\} \in O(W)$ such that $\lambda \in S_3 - R_3$ and $\kappa \in R_3 - S_3$.

Then, the next period (the fourth period) is the following: Player **A** opts for $\rho \neq \nu$, where $\rho, \nu \in \mathsf{X}$. Player **B** opts for $S_4 = \{\rho\} \in O(\psi)$ and $R_4 = \{\nu\} \in O(W)$ such that $\rho \in S_4 - R_4$ and $\nu \in R_4 - S_4$.

Then, the next period (the fifth period) is the following: Player **A** opts for $\rho \neq \lambda$, where $\rho, \lambda \in \mathsf{X}$. Player **B** opts for $S_5 = \{\rho\} \in O(\psi)$ and $R_5 = \{\nu, \lambda\} \in O(W)$ such that $\rho \in S_5 - R_5$ and $\lambda \in R_5 - S_5$.

Then, the next period (the sixth period) is the following: Player **A** opts for $\nu \neq \lambda$, where $\nu, \lambda \in \mathsf{X}$. Player **B** opts for $S_6 = \{\lambda\} \in O(\psi)$ and $R_6 = \{\nu\} \in O(W)$ such that $\lambda \in S_6 - R_6$ and $\nu \in R_6 - S_6$.

Then, $\mathbb{C} = \{(S_1, R_1), ..., (S_6, R_6)\}$ is the winning strategy for Player **B** in this game. Hence, Player **B** wins the game.

*Remark.* A $\pi$-pre-topological space $(\mathsf{X}, \psi, \omega)$ as in Example 2.3, is not $\pi$-pre-$T_1$ because it is not $\pi$-pre-$T_0$.

**Proposition 3.** *Let $(\mathsf{X}, \psi, \omega)$ be a $\pi$-pre-topological space. Then, Player* **B** *has a winning strategy in the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ if and only if $(\mathsf{X}, \psi, \omega)$ is $\pi$-pre-$T_1$.*

*Proof* . "$\Rightarrow$": Let Player **B** have a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$. Then, for any two distinct points $\mathtt{x}_\mathbf{i}, \mathtt{x}_\mathbf{j} \in \mathsf{X}$ will be chosen by Player **A**, Player **B** can find $S \in O(\psi)$ and $R \in O(\omega)$ such that $\mathtt{x}_\mathbf{i} \in S - R$ and $\mathtt{x}_\mathbf{j} \in R - S$. Hence, $(\mathsf{X}, \psi, \omega)$ is a $\pi$-pre-$T_1$ space.

"$\Leftarrow$": Follows from the definition of a $\pi$-pre-$T_1$ space.

**Corollary 4.** *Let $(\mathsf{X}, \psi, \omega)$ be a $\pi$-pre-topological space. Then, Player* **B** *has a winning strategy in the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ if and only if for any two distinct points $\mathtt{x}_\mathbf{i}, \mathtt{x}_\mathbf{j} \in \mathsf{X}$, $F \in C(\psi)$ and $G \in C(\omega)$ exists, such that $\mathtt{x}_\mathbf{i} \in F - G$ and $\mathtt{x}_\mathbf{j} \in G - F$.*

*Proof* . Follows from Proposition 2.12 and the concept of the complement of pre-open sets.

**Corollary 5.** *Let $(\mathsf{X}, \psi, \omega)$ be a $\pi$-pre-topological space. Then, Player* **A** *has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ if and only if $(\mathsf{X}, \psi, \omega)$ is a $\pi$-pre-$T_1$ space.*

*Proof* . "$\Rightarrow$": Let Player **A** have a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$. If $(\mathsf{X}, \psi, \omega)$ is not $\pi$-pre-$T_1$, then, two points exist $\mathtt{x}_\mathbf{i}, \mathtt{x}_\mathbf{j} \in \mathsf{X}$ such that there is no $T \in O(\psi)$ , $U \in O(\omega)$ such that $\mathtt{x}_\mathbf{i} \in T - U$ and $\mathtt{x}_\mathbf{j} \in U - T$. So, Player **A** has a winning strategy. This gives a contradiction with the hypothesis. Thus, $(\mathsf{X}, \psi, \omega)$ is a $\pi$-pre-$T_1$ space.

"$\Leftarrow$": Suppose that $(\mathsf{X}, \psi, \omega)$ is a $\pi$-pre-$T_1$ space. If Player **A** has a winning strategy, then in $\mathbf{r} - th$ period Player **A** opts for two distinct points $\kappa_\mathbf{r}, \rho_\mathbf{r} \in \mathsf{X}$ and Player **B** cannot find $E \in O(\psi)$ , $F \in O(\omega)$ such that $\kappa_\mathbf{r} \in E - F$ and $\rho_\mathbf{r} \in F - E$. This gives a contradiction with the hypothesis. Hence, Player **A** has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$.

**Proposition 4.** *Let $(\mathsf{X}, \psi, \omega)$ be a $\pi$-pre-topological space. Then, Player* **A** *has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ if and only if $(\mathsf{X}, \psi, \omega)$ is not a $\pi$-pre-$T_1$ space.*

*Proof* . "$\Rightarrow$": Let Player **A** have a winning strategy, then, there is $\mathbf{i} - th$ period in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$, where Player **A** opts for two distinct points $\kappa_\mathbf{i}, \rho_\mathbf{i} \in \mathsf{X}$ such that Player **B** cannot find $R_\mathbf{i} \in O(\psi)$ and $S_\mathbf{i} \in O(W)$ such that $\kappa_i \in R_\mathbf{i} - S_\mathbf{i}$ and $\rho_\mathbf{i} \in S_\mathbf{i} - R_\mathbf{i}$. Thus, $(\mathsf{X}, \psi, \omega)$ is not a $\pi$-pre-$T_1$ space.

"$\Leftarrow$": Let $(\mathsf{X}, \psi, \omega)$ not be a $\pi$-pre-$T_1$ space, then, there are two distinct points $\mathtt{x}_\mathbf{i}, \mathtt{x}_\mathbf{j} \in \mathsf{X}$ such that there are no $L \in O(\psi)$ and $M \in O(\omega)$ such that $\mathtt{x}_\mathbf{i} \in L - M$ and $\mathtt{x}_\mathbf{j} \in M - L$. Thus, if $\mathtt{x}_\mathbf{i}$ and $\mathtt{x}_\mathbf{j}$ are the choices of Player **A** in the $\mathbf{r} - th$ period, for some $\mathbf{r}$. In this case, Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$.

**Corollary 6.** *Let $(\mathsf{X},\psi,\omega)$ be a $\pi$-pre-topological space. Then, Player* **B** *has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ if and only if $(\mathsf{X},\psi,\omega)$ is not a $\pi$-pre-$T_1$ space.*

*Proof* . "$\Rightarrow$": Let Player **B** have a losing strategy. If $(\mathsf{X},\psi,\omega)$ is a $\pi$-pre-$T_1$ space, then, for any two distinct points $\mathtt{x_i},\mathtt{x_j}\in\mathsf{X}$, $M\in O(\psi)$ and $N\in O(\omega)$ exists, such that $\mathtt{x_i}\in M-N$ and $\mathtt{x_j}\in N-M$. Thus, Player **B** will win the game. This gives a contradiction with the hypothesis. Therefore, $(\mathsf{X},\psi,\omega)$ is not a $\pi$-pre-$T_1$ space.

"$\Leftarrow$": Suppose that $(\mathsf{X},\psi,\omega)$ is not a $\pi$-pre-$T_1$ space. If Player **B** has a winning strategy, then, for any two distinct points $\mathtt{x_i},\mathtt{x_j}\in\mathsf{X}$ will be chosen by Player **A**, thus, Player **B** can find two pre-open sets $T\in O(\psi)$ and $U\in O(\omega)$ such that $\mathtt{x_i}\in T-U$ and $\mathtt{x_j}\in U-T$. This gives a contradiction with the hypothesis. Therefore, Player **B** has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$.

**Definition 3.** *Let $(\mathsf{X},\psi,\omega)$ be a $\pi$-pre-$T_2$ space. We define a $\pi$-pre-$T_2$ game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ the following: Player* **A** *and Player* **B** *are playing period with each natural number in this game in the* $\mathbf{r}-th$ *period:*
*In the first pace, Player* **A** *opts for $\kappa_{\mathbf{r}}\neq\rho_{\mathbf{r}}$ where $\kappa_{\mathbf{r}},\rho_{\mathbf{r}}\in\mathsf{X}$. In the second pace, Player* **B** *opts for two pre-open subsets $S_{\mathbf{r}}\in O(\psi)$ and $T_{\mathbf{r}}\in O(\omega)$ such that $\kappa_{\mathbf{r}}\in S_{\mathbf{r}}$, $\rho_{\mathbf{r}}\in T_{\mathbf{r}}$ and $S_{\mathbf{r}}\cap T_{\mathbf{r}}=\phi$.*

*Then, Player* **B** *has a winning strategy in the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ if $F=\{(S_1,T_1),(S_2,T_2),...,(S_{\mathbf{r}},T_{\mathbf{r}}),...\}$ is a collection of pre-open sets such that $\forall\kappa_{\mathbf{r}}\neq\rho_{\mathbf{r}}\in\mathsf{X}$ there exist two disjoint pre-open sets $(S_{\mathbf{r}},T_{\mathbf{r}})\in F, S_{\mathbf{r}}\in O(\psi)$ and $T_{\mathbf{r}}\in O(\omega)$ such that $\kappa_{\mathbf{r}}\in S_{\mathbf{r}}$, $\rho_{\mathbf{r}}\in T_{\mathbf{r}}$. otherwise, Player* **A** *wins in the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$. We represent this algorithm for this game in Figure 3.*

*Example 4.* Consider the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$, where $\mathsf{X}=\{\kappa,\rho,\nu,\lambda\}$ the following, from Table 3, $O(\psi)=\{\{\kappa,\rho\},\{\lambda\},\{\kappa\},\mathsf{X},\phi\}$ and $O(\omega)=\{\{\nu,\lambda\},\{\rho,\nu\},\mathsf{X},\phi\}$. Player **A** and Player **B** are playing for six periods in this game. Then, the first period is the following: Player **A** opts for $\kappa\neq\rho$, where $\kappa,\rho\in X$.
Player **B** opts for $S_1=\{\kappa\}\in O(\psi)$ and $R_1=\{\rho,\nu\}\in O(W)$ such that $\kappa\in S_1$, $\rho\in R_1$ and $S_1\cap R_1=\phi$.

Then, the next period (the second period) is the following: Player **A** opts for $\kappa\neq\nu$, where $\kappa,\nu\in X$.
Player **B** opts for $S_2=\{\kappa\}\in O(\psi)$ and $R_2=\{\nu,\lambda\}\in O(W)$ such that $\nu\in S_2$, $\kappa\in R_2$ and $S_2\cap R_2=\phi$.

Then, the next period (the third period) is the following: Player **A** opts for $\kappa\neq\lambda$, where $\kappa,\lambda\in X$.
Player **B** opts for $S_3=\{\kappa\}\in O(\psi)$ and $R_3=\{\nu,\lambda\}\in O(W)$ such that $\lambda\in S_3$ and $\kappa\in R_3$ and $S_3\cap T_3=\phi$.

Then, the next period (the fourth period) is the following: Player **A** opts for $\rho\neq\nu$, where $\rho,\nu\in X$.
Player **B** opts for $S_4=\{\kappa,\rho\}\in O(\psi)$ and $R_4=\{\nu,\lambda\}\in O(W)$ such that $\rho\in S_4$, $\nu\in R_4$ and $S_4\cap R_4=\phi$.

Then, the next period (the fifth period) is the following: Player **A** opts for $\rho\neq\lambda$, where $\rho,\lambda\in X$.
Player **B** opts for $S_5=\{\kappa,\rho\}\in O(\psi)$ and $R_5=\{\nu,\lambda\}\in O(W)$ such that $\rho\in S_5$, $\lambda\in R_5$ and $S_5\cap R_5$.

Then, the next period (the sixth period) is the following: Player **A** opts for $\nu\neq\lambda$, where $\nu,\lambda\in X$.
Player **B** opts for $S_6=\{\lambda\}\in O(\psi)$ and $R_6=\{\nu,\lambda\}\in O(W)$ such that $\lambda\in S_6$, $\nu\in R_6$ and $S_6\cap R_6$.

Then, $F=\{(S_1,R_1),...,(S_6,R_6)\}$ is the winning strategy for Player **B** in this game. Hence, Player **B** wins the game.

*Remark.* Note that a $\pi$-pre-topological space $(\mathsf{X},\psi,\omega)$ in Example 2.3 is not a $\pi$-pre-$T_2$ space.

**Proposition 5.** *Let $(\mathsf{X},\psi,\omega)$ be a $\pi$-pre-topological space. Then, Player* **B** *has a winning strategy in the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ if and only if $(\mathsf{X},\psi,\omega)$ is $\pi$-pre-$T_2$.*

**Table 3:** $\psi(\mathsf{A})$ and $\omega(\mathsf{A})$

| $A$ | $\psi(A)$ | $\omega(A)$ |
|---|---|---|
| $\{\kappa\}$ | $\{\kappa,\rho,\nu\}$ | $\{\kappa,\rho\}$ |
| $\{\rho\}$ | $\{\rho,\nu\}$ | $\mathsf{X}$ |
| $\{\nu\}$ | $\{\kappa,\nu\}$ | $\{\rho,\nu,\lambda\}$ |
| $\{\lambda\}$ | $\mathsf{X}$ | $\{\kappa,\lambda\}$ |
| $\{\kappa,\rho\}$ | $\mathsf{X}$ | $\{\kappa,\rho\}$ |
| $\{\kappa,\nu\}$ | $\{\kappa,\nu,\lambda\}$ | $\{\kappa,\rho,\nu\}$ |
| $\{\kappa,\lambda\}$ | $\mathsf{X}$ | $\{\kappa,\lambda\}$ |

| $A$ | $\psi(A)$ | $\omega(A)$ |
|---|---|---|
| $\{\rho,\nu\}$ | $\mathsf{X}$ | $\mathsf{X}$ |
| $\{\rho,\lambda\}$ | $\{\rho,\nu,\lambda\}$ | $\mathsf{X}$ |
| $\{\nu,\lambda\}$ | $\{\nu,\lambda\}$ | $\{\rho,\nu,\lambda\}$ |
| $\{\kappa,\rho,\nu\}$ | $\{\kappa,\rho,\nu\}$ | $\mathsf{X}$ |
| $\{\kappa,\rho,\lambda\}$ | $\mathsf{X}$ | $\mathsf{X}$ |
| $\{\rho,\nu,\lambda\}$ | $\{\kappa,\nu,\lambda\}$ | $\mathsf{X}$ |
| $\{\kappa,\nu,\lambda\}$ | $\mathsf{X}$ | $\mathsf{X}$ |

*Proof* . "⇒": Let Player **B** have a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$. Then, for any two distinct points $\mathtt{x}_{\mathbf{i}},\mathtt{x}_{\mathbf{j}} \in \mathsf{X}$ will be chosen by Player **A**, Player **B** can find two distinct pre-open sets $S \in O(\psi)$ and $R \in O(\omega)$ such that $\mathtt{x}_{\mathbf{i}} \in S$ and $\mathtt{x}_{\mathbf{j}} \in R$. Hence, $(\mathsf{X},\psi,\omega)$ is a $\pi$-pre-$T_2$ space.

"⇐": Follows from the definition of a $\pi$-pre-$T_2$ space.

**Corollary 7.** *Let $(\mathsf{X},\psi,\omega)$ be a $\pi$-pre-topological space. Then, Player* **A** *has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ if and only if $(\mathsf{X},\psi,\omega)$ is a $\pi$-pre-$T_2$ space.*

*Proof* . "⇒": Let Player **A** have a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$. If $(\mathsf{X},\psi,\omega)$ is not $\pi$-pre-$T_2$, then, two points exist $\mathtt{x}_{\mathbf{i}},\mathtt{x}_{\mathbf{j}} \in \mathsf{X}$ such that there are no distinct two pre-open sets $T \in O(\psi)$ , $U \in O(\omega)$ such that $\mathtt{x}_{\mathbf{i}} \in T$ and $\mathtt{x}_{\mathbf{j}} \in U$. So, Player **A** has a gaining strategy. This gives a contradiction with the hypothesis. Thus, $(\mathsf{X},\psi,\omega)$ is a $\pi$-pre-$T_2$ space.

"⇐": Suppose that $(\mathsf{X},\psi,\omega)$ is a $\pi$-pre-$T_2$ space. If Player **A** has a winning strategy, then, in $\mathbf{j}-th$ period Player **A** opts for two distinct points $\kappa_{\mathbf{j}},\rho_{\mathbf{j}} \in \mathsf{X}$ and Player **B** cannot find two distinct pre-open sets $E \in O(\psi)$ , $\varphi \in O(\omega)$ such that $\kappa_{\mathbf{j}} \in E$ and $\rho_{\mathbf{j}} \in \varphi$. This gives a contradiction with the hypothesis. Hence, Player **A** has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$.

**Proposition 6.** *Let $(\mathsf{X},\psi,\omega)$ be a $\pi$-pre-topological space. Then, Player* **A** *has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ if and only if $(\mathsf{X},\psi,\omega)$ is not a $\pi$-pre-$T_2$ space.*

*Proof* . "⇒": Let Player **A** have a winning strategy, then, there is $\mathbf{i}-th$ period in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$, where Player **A** opts for two distinct points $\kappa_{\mathbf{i}},\rho_{\mathbf{i}} \in \mathsf{X}$ such that Player **B** cannot find two distinct pre-open sets $R_{\mathbf{i}} \in O(\psi)$ and $S_{\mathbf{i}} \in O(W)$, that satisfy $\kappa_{\mathbf{i}} \in R_{\mathbf{i}}$ and $\rho_{\mathbf{i}} \in S_{\mathbf{i}}$. Thus, $(\mathsf{X},\psi,\omega)$ is not a $\pi$-pre-$T_2$ space.

"$\Leftarrow$": Let $(\mathsf{X},\psi,\omega)$ be not a $\pi$-pre-$T_2$ space, then, there are two distinct points $\mathtt{x_i},\mathtt{x_j}\in\mathsf{X}$ such that there are no two distinct pre-open sets $L\in O(\psi)$ and $M\in O(\omega)$ such that $\mathtt{x_i}\in L$ and $\mathtt{x_j}\in M$. Thus, if $\mathtt{x_i}$ and $\mathtt{x_j}$ are the choices of Player **A** in period **i**, for some **i**, then, Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$.

**Corollary 8.** *Let* $(\mathsf{X},\psi,\omega)$ *be a* $\pi$*-pre-topological space. Then, Player* **B** *has a losing strategy in* $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ *if and only if* $(\mathsf{X},\psi,\omega)$ *is not a* $\pi$*-pre-*$T_2$ *space.*

*Proof* . "$\Rightarrow$": Let Player **B** have a losing strategy. If $(\mathsf{X},\psi,\omega)$ is a $\pi$-pre-$T_2$ space, then, for any two distinct points $\mathtt{x_i},\mathtt{x_j}\in\mathsf{X}$, two distinct pre-open sets $M\in O(\psi)$ and $N\in O(\omega)$ exist, such that $\mathtt{x_i}\in M$ and $\mathtt{x_j}\in N$. Thus, Player **B** will win the game, which contradicts the hypothesis. Therefore, $(\mathsf{X},\psi,\omega)$ is not a $\pi$-pre-$T_2$ space.

"$\Leftarrow$": Suppose that $(\mathsf{X},\psi,\omega)$ is not a $\pi$-pre-$T_2$ space. If Player **B** has a winning strategy, then, for any two distinct points $\mathtt{x_i},\mathtt{x_j}\in\mathsf{X}$ will be chosen by Player **A**, Player **B** can find two distinct pre-open sets $T\in O(\psi)$ and $U\in O(\omega)$ such that $\mathtt{x_i}\in T$ and $\mathtt{x_j}\in U$. This gives a contradiction with the hypothesis. Therefore, Player **B** has a losing strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$.

*Remark.* If Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_i)$, $i=1,2$, then, Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_{i-1})$.

*Remark.* If Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_i)$, $i=0,1$, then, Player **A** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_{i+1})$.

*Example 5.* Let $\Gamma_{\mathbf{i}}: P(\mathbb{R})\to P(\mathbb{R})$; $\mathbb{R}$ is the real numbers with usual topology, **i** = 1, 2 be a pre-topologies on $\mathbb{R}$ such that:

i-For every interval $I_1$ which is a subset of $P(\mathbb{R})$ and with right-bounded, the effect of $(\Gamma_1)$ on $I_1$ makes $I_1$ closed from the right. Every subset of $\mathbb{R}$ and any interval with right-unbounded are $\Gamma_1$-closed. Also, from the definition of $\Gamma_1$ we get every singleton is $\Gamma_1$-closed.
ii-For every interval $I_2$ which is a subset of $P(\mathbb{R})$ and with left-bounded, the effect of $(\Gamma_2)$ on $I_2$ makes $I_2$ closed from the left. Every subset of $\mathbb{R}$ and any interval with left-unbounded are $\Gamma_2$-closed. Also, from the definition of $\Gamma_2$ we get every singleton is $\Gamma_2$-closed.

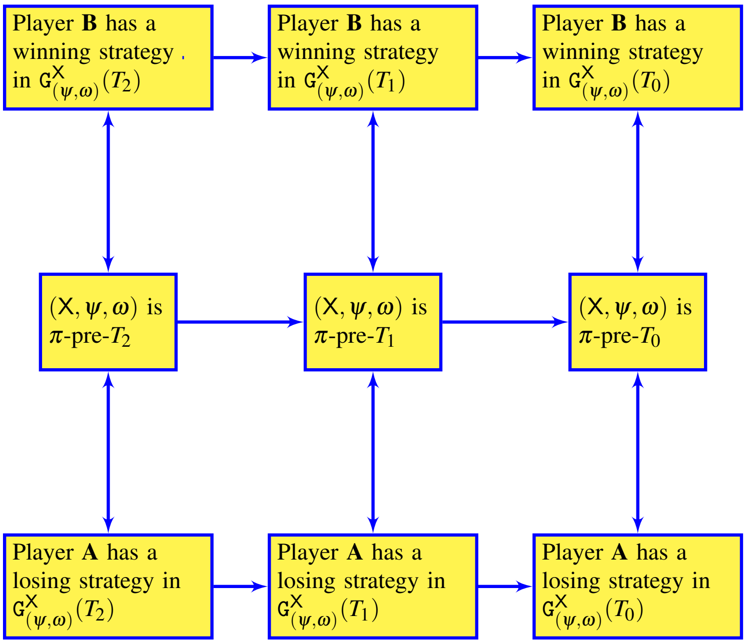
In the $\mathbf{r}-th$ period; $\mathbf{r}=1,2,...$: In the first pace, Player **A**: opts for $\mathtt{x_r}\neq\mathtt{y_r}$ where $\mathtt{x_r},\mathtt{y_r}\in\mathbb{R},\mathtt{x_r}<\mathtt{y_r}$. In the second pace, Player **B** opts for a $\Gamma_2$-pre-open set $\mathsf{H_r}=]-\infty,\mathtt{x_r}]\bigcup]\mathtt{y_r},\infty[\,\in\mathsf{O}(\Gamma_1)$ such that $\mathtt{x_r}\in\mathsf{H_r},\mathtt{y_r}\neq\mathsf{H_r}$. Then, $\mathbb{H}=\{\mathsf{H_1}=]-\infty,\mathtt{x_1}]\bigcup]\mathtt{y_1},\infty[,\mathsf{H_2}=]-\infty,\mathtt{x_2}]\bigcup]\mathtt{y_2},\infty[,...,\mathsf{H_r}=]-\infty,\mathtt{x_r}]\bigcup]\mathtt{y_r},\infty[,...\}$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathbb{R}}_{(\Gamma_1,\Gamma_2)}(T_0)$. Hence, Player **B** gains.

In the $\mathbf{r}-th$ period; $\mathbf{r}=1,2,...$: In the first pace, Player **A** opts for $\mathtt{x_r}\neq\mathtt{y_r}$ where $\mathtt{x_r},\mathtt{y_r}\in\mathbb{R},\mathtt{x_r}<\mathtt{y_r}$. In the second pace, Player **B** opts for $\mathsf{S_r}=]\mathtt{x_r},\infty[\,\in\mathsf{O}(\Gamma_1)$ and $\mathsf{R_r}=]-\infty,\mathtt{y_r}[\,\in\mathsf{O}(\Gamma_2)$ such that $\mathtt{y_r}\in\mathsf{S_r}-\mathsf{R_r}$ and $\mathtt{x_r}\in\mathsf{R_r}-\mathsf{S_r}$. Then, $\mathbb{C}=\{(\mathsf{S_1},\mathsf{R_1})=(]\mathtt{x_1},\infty[,]-\infty,\mathtt{y_1}[),(\mathsf{S_2},\mathsf{R_2})=(]\mathtt{x_2},\infty[,]-\infty,\mathtt{y_2}[),...,(\mathsf{S_r},\mathsf{R_r})=(]\mathtt{x_r},\infty[,]-\infty,\mathtt{y_r}[),...\}$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathbb{R}}_{(\Gamma_1,\Gamma_2)}(T_1)$. Hence, Player **B** wins.

In the $\mathbf{r}-th$ period; $\mathbf{r}=1,2,...$: In the first pace, Player **A** opts for $\mathtt{x_r}\neq\mathtt{y_r}$ where $\mathtt{x_r},\mathtt{y_r}\in\mathbb{R},\mathtt{x_r}<\mathtt{y_r}$. In the second pace, Player **B** opts for $\mathsf{T_r}=]\mathtt{x_r},\infty[\,\in\mathsf{O}(\Gamma_1)$ and $\mathsf{S_r}=]-\infty,\mathtt{x_r}]\in\mathsf{O}(\Gamma_2)$ such that $\mathtt{x_r}\in\mathsf{S_r},\mathtt{y_r}\in\mathsf{T_r}$ and $\mathsf{S_r}\cap$

$\mathsf{T_r} = \emptyset$. Then, $F = \{(\mathsf{T_1}, \mathsf{S_1}) = (]\mathtt{x_1}, \infty[, ]-\infty, \mathtt{x_1}]), (\mathsf{T_2}, \mathsf{S_2}) = (]\mathtt{x_2}, \infty[, ]-\infty, \mathtt{x_2}]), (\mathsf{T_r}, \mathsf{S_r}) = (]\mathtt{x_r}, \infty[, ]-\infty, \mathtt{x_r}]), \ldots\}$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathbb{R}}_{(\Gamma_1,\Gamma_2)}(T_2)$. Hence, Player **B** wins.

The following schema clarifies what we proved.




### *2.1 The relationship between game theory and pretopology in real-life situations:*

We would like to show some real-life examples and explain the link between game theory and pretopology using these examples. We start with the first example in the medical field and the other example in the economic field.

*Example 6.* The following Table 4, consists of information data for a group of 10 patients who are represented by $\mathscr{C} = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}\}$ and the decision to infect a patient with Covid-19 was based on the most common symptoms and exchanges among many patients. Symptoms can be fever, a dry cough, body aches, loss of appetite, shortness of breath, fatigue, mucus, stuffy nose and nausea respectively [30] represented by $\mathscr{A} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$. Also, we define the class for each patient as $[C_i] = \{C_j : C_j$ share 7 or more symptoms with $C_i\}$ from Table 4 we get
$[C_1] = \{C_1, C_4, C_{10}\}$,
$[C_2] = \{C_2, C_3, C_8\}$,
$[C_3] = \{C_2, C_3, C_8\}$,
$[C_4] = \{C_1, C_4, C_6, C_{10}\}$,
$[C_5] = \{C_5\}$,

**Table 4:** Medical information data

| Patients | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | covid-19 |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_1$ | High | Yes | No | No | Yes | Yes | No | Yes | No | Yes |
| $C_2$ | High | No | Yes | Yes | Yes | No | Yes | No | Yes | No |
| $C_3$ | High | Yes | Yes | Yes | Yes | Yes | Yes | No | Yes | Yes |
| $C_4$ | High | Yes | Yes | No | No | Yes | No | Yes | No | Yes |
| $C_5$ | Normal | Yes | Yes | Yes | No | No | Yes | No | Yes | No |
| $C_6$ | High | No | Yes | No | No | Yes | No | Yes | Yes | No |
| $C_7$ | Normal | Yes | No | Yes | No | No | No | No | No | No |
| $C_8$ | High | Yes | Yes | No | Yes | No | Yes | No | Yes | Yes |
| $C_9$ | High | No | Yes | Yes | No | Yes | Yes | Yes | Yes | No |
| $C_{10}$ | High | Yes | No | No | Yes | Yes | No | Yes | No | No |

$[C_6] = \{C_4, C_6, C_9\}$,
$[C_7] = \{C_7\}$,
$[C_8] = \{C_2, C_3, C_8\}$,
$[C_9] = \{C_6, C_9\}$,
$[C_{10}] = \{C_1, C_4, C_{10}\}$.
We can define the pre-topology $\tau$ on $C$ as $\tau(B) = \{C_j : B \cap [C_j] \neq \emptyset, \forall B \in P(C)\}$. Where $B = \{C_2, C_5, C_6, C_7, C_9, C_{10}\}$ is the set of patients without covid-19. We defined the winning and losing strategies as the patient has a winning strategy if he belongs to pre-interior of the subset $B$. Otherwise, the patient has a losing strategy.
Since pre-interior($B$)= $(\tau(B)^c)^c$, where $(B)^c$ is the complement of $B$. Then pre-interior($B$)= $\{C_5, C_7, C_9\}$, this result means that $C_5, C_7$ and $C_9$ only have a winning strategy.

*Example 7.* In this example, we have information data for a group of 5 companies, is represented by $M = \{\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_3, \mathscr{M}_4, \mathscr{M}_5\}$. These companies build houses for sale, and each company has specifications for these houses. specifications can be price, location, green surroundings, area, parking and finishing respectively represented by $S = \{\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \mathscr{S}_4, \mathscr{S}_5, \mathscr{S}_6\}$. Also, we define the class for each company as

**Table 5:** Economic information data

| Company | $\mathscr{S}_1$ | $\mathscr{S}_2$ | $\mathscr{S}_3$ | $\mathscr{S}_4$ | $\mathscr{S}_5$ | $\mathscr{S}_6$ | sales and marketing |
|---|---|---|---|---|---|---|---|
| $\mathscr{M}_1$ | High | Special | Available | Normal | Available | Ultra Lux | High |
| $\mathscr{M}_2$ | Normal | Normal | Available | Normal | Unavailable | Super Lux | High |
| $\mathscr{M}_3$ | Normal | Normal | Unavailable | Normal | Unavailable | Super Lux | Low |
| $\mathscr{M}_4$ | High | Normal | Unavailable | Large | Unavailable | Lux | Low |
| $\mathscr{M}_5$ | High | Special | Available | Large | Available | Ultra Lux | High |

$[\mathscr{M}_i] = \{\mathscr{M}_j : \mathscr{M}_j$ share 4 or more specifications with $\mathscr{M}_i\}$ from Table 5 we get
$[\mathscr{M}_1] = \{\mathscr{M}_1, \mathscr{M}_5\}$, $[\mathscr{M}_2] = \{\mathscr{M}_2, \mathscr{M}_3\}$, $[\mathscr{M}_3] = \{\mathscr{M}_2, \mathscr{M}_3, \mathscr{M}_4\}$, $[\mathscr{M}_4] = \{\mathscr{M}_3, \mathscr{M}_4\}$
and $[\mathscr{M}_5] = \{\mathscr{M}_1, \mathscr{M}_5\}$. We can define the pre-topology $\Gamma$ on $N$ as $\Gamma(N) = \{\mathscr{M}_j : N \cap [\mathscr{M}_j] \neq \emptyset, \forall N \in P(M)\}$. Where $N = \{\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_5\}$ is the set of companies that has high marketing and sales. We defined the winning and losing strategies as, the company has a winning strategy if it belongs to pre-interior of the subset $N$. Otherwise, the company has a losing strategy.
Since pre-interior($N$)= $(\Gamma(N)^c)^c$, where $(N)^c$ is the complement of $N$. Then pre-interior($N$)= $\{\mathscr{M}_1, \mathscr{M}_5\}$, this result means that $\mathscr{M}_1$ and $\mathscr{M}_5$ only have a winning strategy.

## 3 Pre-open Functions with Games

In this section, we apply some properties of functions like pre-open, surjective, injective and pre-continuous to study the effect of these properties with games $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ and $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ on both players' strategies with respect to games $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$, $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_1)$ and $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_2)$.

**Proposition 7.** *Let $\varphi : (\mathsf{X}, \Gamma, \lambda) \to (\mathsf{Y}, \alpha, \beta)$ be a pre-open function. Then,*

*1.If $\varphi$ is surjective and Player* **B** *has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_0)$, then, Player* **B** *has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$.*
*2.If $\varphi$ is bijective and Player* **B** *has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_\mathbf{i})$, then, Player* **B** *has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_\mathbf{i}), \mathbf{i} = 1,2$.*

*Proof* . 1.In the $\mathbf{r}-th$ period, $\mathbf{r} = 1,2,...$, let Player **A** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$ opt two distinct points $\kappa_\mathbf{r}, \tau_\mathbf{r} \in \mathsf{Y}$. Since $\varphi$ is surjective, then, there exists $\varphi^{-1}(\kappa_\mathbf{r}), \varphi^{-1}(\tau_\mathbf{r}) \in \mathsf{X}$ such that $\varphi^{-1}(\kappa_\mathbf{r}) \neq \varphi^{-1}(\tau_\mathbf{r})$. Since Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_0)$, then, there exists a pre-open set $M_\mathbf{r} \in O(\Gamma) \cup O(\lambda)$ which contains one of the two elements $\varphi^{-1}(\kappa_\mathbf{r}), \varphi^{-1}(\tau_\mathbf{r}), \mathbf{r} = 1,2,....$ Since $\varphi$ is pre-open, then, $\varphi(M_\mathbf{r}) \in O(\alpha) \cup O(\beta)$. Thus, Player **B** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$ opts for $\varphi(M_\mathbf{r})$ which contains one of the two elements $\kappa_\mathbf{r}, \tau_\mathbf{r}, \mathbf{r} = 1,2,....$ Therefore, $B = \{\varphi(M_1), ..., \varphi(M_\mathbf{r}), ...\}, \mathbf{r} = 1,2,...$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$, hence, Player **B** wins $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$.
2.For i=1, In the $\mathbf{r}-th$ period, $\mathbf{r} = 1,2,...$, let Player **A** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_1)$ opt two distinct points $\kappa_\mathbf{r}, \tau_\mathbf{r} \in \mathsf{Y}$. Since $\varphi$ is surjective, then, there exists $\varphi^{-1}(\kappa_\mathbf{r}), \varphi^{-1}(\tau_\mathbf{r}) \in \mathsf{X}$ such that $\varphi^{-1}(\kappa_\mathbf{r}) \neq \varphi^{-1}(\tau_\mathbf{r})$. Since Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_1)$, then, two pre-open sets $M_\mathbf{r}, N_\mathbf{r}$, where $M_\mathbf{r} \in O(\Gamma)$ and $N_\mathbf{r} \in O(\lambda)$ exist, such that $\varphi^{-1}(\kappa_\mathbf{r}) \in M_\mathbf{r} - N_\mathbf{r}$ and $\varphi^{-1}(\tau_\mathbf{r}) \in N_\mathbf{r} - M_\mathbf{r}$. Since $\varphi$ is a pre-open function, then, $\varphi(M_\mathbf{r})$ and $\varphi(N_\mathbf{r})$ are pre-open sets, where $\varphi(M_\mathbf{r}) \in O(\alpha)$ and $\varphi(N_\mathbf{r}) \in O(\beta)$. As $\varphi$ is injective, thus, Player **B** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_1)$ opts for $\varphi(M_\mathbf{r})$ and $\varphi(N_\mathbf{r})$ are two pre-open sets such that $\kappa_\mathbf{r} \in \varphi(M_\mathbf{r} - N_\mathbf{r}) = \varphi(M_\mathbf{r}) - \varphi(N_\mathbf{r})$ and $\tau_r \in \varphi(N_r - M_r) = \varphi(N_r) - \varphi(M_r), r = 1,2,....$ Therefore, $B = \{(\varphi(M_1), \varphi(N_1)), ..., (\varphi(M_r), \varphi(N_r)), ...\}, r = 1,2,...$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_1)$, hence, Player **B** wins $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_1)$.
For $\mathbf{i} = 2$, In the $\mathbf{r}-th$ period, $\mathbf{r} = 1,2,...$, let Player **A** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_2)$ opts for two distinct points $\kappa_\mathbf{r}, \tau_\mathbf{r} \in \mathsf{Y}$. Since $\varphi$ is surjective, then, $\varphi^{-1}(\kappa_\mathbf{r}), \varphi^{-1}(\tau_\mathbf{r}) \in \mathsf{X}$ exists, such that $\varphi^{-1}(\kappa_\mathbf{r}) \neq \varphi^{-1}(\tau_\mathbf{r})$. Since Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_2)$, then, two pre-open sets exist $M_\mathbf{r}, N_\mathbf{r}$, where $M_\mathbf{r} \in O(\Gamma)$ and $N_\mathbf{r} \in O(\lambda)$ such that $\varphi^{-1}(\kappa_\mathbf{r}) \in M_\mathbf{r}$, $\varphi^{-1}(\tau_\mathbf{r}) \in N_\mathbf{r}$ and $M_\mathbf{r} \cap N_\mathbf{r} = \phi$. Since $\varphi$ is a pre-open function, then, $\varphi(M_\mathbf{r})$ and $\varphi(N_\mathbf{r})$ are

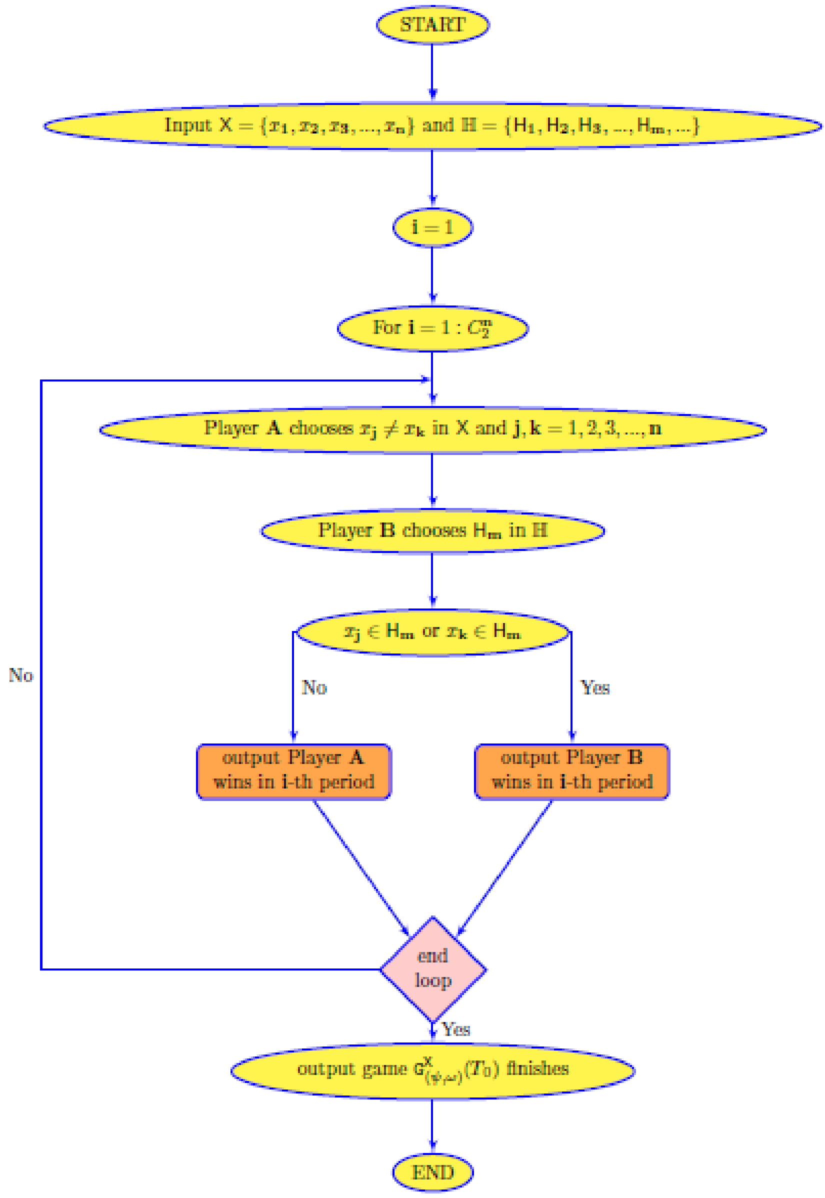



**Fig. 1:** Simple flowchart for the steps of the algorithm for the game $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$

pre-open sets, where $\varphi(M_{\mathbf{r}}) \in O(\alpha)$ and $\varphi(N_{\mathbf{r}}) \in O(\beta)$. As $\varphi$ is injective, thus, Player **B** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_2)$ opts for $\varphi(M_{\mathbf{r}})$ and $\varphi(N_{\mathbf{r}})$ are two pre-open sets such that $\kappa_{\mathbf{r}} \in \varphi(M_{\mathbf{r}})$, $\tau_{\mathbf{r}} \in \varphi(N_{\mathbf{r}})$ and $\varphi(M_{\mathbf{r}} \cap N_{\mathbf{r}}) = \varphi(M_{\mathbf{r}}) \cap \varphi(N_{\mathbf{r}}) = \phi$, $\mathbf{r} = 1, 2, \ldots$. Therefore, $B = \{(\varphi(M_1), \varphi(N_1)), \ldots, (\varphi(M_{\mathbf{r}}), \varphi(N_{\mathbf{r}})), \ldots\}, \mathbf{r} = 1, 2, \ldots$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_2)$, hence, Player **B** wins $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_2)$.

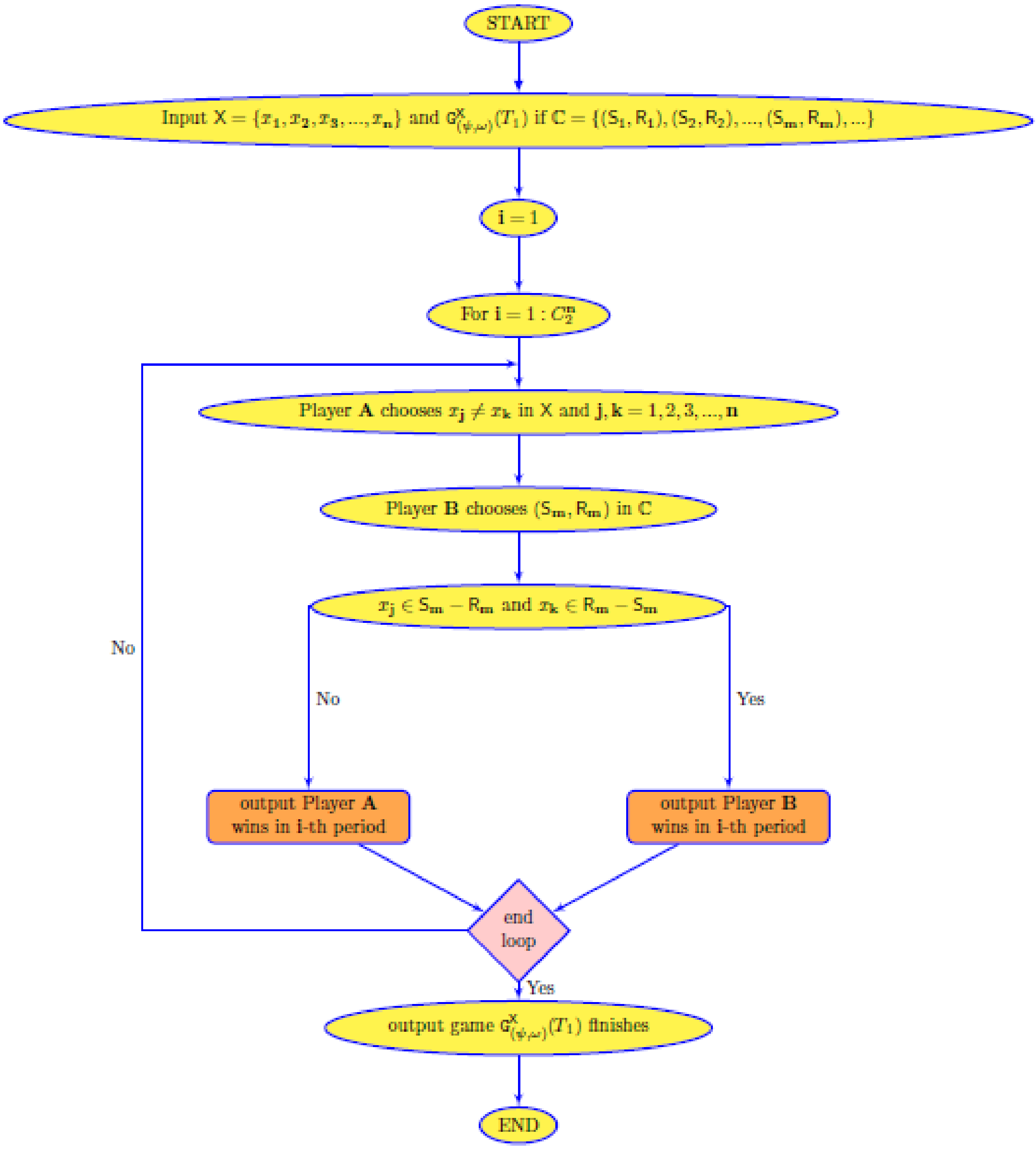
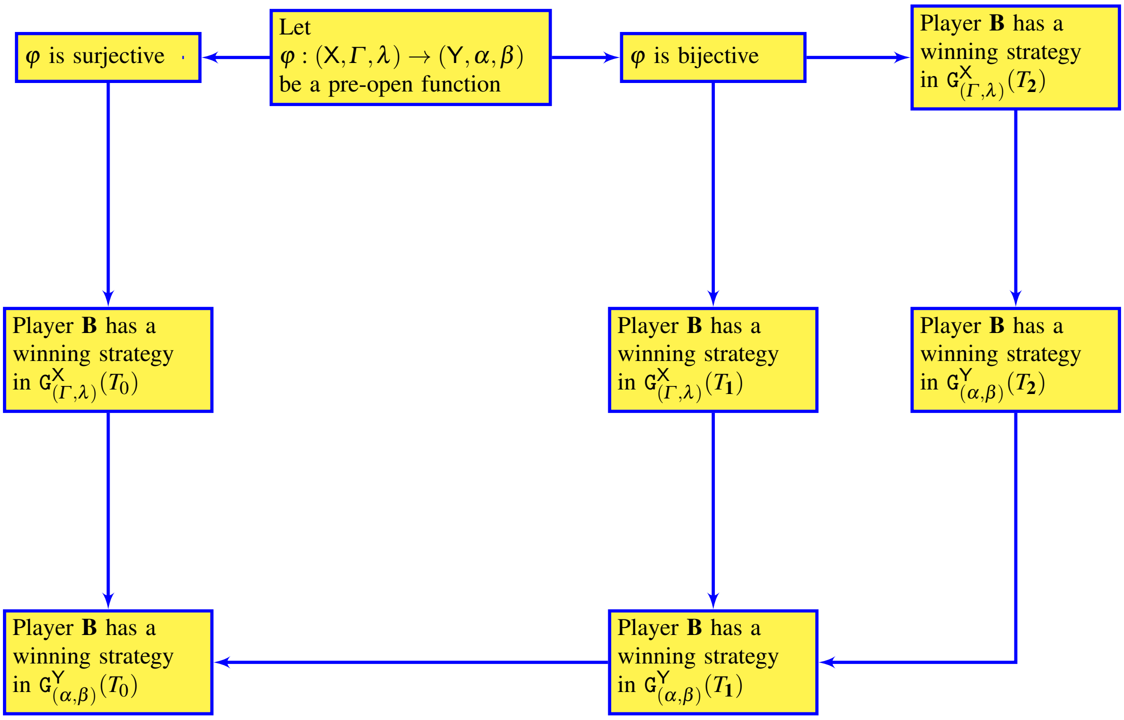



**Fig. 2:** Simple flowchart for the steps of the algorithm for the game $\mathsf{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$

The following schema clarifies what we proved:

**Proposition 8.** *Let $\varphi : (\mathsf{X}, \Gamma, \lambda) \to (\mathsf{Y}, \alpha, \beta)$ be a pre-continuous and injective function. If Player* **B** *has a winning strategy in $\mathsf{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_{\mathbf{i}})$, then, Player* **B** *has a winning strategy in $\mathsf{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_{\mathbf{i}})$,* $\mathbf{i} = 0, 1, 2.$



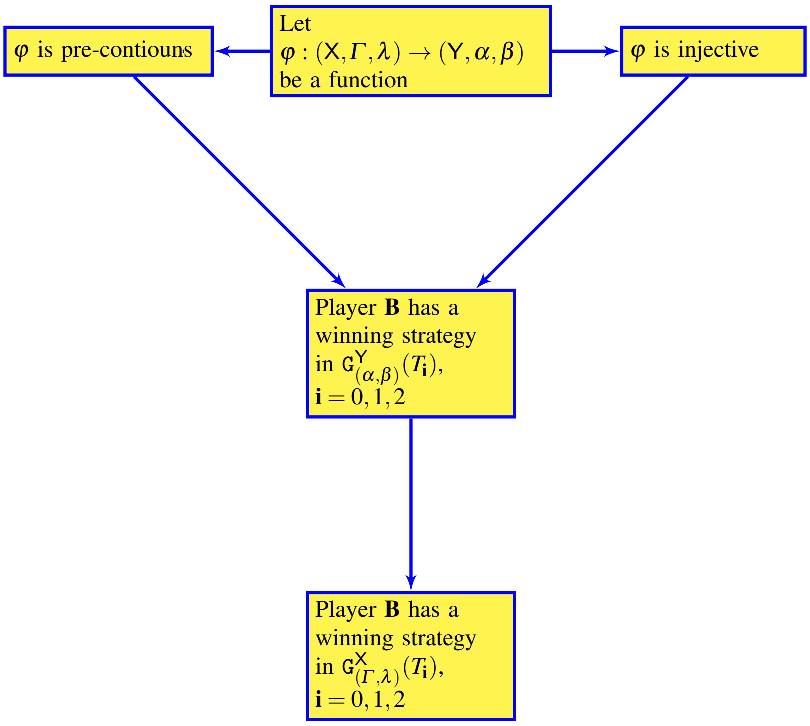

*Proof* . For $\mathbf{i}=0$, In the $\mathbf{r}-th$ period, $\mathbf{r}=1,2,...$, let Player **A** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_0)$ opts for two distinct points $\kappa_{\mathbf{r}},\tau_{\mathbf{r}}\in\mathsf{X}$. Since $\varphi$ is injective, then, $\varphi(\kappa_{\mathbf{r}}),\varphi(\tau_{\mathbf{r}})\in Y$ exists, such that $\varphi(\kappa_{\mathbf{r}})\neq\varphi(\tau_{\mathbf{r}})$. Since Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_0)$, then, there exists a pre-open set $M_{\mathbf{r}}\in O(\alpha)\cup O(\beta)$ which contains one of the two elements $\varphi(\kappa_{\mathbf{r}}),\varphi(\tau_{\mathbf{r}}),\mathbf{r}=1,2,....$ Since $\varphi$ is pre-continuous, then, $\varphi^{-1}(M_{\mathbf{r}})\in O(\Gamma)\cup O(\lambda)$. Thus, Player **B** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_0)$ opts for $\varphi^{-1}(M_{\mathbf{r}})$ which contains one of the two elements $\kappa_{\mathbf{r}},\tau_{\mathbf{r}},\mathbf{r}=1,2,\cdots$. Therefore, $B=\{\varphi^{-1}(M_1),\cdots,\varphi^{-1}(M_{\mathbf{r}}),...\},\mathbf{r}=1,2,...$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_0)$, hence Player **B** wins $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_0)$.

For $\mathbf{i}=1$, In the $\mathbf{r}-th$ period, $\mathbf{r}=1,2,...$, let Player **A** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_1)$ opts for two distinct points $\kappa_{\mathbf{r}},\tau_{\mathbf{r}}\in\mathsf{X}$. Since $\varphi$ is injective, then, $\varphi(\kappa_{\mathbf{r}}),\varphi(\tau_{\mathbf{r}})\in\mathsf{Y}$ exists, such that $\varphi(\kappa_{\mathbf{r}})\neq\varphi(\tau_{\mathbf{r}})$. Since Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_1)$, then, there exist two pre-open sets $M_{\mathbf{r}}\in O(\alpha)$ and $N_{\mathbf{r}}O(\beta)$ such that $\varphi(\kappa_{\mathbf{r}})\in M_{\mathbf{r}}-N_{\mathbf{r}}$ and $\varphi(\tau_{\mathbf{r}})\in N_{\mathbf{r}}-M_{\mathbf{r}},\mathbf{r}=1,2,....$ Since $\varphi$ is pre-continuous, then, $\varphi^{-1}(M_{\mathbf{r}})\in O(\Gamma)$ and $\varphi^{-1}(N_{\mathbf{r}})\in O(\lambda)$. Thus, Player **B** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_1)$ opts for $\varphi^{-1}(M_{\mathbf{r}})$ and $\varphi^{-1}(N_{\mathbf{r}})$ such that $\kappa_{\mathbf{r}}\in\varphi^{-1}(M_{\mathbf{r}}-N_{\mathbf{r}})=\varphi^{-1}(M_{\mathbf{r}})-\varphi^{-1}(N_{\mathbf{r}})$ and $\tau_{\mathbf{r}}\in\varphi^{-1}(N_{\mathbf{r}}-M_{\mathbf{r}})=\varphi^{-1}(N_{\mathbf{r}})-\varphi^{-1}(M_{\mathbf{r}}),\mathbf{r}=1,2,....$ Therefore, $B=\{(\varphi^{-1}(M_1),\varphi^{-1}(N_1)),...,(\varphi^{-1}(M_{\mathbf{r}}),\varphi^{-1}(N_{\mathbf{r}})),...\},\mathbf{r}=1,2,...$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_1)$, hence, Player **B** wins $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_1)$.

For $\mathbf{i}=2$, In the $\mathbf{r}-th$ period, $\mathbf{r}=1,2,...$, let Player **A** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_2)$ opts for two distinct points $\kappa_{\mathbf{r}},\tau_{\mathbf{r}}\in\mathsf{X}$. Since $\varphi$ is injective, then, $\varphi(\kappa_{\mathbf{r}}),\varphi(\tau_{\mathbf{r}})\in\mathsf{Y}$ exists, such that $\varphi(\kappa_{\mathbf{r}})\neq\varphi(\tau_{\mathbf{r}})$. Since Player **B** has a winning strategy in $\mathtt{G}^{\mathsf{Y}}_{(\alpha,\beta)}(T_2)$, then, there exist two pre-open sets $M_{\mathbf{r}}\in O(\alpha)$ and $N_{\mathbf{r}}\in O(\beta)$ such that $\varphi(\kappa_{\mathbf{r}})\in M_{\mathbf{r}}$, $\varphi(\tau_{\mathbf{r}})\in N_{\mathbf{r}}$ and $M_{\mathbf{r}}\cap N_{\mathbf{r}}=\phi,\mathbf{r}=1,2,....$ Since $\varphi$ is pre-Continuous, then, $\varphi^{-1}(M_{\mathbf{r}})\in O(\Gamma)$ and $\varphi^{-1}(N_{\mathbf{r}})\in O(\lambda)$. Thus, Player **B** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_2)$ opts for $\varphi^{-1}(M_{\mathbf{r}})$ and $\varphi^{-1}(N_{\mathbf{r}})$ such that $\kappa_{\mathbf{r}}\in\varphi^{-1}(M_{\mathbf{r}})$, $\tau_{\mathbf{r}}\in\varphi^{-1}(N_{\mathbf{r}})$ and $\varphi^{-1}(M_{\mathbf{r}})\cap\varphi^{-1}(N_{\mathbf{r}})=\phi,\mathbf{r}=$

$1,2,....$ Therefore, $B=\{(\varphi^{-1}(M_1),\varphi^{-1}(N_1)),...,(\varphi^{-1}(M_\mathbf{r}),\varphi^{-1}(N_\mathbf{r})),...\}, \mathbf{r}=1,2,...$ is the winning strategy for Player **B** in $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_2)$, hence, wins $\mathtt{G}^{\mathsf{X}}_{(\Gamma,\lambda)}(T_2)$.

The following schema clarifies what we proved:




## 4 Conclusion

We introduce the notions of infinitely long games $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_0)$, $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_1)$ and $\mathtt{G}^{\mathsf{X}}_{(\psi,\omega)}(T_2)$ and study the relationship between these games and both players' strategies. Also, we used the properties of functions to get the relation between the three kinds of games with respect to different $\pi$-pre-topological spaces and players' strategies. In future studies, we introduce another link between game theory and topological structures, such as fuzzy and rough topological spaces with different examples like computer science and biology.

## Availability of data and material

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## Conflict of interest

The author declares that the have no competing interests.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## Authors' contributions

The authors contribute equally to the work.


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