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# Fuzzy Topological Partial Groups

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**Abstract:** In this paper, we present a new concept, called fuzzy topological partial group, discussing some of its fundamental properties. We also examine some categorical properties of this notion.

**Keywords:** Fuzzy Topology; Fuzzy Group; Fuzzy Partial Group; Fuzzy Topological Group; Topological Partial Group.

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## 1. Introduction and Preliminaries

The notion of fuzzy sets, originally presented by Zadeh [1], was used to define fuzzy topological spaces [2-4]. Later, Rosenfeld [5] presented the fuzzification of groups, and Foster [6] followed by introducing fuzzy topological groups. Also, the fuzzification of partial groups was introduced by [7], using T-partial monoids on $I = [0,1]$.

Abd-Allah et al [8] presented the notion of topological partial groups, which we give the abbreviation 'TPG'. In the current paper, we examine the fuzzification of TPGs, using the definition of fuzzy partial groups with the T-operation as the T−norm minimum. Also, we introduce the fundamental theorems of isomorphisms of fuzzy TPGs. Finally, the product of fuzzy TPGs was introduced.

We provide basic definitions and necessary results that are utilized throughout the paper. We consider $I = [0,1]$ and $P_j$ are the projection mappings $\forall j \in J$ in this paper.

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**Definition 1.** **[1]** Let $X$ be a set, and $\alpha : X \longrightarrow I$ be a mapping. We call $\alpha$ a fuzzy set, that associates a membership degree $\alpha(x)$ to every element $x \in X$.

**Definition 2.** **[4]** Let $X$ be a set, and $\delta$ be a collection of fuzzy subsets of $X$. We call $(X, \delta)$ a fuzzy topological space if:

1. $I_c \in \delta \quad \forall c \in I$, where $I_c$ is a constant fuzzy set,
2. If $\mu, \nu \in \delta$, then $\mu \cap \nu \in \delta$,
3. If $\mu_j \in \delta \quad \forall j \in J$, then $\bigcup_{j \in J} \mu_j \in \delta$.

We refer to the elements of $\delta$ as fuzzy open sets, and the fuzzy topological space will be called FTS for short.

**Definition 3.** **[3]** Let $\{(X_j, \delta_j), j \in J\}$ be a collection of FTS's, $X = \prod_{j \in J} X_j$, and let $\delta$ be the smallest topology on $X$ that makes $P_j$ fuzzy continuous $\forall j \in J$. Then, we call $(X, \delta)$ the product FTS.

**Definition 4.** **[6]** Let $(X, \delta)$ be an FTS, and let $\alpha$ be a fuzzy subset of $X$. Then, $\delta_\alpha = \{\mu \cap \alpha : \mu \in \delta\}$ is called the induced fuzzy topology on $\alpha$, and we call $(\alpha, \delta_\alpha)$ a fuzzy subspace of $(X, \delta)$.

**Proposition 1.** **[6]** Let $\{(X_i, \delta_i), i = 1, 2, \ldots, n\}$, be a finite collection of FTS's, and let $(X, \delta)$ be their product FTS. Let $\alpha_i$ be a fuzzy subset of $X_i$, and $\delta_{\alpha_i}$ be the induced fuzzy topology on $\alpha_i, \forall i = 1, 2, \ldots, n$. Then, the product of the fuzzy subspaces $\{(\alpha_i, \delta_{\alpha_i}), i = 1,2, \ldots, n\}$, is a fuzzy subspace of $(X, \delta)$.

**Proposition 2.** **[3]** Let $\{(X_j, \delta_j), j \in J\}$ be a collection of FTS's, and let $(X, \delta)$ be their product FTS, and let the mapping $\phi : (Y, \omega) \longrightarrow (X, \delta)$. Then, The mapping $\phi$ is fuzzy continuous if and only if the mapping $P_j \phi$ is fuzzy continuous $\forall j \in J$.

**Definition 5.** **[6]** Let $(X, \delta)$ be an FTS, $\phi : X \longrightarrow Y$. The fuzzy image topology $\phi[\delta] = \{\mu \in Y : \phi^{-1}[\mu] \in \delta\}$ is the largest fuzzy topology on $Y$ which makes $\phi$ fuzzy continuous.

**Definition 6.** **[6]** Let $(Y, \omega)$ be an FTS, $\phi : X \longrightarrow Y$. The fuzzy inverse image topology $\phi^{-1}[\omega] = \{\phi^{-1}[\mu] : \mu \in \omega\}$ is the smallest fuzzy topology on $Y$ which makes $\phi$ fuzzy continuous.

**Definition 7.** **[5]** Let $X$ be a group, and $\gamma$ be a fuzzy subset of $X$. We call $\gamma$ a fuzzy group if:

1. $\gamma(xy) \geq \min\{\gamma(x), \gamma(y)\} \forall x, y \in X$;
2. $\gamma(x^{-1}) \geq \gamma(x) \; \forall x \in X$.

**Definition 8.** **[5]** Let $\alpha$ be a fuzzy subset of a set $X$. Then, $\alpha$ has the sub property if for every $A \subseteq X, \exists a_0 \in A$ such that $\alpha(a_0) = \sup_{\alpha \in A} \alpha(a)$.

**Definition 9.** **[5]** Let $\gamma$ be a fuzzy group of a group $X$. $\gamma$ is called $\phi$-invariant if $\forall x_1, x_2 \in X$ such that $\phi(x_1) = \phi(x_2)$, then $\gamma(x_1) = \gamma(x_2)$.

**Proposition 3.** **[5]** Let $\gamma$ be a fuzzy group in a group $X$, and let the mapping $\phi : X \longrightarrow Y$ be a group homomorphism. Then, $\phi[\gamma]$ is a fuzzy group in $Y$ provided that $\gamma$ is $\phi$-invariant.

**Definition 10.** **[6]** Let $\gamma$ be a fuzzy group of a group $X$. Let $(X, \delta)$ be an FTS, $(\gamma, \delta_\gamma)$ be the fuzzy subspace. Then, we call $\gamma$ a fuzzy topological group if the following mappings are fuzzy continuous:

1. Product mapping $\beta_\gamma : \gamma \times \gamma \longrightarrow \gamma; (x, y) \longmapsto xy$.
2. Inverse mapping $\sigma_\gamma : \gamma \longrightarrow \gamma; x \longmapsto x^{-1}$.

**Definition 11.** **[6]** Let $\gamma$ be a fuzzy group of a group $X$, and let $a \in X$. Then, the two mappings $R_a : \gamma \longrightarrow \gamma; x \longmapsto xa \text{ and } L_a : \gamma \longrightarrow \gamma; x \longmapsto ax$, are the right transformation and the left transformation, respectively.

**Definition 12.** **[8,9]** Let $X$ be a semigroup. Then, $X$ is a partial group if:

1. for all $x \in X$ there exists a partial identity $e_x$,
2. for all $x \in X$ there exists a partial inverse $x^{-1}$.
3. The mapping $e_X : X \longrightarrow X; x \longmapsto e_x$ is a semigroup homomorphism.
4. The mapping $\sigma_X : X \longrightarrow X;\ x \longmapsto x^{-1}$ is a semigroup antihomomorphism.

Let $E(X) = \{e_x, x \in X\}, X_a = \{x \in X: e_x = e_a\}$.

**Definition 13.** **[8,9]** Let $X$ be a partial group, and $(X, \delta)$ be a topological space. Then, we call $X$ a TPG if the following mappings are continuous:

1. Product mapping $\beta_X : X \times X \longrightarrow X; (x, y) \longmapsto xy$.
2. Partial inverse mapping $\sigma_X : X \longrightarrow X; x \longmapsto x^{-1}$.
3. Partial identity mapping $e_X : X \longrightarrow X; x \longmapsto e_x$.

## 2 Fuzzy Topological Partial Groups

In the current section, we present the concept of the fuzzy TPG and examine the fundamental properties of this concept.

**Definition 14.** Let $X$ be a partial group, and $\xi$ be a fuzzy subset of $X$. We call $\xi$ a fuzzy partial group in $X$ if $\forall x, y \in X$,

1. $\xi(xy) \geq \min\{\xi(x), \xi(y)\}$;
2. $\xi(x^{-1}) = \xi(x)$;
3. $\xi\ (e_x)$ is a constant on $I$.

We note that condition (iii) implies that $\xi(e_x) = k$, where $k = \sup_{\alpha \in A} \xi(x)$.

Throughout this paper, $X, Y$ are partial groups and $\xi, \zeta$ are fuzzy partial groups of $X, Y$, respectively.

**Proposition 4.** 1. $\beta_X[\xi \times \xi] \subseteq \xi$;

2. $\sigma_X[\xi] \subseteq \xi$;
3. $e_X[\xi] \subseteq \xi$.

*Proof.* Let $x \in X$. Then:

1. $$\beta_X[\xi \times \xi](x) = \sup_{(z_1,z_2) \in \beta_X^{-1}(x)} (\xi \times \xi)(z_1, z_2)$$
$$= \sup_{(z_1,z_2) \in \beta_X^{-1}(x)} \min\{\xi(z_1), \xi(z_2)\}$$
$$\leq \sup_{(z_1,z_2) \in \beta_X^{-1}(x)} \xi(z_1 z_2) = \xi(x)$$

   Hence, $\beta_X[\xi \times \xi] \subseteq \xi$,

2. Since $\xi(x) = \xi(x^{-1})$, then $\sigma_X\,[\xi] \subseteq \xi$.
3. $$e_X[\xi](x) = \begin{cases} \sup_{t \in e_X^{-1}(x)} \xi(t) & \text{if } e_X^{-1}(x) \neq \Phi \\ 0 & \text{if } e_X^{-1}(x) = \Phi \end{cases}$$
$$= \begin{cases} \sup_{t \in X_x} \xi(t) & \text{if } x \in E(X) \\ 0 & \text{if } x \notin E(X) \end{cases}$$

   If $x \in E(X)$, then we have $\sup_{t \in X_x} \xi(t) = \xi(x)$. Hence, $e_X[\xi] \subseteq \xi$.

**Definition 15.** Let $(X, \delta)$ be an FTS, and $(\xi, \delta_\xi)$ be the fuzzy subspace. Then, we call $\xi$ a fuzzy topological partial group if the following mappings are fuzzy continuous:

1. Product mapping $\beta_\xi : \xi \times \xi \longrightarrow \xi; (x, y) \longmapsto xy$.
2. Partial inverse mapping $\sigma_\xi : \xi \longrightarrow \xi;\ x \longmapsto x^{-1}$.
3. Partial identity mapping $e_\xi : \xi \longrightarrow \xi; x \longmapsto e_x$.

From now on, we will give the fuzzy topological partial group the abbreviation FTPG. From the above definition, we can say that a fuzzy topological group is an FTPG, but the converse isn't necessarily true. We show that through the next two examples.

***Example 1***. Let $(X, \delta)$ be an FTS, such that $X = \{e, p, q\}$ is a group that is isomorphic with $\mathbb{Z}_3$, and $\delta = \{I_c : c \in [0,1]\}$. Let $\gamma = \{e_{0.7}, p_{0.5}, q_{0.5}\}$. It can be verified easily that $\gamma$ is a fuzzy group of $X$. Then, $\gamma$ with the fuzzy topology $\delta_\gamma = \{I_k : k \in [0,0.5], \{e_c, p_{0.5}, q_{0.5}\} : c \in ]0.5,0.7]\}$ is clearly a fuzzy topological group.

***Example 2***. Let $X_e = \{e, p\}$ and $X_m = \{q, r\}$ be both isomorphic to $\mathbb{Z}_2$. Then, $X = \{e, p, q, r\}$ is a partial group with the shown multiplication table.

| $\cdot$ | $e$ | $p$ | $q$ | $r$ |
|---|---|---|---|---|
| $e$ | $e$ | $p$ | $e$ | $p$ |
| $p$ | $p$ | $e$ | $p$ | $e$ |
| $q$ | $e$ | $p$ | $q$ | $r$ |
| $r$ | $p$ | $e$ | $r$ | $q$ |

Let $\xi = \left\{e_1, p_{\frac{3}{4}}, q_1, r_{\frac{1}{4}}\right\}$. Then, we can easily show that $\xi$ is a fuzzy partial group of $X$. Let $(X, \delta)$ be an FTS, where $\delta = \{I_c : c \in [0,1]\}$. Then, $\delta_\xi = \left\{I_k : k \in \left[0, \frac{1}{4}\right], \left\{e_c, p_c, q_c, r_{\frac{1}{4}}\right\} : c \in \left]\frac{1}{4}, \frac{3}{4}\right], \left\{e_c, p_{\frac{3}{4}}, q_c, r_{\frac{1}{4}}\right\} : c \in \left]\frac{1}{4}, 1\right]\right\}$. Hence, $\xi$ with the fuzzy topology $\delta_\xi$ is an FTPG, but not a fuzzy topological group.

From now on, $\xi$ is an FTPG of $X$.

**Proposition 5.** The diagonal mapping $\Delta : \xi \longrightarrow \xi \times \xi; x \longmapsto (x, x)$ is fuzzy continuous.

*Proof.* Clearly, $\Delta[\xi] \subseteq \xi \times \xi$. Since $P_j\Delta = I$, where $I$ is the identity mapping, then $\Delta$ is fuzzy continuous from Proposition 2.

**Proposition 6.** Let $(X, \delta)$ be an FTS, and $(\xi, \delta_\xi)$ be the fuzzy subspace. Then, $\xi$ is an FTPG if and only if the mapping $\phi : \xi \times \xi \longrightarrow \xi; (x, y) \longmapsto xy^{-1}$ is fuzzy continuous.

*Proof.* The fuzzy continuity of $\phi$ is obvious, since $\phi = \beta_\xi\ (I \times \sigma_\xi)$. Conversely, since $e_\xi = \phi\Delta$, then $e_\xi$ is fuzzy continuous. Also, the fuzzy continuity of $\sigma_\xi$ is true since $\sigma_\xi = \phi(e_\xi \times I)\Delta$. Finally, since $\beta_\xi = \phi(I \times \sigma_\xi)$, then $\beta_\xi$ is fuzzy continuous.

We observe that if $a \in X$, then $R_a[\xi]$, $L_a[\xi]$ may not be fuzzy subsets of $\xi$, and consequently, $R_a$ and $L_a$ may not be fuzzy continuous on $\xi$. We show that in the next example.

***Example 3***. With reference to Example 2, $R_p[\xi] = \left\{e_{\frac{3}{4}}, p_1, q_0, r_0\right\} \not\subset \xi$.

To resolve the above deficiency, we define the following set:

$\xi_e = \{x \in X : \xi(x) = \xi(e) \text{ where } e \in E(X)\}$.

**Proposition 7.** $R_a[\xi] \subseteq \xi, L_a[\xi] \subseteq \xi, \forall a \in \xi_e$, where $R_a, L_a$ are the right transformation and the left transformation mappings on $X$, respectively.

*Proof.* Let $x \in X$ and $a \in \xi_e$. Then,

$$R_a[\xi](x) = \sup_{z \in R_a^{-1}(x)} \xi(z) = \begin{cases} \sup\limits_{R_a(z)=x} \xi(z) & \text{if } R_a^{-1}(x) \neq \Phi \\ 0 & \text{if } R_a^{-1}(x) = \Phi \end{cases}$$

Since $za = x$, then $\xi(x) \geq \min\{\xi(z), \xi(a)\} = \xi(z)$, and so, $R_a[\xi](x) \leq \xi(x)$. Hence, $R_a[\xi] \subseteq \xi$. Similarly, $L_a[\xi] \subseteq \xi$.

**Proposition 8.** The mappings $R_a : \xi \longrightarrow \xi, L_a : \xi \longrightarrow \xi$ are fuzzy continuous $\forall a \in \xi_e$.

*Proof.* The constant mapping $i_a : \xi \longrightarrow \xi; x \longmapsto a$ is fuzzy continuous since $i_a[\xi] \subseteq \xi$, and $\forall \mu \in \delta_\xi, i_a^{-1}[\mu](x) = \mu(a) = k, k \in I$. Since $R_a = \beta(I \times i_a)\Delta$, then $R_a$ is clearly fuzzy continuous. The proof that $L_a$ is also fuzzy continuous can be carried out in a similar way.

***Example 4***. With reference to Example 2, $R_e[\xi] = \left\{e_1, p_{\frac{3}{4}}, q_0, r_0\right\} \subset \xi$. Also, $R_q[\xi] = \left\{e_1, p_{\frac{3}{4}}, q_1, r_{\frac{1}{4}}\right\} = \xi$. The fuzzy continuity of both $R_e, R_q$ can be easily verified.

## 3 Homomorphism of Fuzzy Topological Partial Groups

Through the current section, we examine the image and preimage of a fuzzy partial group and an FTPG under a homomorphism. From now on, $\phi : X \longrightarrow Y$ is a partial group homomorphism.

**Proposition 9.** If $\zeta$ is a fuzzy partial group in $Y$, then $\phi^{-1}[\zeta]$ is also a fuzzy partial group in $X$.

*Proof.* Take $x, y \in X$. Now,

1. $\phi^{-1}[\zeta](xy) \geq \min\{\zeta(\phi(x)), \zeta(\phi(y))\} = \min\{\phi^{-1}[\zeta](x), \phi^{-1}[\zeta](y)\}$.
2. $\phi^{-1}[\zeta](x^{-1}) = \zeta(\phi(x)^{-1}) = \zeta(\phi(x)) = \phi^{-1}[\zeta](x)$.
3. $\phi^{-1}[\zeta](e_x) = \zeta(\phi(e_x)) = \zeta(e_{\phi(x)}) = k$.

**Proposition 10.** If $\xi$ is a fuzzy partial group in $X$, where $\xi$ has the sup property, then $\phi[\xi]$is also a fuzzy partial group in $Y$.

*Proof.* Take $x, y \in X, \hat{x} \in \phi^{-1}(\phi(x)), \hat{y} \in \phi^{-1}(\phi(y))$, where $\mu(\hat{x}) = \sup\limits_{w \in \phi^{-1}(\phi(x))} \xi(w)$ and $\mu(\hat{y}) = \sup\limits_{w \in \phi^{-1}(\phi(y))} \xi(w)$. Then,

1. $\phi[\xi](\phi(x)\phi(y)) = \sup_{t \in \phi^{-1}(\phi(x)\phi(y))} \xi(t)$
$\geq \min\{\xi(\hat{x}), \xi(\hat{y})\}.$
$\geq \min\{\phi(\xi)(\phi(x)), \phi(\xi)(\phi(y))\}.$
2. $\phi[\xi](\phi(x)^{-1}) = \sup_{t \in \phi^{-1}(\phi(x)^{-1})} \xi(t) = \xi(\hat{x}^{-1}) = \xi(\hat{x}) = \phi(\xi)(\phi(x))$
3. $\phi[\xi](e_{\phi(x)}) = \sup_{t \in \phi^{-1}(\phi(e_x))} \xi(t) = \xi(e_{\hat{x}}) = k.$

**Proposition 11.** If $\xi$ is a fuzzy partial group in $X$, then $\phi[\xi]$ is also a fuzzy partial group in $Y$ provided that $\xi$ is $\phi$-invariant.

*Proof.* Take $x, y \in X$. Now, since $\xi$ is $\phi$-invariant, then it can be easily proved that $\phi[\xi](\phi(x)) = \xi(x)$. Now,

1. $\phi[\xi](\phi(x)\phi(y)) = \xi(xy) \geq \min\{\phi[\xi](\phi(x)), \phi[\xi](\phi(y))\}.$
2. $\phi[\xi](\phi(x)^{-1}) = \xi(x^{-1}) = \xi(x) = \phi[\xi](\phi(x)).$
3. $\phi[\xi](e_{\phi(x)}) = \xi(e_x) = k.$

**Proposition 12.** Let $(X, \delta), (Y, \omega)$ be two FTS's, such that $\delta = \phi^{-1}[\omega]$, and let $\zeta$ be an FTPG in $Y$. Then, $\phi^{-1}[\zeta]$ is also an FTPG in $X$.

*Proof.* Let $\psi : \phi^{-1}[\zeta] \times \phi^{-1}[\zeta] \longrightarrow \phi^{-1}[\zeta]\, ; (x_1, x_2) \longmapsto x_1 {x_2}^{-1}$. The fuzzy continuity of $\phi$ is obvious since $\phi^{-1}[\omega]$ is the fuzzy inverse topology, so the mapping $\phi : \phi^{-1}[\zeta] \longrightarrow \zeta$ is fuzzy continuous, and consequently, $\phi \times \phi : \phi^{-1}[\zeta] \times \phi^{-1}[\zeta] \longrightarrow \zeta \times \zeta$ is also fuzzy continuous. Now, take $\mu \in \delta_{\phi^{-1}[\zeta]}$, then $\exists \nu \in \omega_\zeta$ where $\phi^{-1}[\nu] = \mu$, and so, $\psi^{-1}[\mu](x_1, x_2) = \nu(\phi(x_1)\phi(x_2)^{-1})$ for all $(x_1, x_2) \in X \times X$. Since $\zeta$ is an FTPG, then $\eta : \zeta \times \zeta \longrightarrow \zeta; (y_1, y_2) \longmapsto y_1 {y_2}^{-1}$ is fuzzy continuous. Now, $\nu(\phi(x_1)\phi(x_2)^{-1}) = (\phi \times \phi)^{-1}[\eta^{-1}[\nu]](x_1, x_2) \in \delta_{\phi^{-1}[\zeta]} \times \delta_{\phi^{-1}[\zeta]}$. Hence, $\psi^{-1}[\mu] \cap (\phi^{-1}[\zeta] \times \phi^{-1}[\zeta]) = (\phi \times \phi)^{-1}[\eta^{-1}[\nu]] \cap (\phi^{-1}[\zeta] \times \phi^{-1}[\zeta]) \in \delta_{\phi^{-1}[\zeta]} \times \delta_{\phi^{-1}[\zeta]}$. Therefore, $\psi$ is fuzzy continuous.

**Proposition 13.** Let $(X, \delta), (Y, \omega)$ be two FTS's, such that $\mu = \phi[\delta]$, and $\phi$ is fuzzy open. Let $\xi$ be an FTPG in $X$. Then, $\phi[\xi]$ is an FTPG in $Y$ provided that $\xi$ is $\phi$-invariant.

*Proof.* $\phi[\xi]$ is a fuzzy partial group by Proposition 4.3. Let $\mu' \in \delta_\xi$, then $\exists \mu \in \delta$ where $\mu' = \mu \cap \xi$ and by $\phi$-invariance of $\xi$, it can be shown that $\phi[\mu'] = \phi[\mu] \cap \phi[\xi] \in \omega_{\phi[\xi]}$, and therefore, $\phi : \xi \longrightarrow \phi[\xi]$ is fuzzy open, and so is $\phi \times \phi : \xi \times \xi \longrightarrow \phi[\xi] \times \phi[\xi]$. Also, it is given that $\phi : \xi \longrightarrow \phi[\xi]$ is fuzzy continuous. Since $\xi$ is an FTPG, then $\eta : \xi \times \xi \longrightarrow \xi\, ; (x_1, x_2) \longmapsto x_1 {x_2}^{-1}$ is fuzzy continuous. Let $\psi : \phi[\xi] \times \phi[\xi] \longrightarrow \phi[\xi]\, ; (y_1, y_2) \longmapsto y_1 {y_2}^{-1}$, and let $\nu' \in \omega_{\phi[\xi]}$. Then, $(\phi \times \phi)^{-1}\psi^{-1}[\nu'](x_1, x_2) = \eta^{-1}\phi^{-1}[\nu] \in \delta_\xi \times \delta_\xi$. By $\phi$ -

invariance of $\xi$, we have $(\phi \times \phi)^{-1}[\psi^{-1}[\nu'] \cap (\phi[\xi] \times \phi[\xi])] = (\phi \times \phi)^{-1}\psi^{-1}[\nu'] \cap (\xi \times \xi) \in \delta_\xi \times \delta_\xi$. Now, $(\phi \times \phi)(\phi \times \phi)^{-1}[\psi^{-1}[\nu'] \cap (\phi[\xi] \times \phi[\xi])] = \psi^{-1}[\nu'] \cap (\phi[\xi] \times \phi[\xi]) \in \delta_{\phi[\xi]} \times \delta_{\phi[\xi]}$, as $\phi \times \phi$ is a fuzzy open mapping. Therefore, $\psi$ is finally fuzzy continuous.

## 4 Quotient Fuzzy Topological Partial Group

Throughout this section, $N$ is normal subpartial group of $X, p : X \longrightarrow X/N; x \longmapsto xN$, is the quotient mapping.

**Proposition 14.** If $\xi$ is constant on $N$, then $\xi$ is $p$-invariant.

*Proof.* Since $\xi$ is constant on $N$, then $\xi(z) = c \; \forall \; z \in N$, and since $N$ is normal subpartial group of $X$, then $e_x \in N \; \forall x \in X$, and hence $\xi(e_x) = c$. Assume $x, y \in X$ satisfying $p(x) = p(y)$. So, $xN = yN$, which implies that $x \in yN$, hence $x = yz$ for some element $z \in N$. Now, $\xi(x) \geq \min\{\xi(z), \xi(y)\} = \min\{c, \xi(y)\} = \xi(y)$. In a similar manner, $\xi(y) \geq \xi(x)$ also, and so, $\xi(x) = \xi(y)$.

From Proposition 11, It can be said that $p[\xi] = \xi/N$ is a fuzzy partial group in $X/N$, and we will call it the fuzzy quotient partial group.

**Proposition 15.** Let $(X, \delta)$ be an FTS, and $p[\delta]$ be the fuzzy topology on $X/N$. If $\xi$ is constant on $N$, then $\xi/N$ is an FTPG in $X/N$.

*Proof.* From Propositions 14 and 13, $\xi/N$ is clearly an FTPG.

We will call $p[\delta]$ the quotient fuzzy topology, and $\xi/N$ will be called the quotient FTPG.

**Proposition 16.** Let $(X, \delta), (Y, \omega)$ be two FTS's, and $\phi$ be a mapping that is fuzzy continuous and fuzzy open. Let $\xi$ be constant on $ker\phi$, and let $X/ker\phi$ be given the quotient fuzzy topology. We therefore have that:

1. $\xi/\ker\phi$ and $\phi[\xi]$ are FTPGs in $X/\ker\phi$ and $Y$ respectively.
2. If $f : X/ker\,\phi \longrightarrow Y$ is the canonical isomorphism, then $f : \xi/ker\,\phi \longrightarrow \phi[\xi]$ is a fuzzy homeomorphism.

*Proof.* 1. From Proposition 15, $\xi/\ker\phi$ is clearly an FTPG in $X/\ker\phi$. Let $\nu \in \phi[\delta]$, then since $\phi[\delta]$ is the fuzzy image topology, we know that $\phi^{-1}[\nu] \in \delta$. It is easy to prove that for any fuzzy set $\nu$, $\phi[\phi^{-1}[[\nu]] = \nu$. And since $\phi$ is fuzzy open, then $\nu \in \omega$. Now, let $\nu \in$

ω, by the fuzzy continuity of $\phi$, we have $\phi^{-1}[\nu] \in \delta$, and so $\nu \in \phi[\delta]$. Hence, $\mu = \phi[\delta]$. By Proposition 4.5, $\phi[\xi]$ is then an FTPG in $Y$.

2. Let $p : X \longrightarrow X/\ker\phi$ be the quotient mapping, and $\nu' \in \omega_{\phi[\xi]}$. Then, $\phi^{-1}[\nu'] \in \delta_\xi$ given the fuzzy continuity of $\phi$, and since the quotient mapping $p$ is fuzzy open, then $p\phi^{-1}[\nu']$ is also fuzzy open in $\xi/\ker\phi$. Hence, $f$ is fuzzy continuous since $p\phi^{-1} = f^{-1}$. Now, let $\mu'$ be a fuzzy open set in $\xi/\ker\phi$, so $p^{-1}[\mu'] \in \delta_\xi$, and given that $\phi$ is fuzzy open, therefore $\phi p^{-1}[\mu'] \in \omega_{\phi[\xi]}$. Finally, since $\phi p^{-1} = f$, then, $f$ is fuzzy open and therefore a fuzzy homeomorphism.

## 5 Product Fuzzy Topological Partial Group

Consider that $X$ is the product partial group of the partial groups $\{X_i, i = 1,2,\dots,n\}$, and $\xi = \prod_{i=1}^{n}\xi_i$ , where $\xi_i$ is a fuzzy partial group in $X_i$ $\forall i = 1,2,\dots,n$, and $\xi(x) = \min\{\xi_1(x_1), \xi_2(x_2), \dots, \xi_n(x_n)\}$, where $x = (x_1, x_2, \dots, x_n)$.

**Proposition 17.** $\xi$ is a fuzzy partial group in $X$.

*Proof.* Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in X$. Then,

$$\begin{aligned}\xi(xy^{-1}) &= \xi_2(x_1y_1^{-1}, x_2y_2^{-1}, \dots, x_ny_n^{-1})\\ &= \min\{\xi_1(x_1y_1^{-1}), \xi_1(x_2y_2^{-1}), \dots, \xi_1(x_ny_n^{-1})\}\\ &\geq \min\{\min\{\xi_1(x_1), \xi_1(y_1)\}, \min\{\xi_2(x_2), \xi_2(y_2)\}, \dots, \min\{\xi_n(x_n), \xi_n(y_n)\}\}\\ &= \min\{\min\{\xi_1(x_1), \xi_2(x_2), \dots, \xi_n(x_n)\}, \min\{\xi_1(y_1), \xi_2(y_2), \dots, \xi_n(y_n)\}\}\\ &= \min\{\xi(x), \xi(y)\}.\end{aligned}$$

And also,
$$\begin{aligned}\xi(e_x) &= \xi(e_{x_1}e_{x_2}, \dots, e_{x_n})\\ &\geq \min\{\xi_1(e_{x_1}), \xi_2(e_{x_2}), \dots, \xi_n(e_{x_n})\}\\ &= \min\{k_1, k_2, \dots, k_n\}\\ &= k, \text{where } k = k_i \text{ for some } i.\end{aligned}$$

Let $(X_i, \delta_i)$ be an FTS, and $\xi_i$ be an FTPG in $X_i$, where $i = 1,2,\dots,n$. And let the fuzzy topology on $X = \prod_{i=1}^{n} X_i$ , $\xi = \prod_{i=1}^{n}\xi_i$ be respectively the product fuzzy topology $\delta$ and the induced fuzzy topology $\delta_\xi$.

**Proposition 18.** The product fuzzy partial group $\xi$ is an FTPG in $X$.

*Proof.* The proof of the fuzzy continuity of $\phi : (x, y) \longmapsto xy^{-1}$ is enough to prove the required. Let $\psi : (x, y) \longmapsto ((x_1, y_1), \dots, (x_n, y_n))$, then from Proposition 2.2, $\psi$ is fuzzy continuous.

Let $\eta : \big((x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\big) \longmapsto (x_1y_1^{-1},x_2y_2^{-1},\dots,x_ny_n^{-1})$, then since $\eta$ is the product of $\phi_i : (x_i,y_i) \longmapsto x_iy_i^{-1}$, then it is also fuzzy continuous. So, the mapping $\phi$ is clearly fuzzy continuous as $\phi = \eta\psi$.

We refer to $\xi = \prod_{i=1}^{n} \xi_i$ as a product FTPG.

**Proposition 19.** Let $N_i$ be normal subpartial group of $X_i$, and $\xi_i$ an FTPG in $X_i$ such that $\xi_i$ is a constant fuzzy set on $N_i$. Let the fuzzy topologies on $X/N$, where $N = \prod_{i=1}^{n} N_i$, and $X_i/N_i, i = 1,2,\dots,n$, be the respective quotient fuzzy topologies, and the fuzzy topologies on $X = \prod_{i=1}^{n} X_i$ and $\prod_{i=1}^{n} X_i/N_i$ be the respective product fuzzy topologies. Let $\xi = \prod_{i=1}^{n} \xi_i$ be the product FTPG in $X$. If $\iota : X/N \longrightarrow \prod_{i=1}^{n} X_i/N_i$ is the canonical isomorphism, then $\iota : \xi/N \longrightarrow \prod_{i=1}^{n} \xi_i/N_i$ is a fuzzy homeomorphism.

*Proof.* Let the quotient mappings on $X, X_i$ be $p, p_i$ respectively. Let $\prod_{i=1}^{n} p_i : X \longrightarrow \prod_{i=1}^{n} X_i/N_i$. Thus, $\iota p = \prod_{i=1}^{n} p_i$. We have

$$
\begin{aligned}
(\xi/N)(xN) = (\xi)(x) &= \left(\prod_{i=1}^{n} \xi_i\right)(x_1,\dots,x_n) \\
&= \min\{\xi_1(x_1),\dots,\xi_n(x_n)\} \\
&= \min\{(\xi_1/N_1)(x_1N_1),\dots,(\xi_n/N_n)(x_nN_n)\} \\
&= \left(\prod_{i=1}^{n}(\xi_i/N_i)\right)(\iota[xN]) \;\; \forall xN \in X/N
\end{aligned}
$$

By Propositions 15 and 18, $\xi/N$ and $\prod_{i=1}^{n}(\xi_i/N_i)$ are FTPGs. Assume that $\nu$ is a fuzzy open set in $\prod_{i=1}^{n}(\xi_i/N_i)$. Since $\prod_{i=1}^{n} p_i$ is fuzzy continuous, then $(\prod_{i=1}^{n} p_i)^{-1}[\nu]$ is fuzzy open in $\xi$, and since $p$ is fuzzy open, then $p(\prod_{i=1}^{n} p_i)^{-1}[\nu]$ is fuzzy open in $\xi/N$. Since $\iota^{-1}[\nu] = p(\prod_{i=1}^{n} p_i)^{-1}[\nu]$, then $\iota$ is fuzzy continuous. Conversely, take a fuzzy open set $\mu$ in $\xi/N$, so $p^{-1}[\mu]$ is a fuzzy open set in $\xi$, since the quotient mapping is fuzzy continuous, and hence, $(\prod_{i=1}^{n} p_i)(p^{-1}[\mu])$is fuzzy open in $\prod_{i=1}^{n}(\xi_i/N_i)$, since $\prod_{i=1}^{n} p_i$ is the product of fuzzy open mappings, and is therefore fuzzy open. Since $\iota[\mu] = (\prod_{i=1}^{n} p_i)(p^{-1}[\mu])$, therefore $\iota$ is fuzzy open. Finally, $\iota$ is then a fuzzy homeomorphism.

## Availability of data and material

## Conflict of interest

The author declares that he has no competing interests.

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**References**

[1] L. A. Zadeh, Fuzzy sets, Inform. Contr. **8**, 338-353 (1965).

[2] C. L. Chang, Fuzzy topological space, *J. Math. Anal. Appl*. **24** 182-190 (1968).

[3] C. K. Wong, Fuzzy topology: Product and quotient theorems, *J. Math. Anal. Appl*. **45**, 512-521. (1974)

[4] R. Lowen, Fuzzy topological spaces and fuzzy compactness, *J. Math. Anal. Appl*. **56**, 621-633 (1976).

[5] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl*. **35**, 512-517 (1971).

[6] David H. Foster, Fuzzy topological groups, *J. Math. Anal. Appl*. **67**, 549-564 (1979).

[7] A. M. Abd-Allah, R. A. K. Omar, Fuzzy partial groups, Fuzzy Sets and Systems, **82** 369-374 (1996).

[8] A. M. Abd-Allah, A. I. Aggour, A. Fathy, Strong semilattices of topological groups, *Journal of Egyptian Mathematical Society*. **24** (4), 597-602 (2016).

[9] A. M. Abd-Allah, A. I. Aggour and A. Fathy, k -partial groups, *Journal of Egyptian Mathematical Society*. **25** (3), 276-278 (2017).