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Original Article

Dynamical analysis of a Cournot duopoly model

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Abstract In this paper, two different mechanisms are used to study a homogeneous Cournot duopoly in a market characterized by the downward sloping and concave price function. Two firms, which have constant marginal costs, use adaptive, low-rationality mechanisms to adjust their production levels toward equilibrium. In particular, the stability of the equilibrium for two different mechanisms is studied. However, complex dynamics arise, especially when the reaction coefficient increases. Finally, we compare the obtained results of the two models.

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1. Introduction

The economic competition among few firms goes back to Cournot [1] who introduced a model of imperfect competition between firms, and by now it become a central concept in the field of economical market. Cournot suggested quantities as strategic variables, so that firms adapt the production levels in order to obtain their optimal profits. A different approach was proposed in 1883 by Bertrand in [2], in which he proposed that firms can instead compete with respect to prices. Since these two important papers, a huge number of researches focused on the oligopoly modeling. Quite early studies on such economic competition have suggested that complex dynamic behaviors such bifurcation and chaos may arise [3–20].

Bounded rationality and Puu's incomplete information are two different approaches that have been recently used to study monopoly and duopoly markets. Bounded rational players (firms) update their production strategies based on discrete time periods and by using a local estimate of the marginal profit. With such local adjustment mechanism, the players are not requested to have a complete knowledge of the demand and the cost functions [15], as all they need to know is how the market will response to small production changes, in order to adjust their production levels by means of a local estimate of the marginal profit. On the other hand, Puu's [21] has recently introduced the so-called Puu's incomplete information in which,

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more realistically, a firm does not need to know the local slope of the profit function to choose the quantity to produce in the next time step [4,6]. Instead all it needs is its profit and the quantities produced in the past two times.

Recently, some papers [7,16] have shown that complex dynamical characteristics such as bifurcation and chaos, in a monopoly if a market characterized by gradient rule is considered, can be achieved if a market characterized by a simple demand function that has no inflection point is considered. Askar [22] has studied the Cournot duopoly game with a cubic demand function and considering boundedly rational firms and Puu's incomplete information. In this paper, we extend the investigations of Askar [22] to more general demand function, considering a different adjustment mechanism too.

The paper is organized as follows: In Section 2, the bounded rationality version of the model and its analysis are illustrated and discussed in detail. In Section 3, Puu's incomplete information version of the model and its analysis are illustrated and discussed. Finally, conclusions are presented.

2. The bounded rationality version of the model

In this work, we consider a more general form of downward sloping and concave inverse demand function with respect to that presented in [16,22]

$$P(Q^t) = a - b\,(Q^t)^{2n+1}, \quad n \in \mathbb{N}, \tag{1}$$

where P is the commodity price, $Q^t = q_1^t + q_2^t$ is the aggregated quantity in which q_i^t is the ith quantity of commodity produced by the ith firm, $i = 1, 2$, while the parameters a, b are positive constants. We remark that function $P(Q_t)$ has no inflection points. Also, for a cubic demand function and a monopolistic market, it was shown in [7,16] that the Nash equilibrium points loses its stability through a period doubling bifurcation which leads to chaos. We assume that the market has only two firms with linear cost function

$$C_i(q_1^t, q_2^t) = c\,q_i^t, \quad i = 1, 2, \tag{2}$$

where $c > 0$ is the constant marginal cost.

Each firm wants to maximize its profit

$$\begin{aligned} \Pi_1(q_1^t, q_2^t) &= (a - c - b\,(Q^t)^{2n+1})\,q_1^t, \\ \Pi_2(q_1^t, q_2^t) &= (a - c - b\,(Q^t)^{2n+1})\,q_2^t. \end{aligned} \tag{3}$$

and to do this, they use a gradient mechanism. A positive (negative) variation of the profits will induce a change in the quantity in the same (opposite) direction from that of the previous period. The resulting quantity adjustment mechanism can be described by the following dynamical system

$$q_i^{t+1} = q_i^t + \alpha_i(q_i^t)\,\frac{\partial\,\Pi(q_1^t, q_2^t)}{\partial\, q_i^t}, \quad i = 1, 2, \tag{4}$$

similar to those used in [16]. Function $\alpha_i(q_i^t)$ represents the speed of adjustment and it is a positive function which gives the extent of production variation of the ith firm following a given profit signal. Moreover it captures the fact that relative effort variations are proportional to the marginal profit. Here, we assume that $\alpha_i(q_i^t) = k_i q_i^t$, where k_i is a positive constant. Substituting Eq. (3) in Eq. (4), we get the following two-dimensional nonlinear dynamical system

$$\begin{aligned} q_1^{t+1} &= q_1^t + k_1\,q_1^t\,(a - c - b\,(Q^t)^{2n}\,(2\,(n+1)\,q_1^t + q_2^t)), \\ q_2^{t+1} &= q_2^t + k_2\,q_2^t\,(a - c - b\,(Q^t)^{2n}\,(q_1^t + 2\,(n+1)\,q_2^t)). \end{aligned} \tag{5}$$

This system has the following equilibrium points

$$E_0 = (0,\ 0),\ E_1 = \left(\left(\frac{a-c}{2\,(n+1)\,b}\right)^{\frac{1}{2n+1}},\ 0\right),$$
$$E_2 = \left(0,\ \left(\frac{a-c}{2\,(n+1)\,b}\right)^{\frac{1}{2n+1}}\right),$$
$$E_3 = \left(\left(\frac{a-c}{2^{2n}\,(2n+3)\,b}\right)^{\frac{1}{2n+1}},\ \left(\frac{a-c}{2^{2n}\,(2n+3)\,b}\right)^{\frac{1}{2n+1}}\right),$$

the positivity which is guaranteed by $a > c$.

Proposition 2.1. *Steady states E_0, E_1 and E_2 are unstable equilibrium points of system (5), whereas the steady state E_3 is locally asymptotically stable if $k_1 + k_2 < \frac{4\,(2n+3)}{(n+2)\,(2n+1)\,(a-c)}$ and $k_1 k_2 < \frac{4\,(2n+3)}{(2n+1)^2\,(a-c)^2}$.*

Proof. The proof of the above proposition is based on the standard analysis of eigenvalues for more details we refer to [23]. The Jacobian matrix of system (5) is

$$J(q_1, q_2) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},$$

where

$$\begin{aligned} J_{11} &= 1 + k_1\left(a - c - b\,Q^{2n-1}\left((2n+2)^2\,q_1^2 \right.\right. \\ &\quad \left.\left. +(6n+5)\,q_1\,q_2 + q_2^2\right)\right), \\ J_{12} &= -(2n+1)\,k_1\,b\,Q^{2n-1}\left((2n+1)\,q_1^2 + q_1\,q_2\right), \\ J_{21} &= -(2n+1)\,k_2\,b\,Q^{2n-1}\left(q_1\,q_2 + (2n+1)\,q_2^2\right), \\ J_{22} &= 1 + k_2\left(a - c - b\,Q^{2n-1}\left(q_1^2 + (6n+5)\,q_1\,q_2 \right.\right. \\ &\quad \left.\left. +(2n+2)^2\,q_2^2\right)\right). \end{aligned} \tag{6}$$

At the equilibrium point E_0, the Jacobian matrix becomes

$$J(0,\ 0) = \begin{bmatrix} 1 + k_1\,(a-c) & 0 \\ 0 & 1 + k_2\,(a-c) \end{bmatrix},$$

whose eigenvalues are $\lambda_1 = 1 + k_1\,(a-c) > 1$, $\lambda_2 = 1 + k_2\,(a-c) > 1$. Thus, the equilibrium point E_0 is unstable.

At the equilibrium point E_1, the Jacobian matrix is

$$J(\overline{q}_1,\ 0) = \begin{bmatrix} 1 - (2n+1)\,k_1\,(a-c) & \frac{-(2n+1)^2\,k_1\,(a-c)}{2\,(n+1)} \\ 0 & 1 + \frac{(2n+1)\,k_2\,(a-c)}{2\,(n+1)} \end{bmatrix},$$

with eigenvalues $\lambda_1 = 1 - (2n+1)\,k_1\,(a-c) < 1$ and $\lambda_2 = 1 + \frac{(2n+1)\,k_2\,(a-c)}{2\,(n+1)} > 1$, meaning that the equilibrium point E_1 is unstable. For E_2, the same considerations and arguments hold exchanging indices 1, 2.

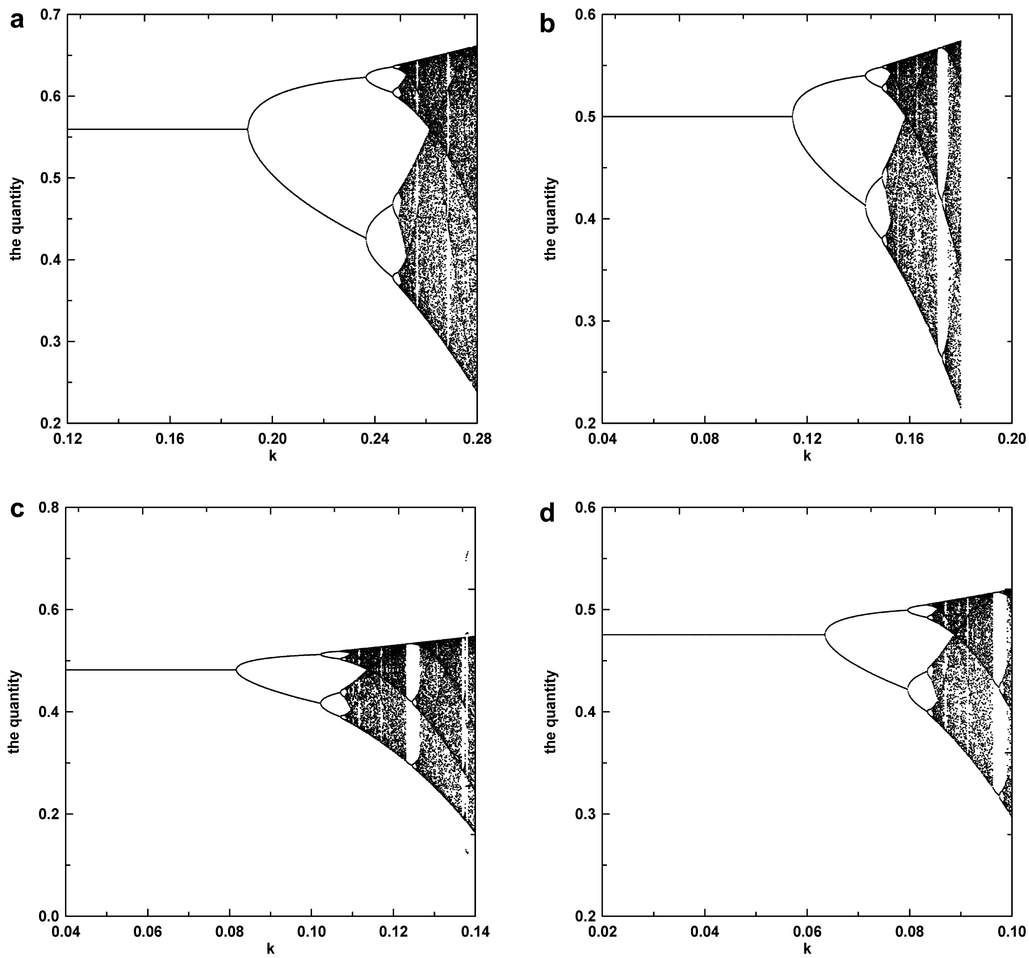

Fig. 1 The complex behavior of the model at the quantities $a = 4.0,\ b = 0.6,\ c = 0.5$ and (a) $n = 1$, (b) $n = 2$, (c) $n = 3$ and (d) $n = 4$.

To study the stability of the equilibrium point E_3, we notice that evaluating J at it, we obtain

$$J(\overline{q}_1, \overline{q}_2) = \begin{bmatrix} 1 - \frac{(n+2)\,(2n+1)\,k_1\,(a-c)}{2n+3} & -\frac{(n+1)\,(2n+1)\,k_1\,(a-c)}{2n+3} \\ -\frac{(n+1)\,(2n+1)\,k_2\,(a-c)}{2n+3} & 1 - \frac{(n+2)\,(2n+1)\,k_2\,(a-c)}{2n+3} \end{bmatrix},$$

which has the characteristic equation $\lambda^2 - \beta\,\lambda + \gamma = 0$, where $\beta = 2 - \frac{(n+2)\,(2n+1)\,(a-c)\,(k_1+k_2)}{2n+3}$ and $\gamma = 1 - \frac{(n+2)\,(2n+1)\,(a-c)\,(k_1+k_2)}{2n+3} + \frac{(2n+1)^2\,(a-c)^2\,(k_1\,k_2)}{2n+3}$. This equation has eigenvalues less than one under the condition $2 < 1 + \gamma < abs(\beta)$. Thus the conditions of stability for the equilibrium point E_3 become $k_1 + k_2 < \frac{4\,(2n+3)}{(n+2)\,(2n+1)\,(a-c)}$ and $k_1\,k_2 < \frac{4\,(2n+3)}{(2n+1)^2\,(a-c)^2}$.

To investigate the dynamical behavior of system (5) when the equilibrium loses its stability, some numerical simulations are presented. Setting $a = 4.0$, $b = 0.6$, $c = 0.5$, it is shown in Fig. 1 that the equilibrium point E_3 loses its stability through period doubling bifurcation which leads to chaos when the value of k increases. This behavior is the same for $n = 1, 2, 3, 4$ as shown in Fig. 1(a)–(d) in which we can see that instability arises for the reaction coefficient $k \approx 0.19, 0.12, 0.08, 0.06$, respectively. A similar behavior is shown in Fig. 2, in which we set $a = 1.0$, $b = 0.3$, $c = 0.1$. For $n = 1, 2, 3, 4$ as shown in Fig. 2(a)–(d), the values of the reaction coefficient for which equilibrium loses its stability are $k \approx 0.74, 0.44, 0.31, 0.24$, respectively. Note that, the value of k at which the system loses its stability is inversely proportional to the value of n, this means that k decreases when n increases. Conversely, parameter b has effect on the equilibrium points but has no effect on their stability. □

3. Puu's incomplete information version of the model

In concrete economic contexts, the firms might not know the profit function to estimate the quantities produced in the next step [3,6]. Actually all they need are the profits they achieved and the produced quantities in the last two time steps. Next period strategy can be established using the so called Puu's incomplete information approach, namely by means of the rule of thumb mechanism described by

$$q_i^{t+1} = q_i^t + \alpha_i(q_i^t)\,\frac{\Pi_i^t - \Pi_i^{t-1}}{q_i^t - q_i^{t-1}}, \quad i = 1,\ 2. \tag{7}$$

Recently, Ahmed et al. [5] showed that systems based on approaches similar to [2] suffer from numerical instabilities when the dynamic approaches the equilibrium. Moreover such systems exhibit serious instabilities in the case of duopoly. Again, $\alpha_i(q_i^t)$ represents the speed of adjustment and we will assume that $\alpha_i(q_i^t) = k_i q_i^t$, where k_i is a positive constant.

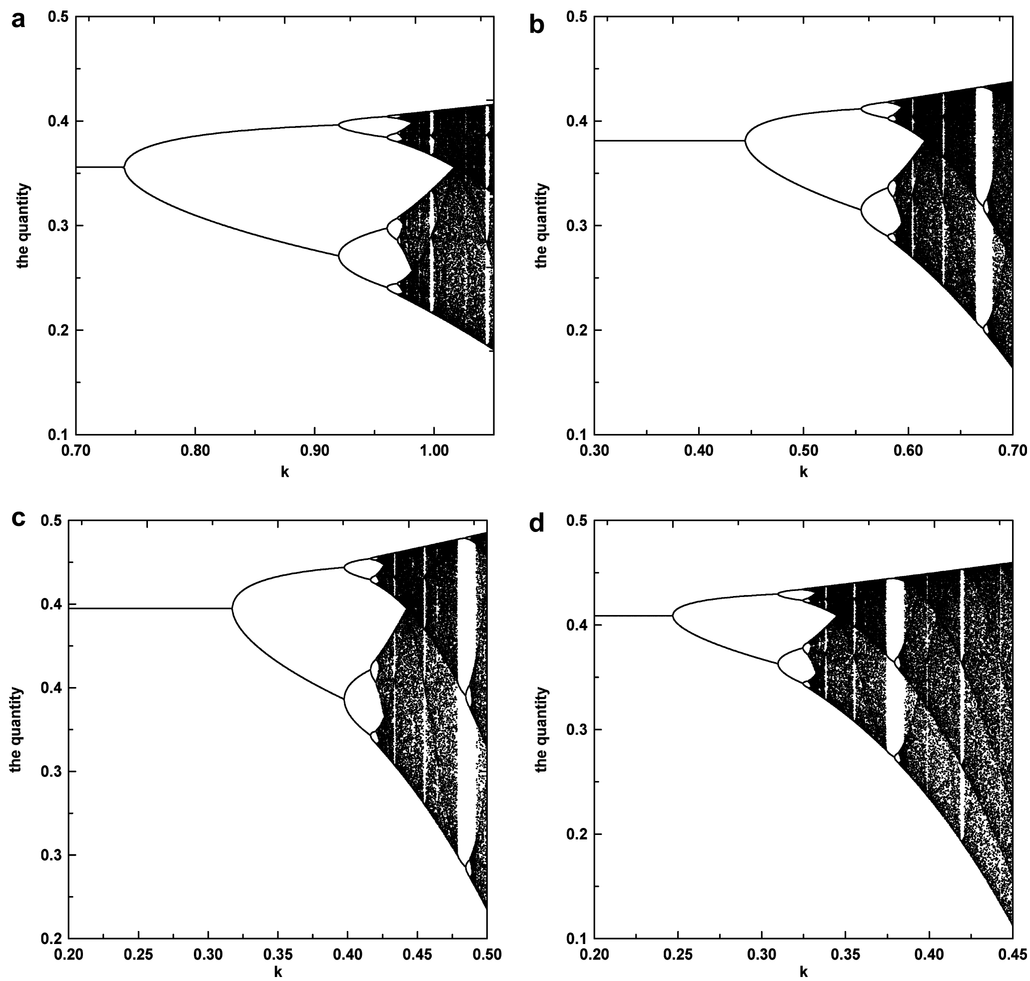

Fig. 2 The complex behavior of the model at the quantities $a = 1.0$, $b = 0.3$, $c = 0.1$ and (a) $n = 1$, (b) $n = 2$, (c) $n = 3$ and (d) $n = 4$.

Here, to simplify the analysis, we assume that the two firms strategies are sufficiently close, so that we can set $q_1^t \approx q_2^t$ and $q_1^{t-1} \approx q_2^{t-1}$. Thus, from Eqs. (3) and (7), the corresponding dynamical system is

$$q_i^{t+1} = q_i^t + k_i q_i^t \Big(a - c - 2^{2n+1} b \times \frac{((q_i^t)^{n+1} + (q_i^{t-1})^{n+1})\,((q_i^t)^{n+1} - (q_i^{t-1})^{n+1})}{q_i^t - q_i^{t-1}}\Big), \quad i = 1, 2. \tag{8}$$

Askar [7] has studied this system for $n = 1$. Here, we will study this system for $n = 2$, obtaining

$$\begin{aligned} q_1^{t+1} &= q_1^t + k_1 q_1^t \,(a - c - 2^5 b\,((q_1^t)^3 + (q_1^{t-1})^3)\,((q_1^t)^2 + q_1^t q_1^{t-1} + (q_1^{t-1})^2)), \\ q_2^{t+1} &= q_2^t + k_2 q_2^t \,(a - c - 2^5 b\,((q_2^t)^3 + (q_2^{t-1})^3)\,((q_2^t)^2 + q_2^t q_2^{t-1} + (q_2^{t-1})^2)). \end{aligned} \tag{9}$$

This system has the following equilibrium points

$$e_0 = (0,\, 0),\; e_1 = \left(\left(\frac{a-c}{192\,b}\right)^{\frac{1}{5}},\, \left(\frac{a-c}{192\,b}\right)^{\frac{1}{5}}\right).$$

Proposition 3.1. *Steady state e_0 is unstable equilibrium point for the system (9), whereas the steady state e_1 is locally asymptotically stable if* $k_1, k_2 < \frac{2}{5\,(a-c)}$.

Proof. The proof of the above proposition is based on the standard analysis of eigenvalues for more details we refer to [23]. Since $q_1^t \approx q_2^t$ and $q_1^{t-1} \approx q_2^{t-1}$, then the Jacobian matrix of the system (9) becomes

$$J(q_1, q_2) = \begin{bmatrix} 1 + k_1\,[a - c - (32)\,(36\,\overline{q}_1^{\,5})] & 0 \\ 0 & 1 + k_2\,[a - c - (32)\,(36\,\overline{q}_2^{\,5})] \end{bmatrix}.$$

At the point e_0, the Jacobian matrix becomes

$$J(0,\, 0) = \begin{bmatrix} 1 + k_1\,(a - c) & 0 \\ 0 & 1 + k_2\,(a - c) \end{bmatrix},$$

whose eigenvalues are $\lambda_1 = 1 + k_1\,(a - c) > 1$, $\lambda_2 = 1 + k_2\,(a - c) > 1$. Thus, the steady state e_0 is unstable.

To study the stability of the steady state e_1, we notice that evaluating J in it we obtain

$$J(\overline{q}_1, \overline{q}_2) = \begin{bmatrix} 1 - 5\,k_1\,(a - c) & 0 \\ 0 & 1 - 5\,k_2\,(a - c) \end{bmatrix},$$

with eigenvalues $\lambda_1 = 1 - 5\,k_1\,(a - c)$, $\lambda_2 = 1 - 5\,k_2\,(a - c)$. Thus, the condition of stability becomes $k_1, k_2 < \frac{2}{5\,(a-c)}$.

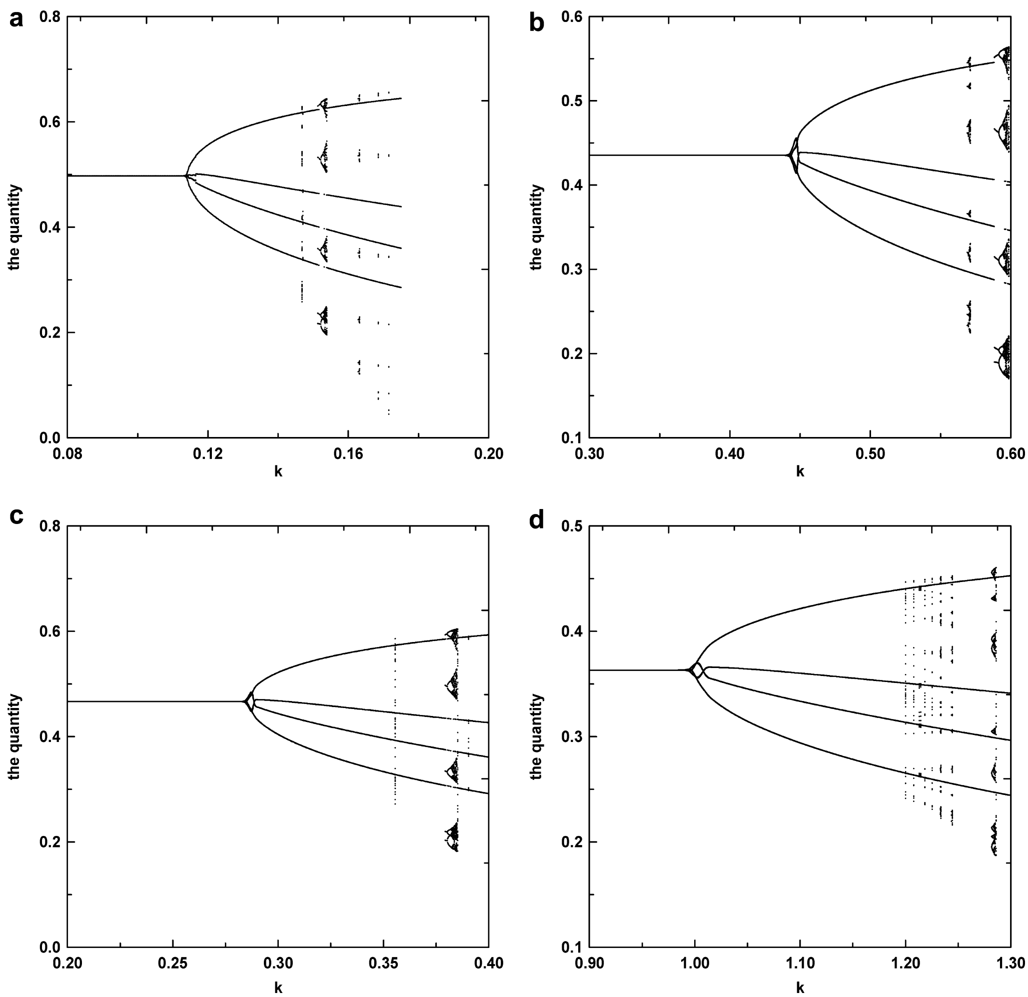

Fig. 3 The complex behavior of the model at $n = 2$ and the quantities (a) $a = 4.0$, $b = 0.6$, $c = 0.5$, (b) $a = 1.0$, $b = 0.3$, $c = 0.1$, (c) $a = 1.5$, $b = 0.33$, $c = 0.1$, and (d) $a = 0.5$, $b = 0.3$, $c = 0.1$.

The dynamical behavior of system (9) is shown in Fig. 3. It is easy to see that the quantities move from stability through a sequence of period doubling bifurcation to chaos. Note that, in Fig. 3(a), where the system parameters values are $a = 4.0$, $b = 0.6$, $c = 0.5$, as in Fig. 3(b), the equilibrium point loses its stability at the same value $k < \frac{2}{(2n+1)(a-c)}$. This means that the two approaches of bounded rationality and Puu's incomplete information are nearly give the same behavior (at least in the case $n = 2$). This behavior repeated in Fig. 3(b)–(d) for (b) $a = 1.0$, $b = 0.3$, $c = 0.1$, (c) $a = 1.5$, $b = 0.33$, $c = 0.1$, and (d) $a = 0.5$, $b = 0.3$, $c = 0.1$. □

4. Conclusion

In this paper, two different kinds of repeated games are introduced, which are based on a gradient adjustment mechanism and Puu's incomplete information approaches. A demand function without inflection points is used. By using rationality process firms do not need to solve any optimization problem but they adjust their production on the base of the estimation of the marginal profit. using Puu's approach, firms only need to know their profits and the quantities produced in the past two time steps. We obtained the equilibrium points of each case, which correspond to the profit maximizing quantities and found the local stability conditions of them. Complex dynamics arose when the reaction coefficient parameter was increased. We compared the properties of the two models under the two approaches. The paper generalized the results of other authors that consider similar processes.

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