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Original Article

A novel fractional technique for the modified point kinetics equations☆

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Abstract A fractional model for the modified point kinetics equations is derived and analyzed. An analytical method is used to solve the fractional model for the modified point kinetics equations. This methodical technique is based on the representation of the neutron density as a power series of the relaxation time as a small parameter. The validity of the fractional model is tested for different cases of step, ramp and sinusoidal reactivity. The results show that the fractional model for the modified point kinetics equations is the best representation of neutron density for subcritical and supercritical reactors.

2010 Mathematics Subject Classifications: 81V35; 82D75; 34A08

1. Introduction

The neutron density and the precursor concentrations of delayed neutrons at the center of a homogeneous nuclear reactor are described by a system of stiff coupled linear and/or nonlinear differential equations. An important peculiarity of the reactor kinetics is the stiffness of the system. A host of mathematical methods are developed to solve this system as a function of neutron density with different energy groups of delayed neutrons. Although, it is still currently requires more effort from the scientists to develops a new mathematical techniques and computational scheme to overcome this problem. The continuous indication of the neutron density and its rate of change are important for the safe startup, accurate determination of reactivity effects and operation of reactors. Recently, the interest of the nuclear reactor scientists is the development and analysis of different versions and approximations of the fractional-order point reactor kinetics model for a nuclear reactor "fractional neutron point kinetics equations (FNPKE)". For example: Espinosa-Paredes, Polo-Labarrios, et al. [1–7]; Ray and Patra [8]; Nowak

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et al. [9,10]; and Schramm et al. [11]. Espinosa-Paredes et al. [1] is the first scientific group derived the FNPKE. Aboanber and Nahla commented on the paper [1] through letter to editor [12,13]. In this letter, the corrected form for the fractional neutron point kinetics equations is developed. In this work we focus on the derivation of the fractional modified point kinetics equations (FMPKE) and its analytical solution. The developed solution of the FMPKE is based on the representation of the neutron density as a power series in terms of the relaxation time as a small parameter, which is less than 10^{-3} (s).

The presented paper is organized as follows: Section 2 contains the derivation of the fractional modified point kinetics equations. An analytical method based on the representation of the neutron density as a power series of a small parameter is presented in Section 3. A comparison between the neutron density of the fractional modified point kinetics equations and the point kinetics equations is discussed in Section 4. The conclusion is suggested in Section 5.

2. Fractional modified point kinetics equations (FMPKE)

The explicit forms taken by the neutron conservation equation and the corresponding equation for the current density in the one-speed case are [14]

$$\frac{1}{v}\frac{\partial}{\partial t}\Phi(\mathbf{r},t)+\nabla.\mathbf{J}(\mathbf{r},t)+\Sigma_t\Phi(\mathbf{r},t)=\Sigma_s\Phi(\mathbf{r},t)+S(\mathbf{r},t), \quad (1)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)+\nabla.\int_{4\pi}\hat{\Omega}\hat{\Omega}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}+\Sigma_t\mathbf{J}(\mathbf{r},t) =\bar{\mu}_0\Sigma_s\mathbf{J}(\mathbf{r},t)+S_1(\mathbf{r},t) \quad (2)$$

where $\varphi(\mathbf{r},\hat{\Omega},t)$ is the time dependent angular flux, as a function of space $\mathbf{r}$ and neutron direction of motion $\hat{\Omega}$, $\Phi(\mathbf{r},t)\equiv\int_{4\pi}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}$ is zero moment of the angular flux represents the neutron density, whereas the first moment of the angular flux $\mathbf{J}(\mathbf{r},t)\equiv\int_{4\pi}\hat{\Omega}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}$ represents the neutron current density, v is the neutron speed, Σ_t is the total cross section, Σ_s is the scattering cross section, $\bar{\mu}_0=\langle\hat{\Omega}.\hat{\hat{\Omega}}\rangle$ is the average scattering angle cosine, $S(\mathbf{r},t)\equiv\int_{4\pi}S(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}$ is the neutron source term and $S_1(\mathbf{r},t)\equiv\int_{4\pi}\hat{\Omega}S(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}$ is first moment of the neutron source.

Assume that the angular flux is only linearly anisotropic and the neutron source is isotropic, that is $\nabla.\int_{4\pi}\hat{\Omega}\hat{\Omega}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}=\frac{1}{3}\nabla\Phi(\mathbf{r},t)$ and $S_1(\mathbf{r},t)=0$. The system of Eqs. (1) and (2) take the following form

$$\frac{1}{v}\frac{\partial\Phi(\mathbf{r},t)}{\partial t}+\nabla\cdot\mathbf{J}(\mathbf{r},t)+\Sigma_a\Phi(\mathbf{r},t)=S(\mathbf{r},t) \quad (3)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)+\Sigma_{tr}\mathbf{J}(\mathbf{r},t)+\frac{1}{3}\nabla\Phi(\mathbf{r},t)=0 \quad (4)$$

where $\Sigma_a=\Sigma_t-\Sigma_s$ is the time dependent absorption cross section, Σ_t is the total cross section, Σ_s is the scattering cross section, $\Sigma_{tr}=\Sigma_t-\bar{\mu}_0\Sigma_s$ is the transport cross section.

Eqs. (3) and (4) representing the P_1 approximation (i.e. the one-speed approximation). To simplify this system of equations, Fick's suggested for Eq. (4) that the time derivative $\frac{1}{v}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$ can be neglected in comparing with the other terms [14], which contradicts Cattaneo's law [15]. Let us divide Eq. (4) by Σ_{tr} and taking the fractional derivative on the first term as follows [16,17]:

$$\tau^\kappa\frac{\partial^\kappa}{\partial t^\kappa}\mathbf{J}(\mathbf{r},t)+\mathbf{J}(\mathbf{r},t)=-D\nabla\Phi(\mathbf{r},t) \quad (5)$$

where $D=\frac{1}{3\Sigma_{tr}}$ is the neutron diffusion coefficient and $\tau=\frac{1}{v\Sigma_{tr}}=\frac{3D}{v}$ is the relaxation time.

Substituting from Eq. (5) into Eq. (3) yields

$$\tau^\kappa\frac{\partial^\kappa}{\partial t^\kappa}\left[\frac{1}{v}\frac{\partial}{\partial t}\Phi(\mathbf{r},t)+\Sigma_a\Phi(\mathbf{r},t)-S(\mathbf{r},t)\right]+\frac{1}{v}\frac{\partial}{\partial t}\Phi(\mathbf{r},t) +\Sigma_a\Phi(\mathbf{r},t)-S(\mathbf{r},t)-D\nabla^2\Phi(\mathbf{r},t)=0 \quad (6)$$

The neutron diffusion equations with delayed neutrons are obtained by adding the source term of reactor kinetics with I groups of delayed neutrons as follow:

$$\tau^\kappa\frac{\partial^\kappa}{\partial t^\kappa}\left[\frac{1}{v}\frac{\partial}{\partial t}\Phi(\mathbf{r},t)+(\Sigma_a-(1-\beta)\nu\Sigma_f)\Phi(\mathbf{r},t) -\sum_{i=1}^{I}\lambda_iC_i(\mathbf{r},t)\right]+\frac{1}{v}\frac{\partial}{\partial t}\Phi(\mathbf{r},t)+[\Sigma_a-(1-\beta)\nu\Sigma_f]\Phi(\mathbf{r},t) -\sum_{i=1}^{I}\lambda_iC_i(\mathbf{r},t)-D\nabla^2\Phi(\mathbf{r},t)=0 \quad (7)$$

where $S(\mathbf{r},t)=(1-\beta)\nu\Sigma_f\Phi(\mathbf{r},t)+\sum_{i=1}^{I}\lambda_iC_i(\mathbf{r},t)$, $\beta=\sum_{i=1}^{I}\beta_i$ is total fraction of delayed neutrons, β_i is the fraction of i-group of delayed neutrons, ν is the mean number of fission neutrons, Σ_f is the fission cress section, λ_i is the decay constant of i-group of delayed neutrons, I is the total number of delayed neutron groups and $C_i(\mathbf{r},t)$ is the precursor concentrations of i-group of delayed neutrons which satisfy the following equations

$$\frac{\partial C_i(\mathbf{r},t)}{\partial t}=\beta_i\nu\Sigma_f\Phi(\mathbf{r},t)-\lambda_iC_i(\mathbf{r},t), \quad i=1,2,\ldots,I. \quad (8)$$

Consider that:

$$\Phi(\mathbf{r},t)=vp(t)\phi(\mathbf{r}), \quad C_i(\mathbf{r},t)=c_i(t)\phi(\mathbf{r}) \quad (9)$$

where $\phi(\mathbf{r})$ is the fundamental function, which is obtained from the diffusion equation:

$$\nabla^2\phi(\mathbf{r})+B_g^2\phi(\mathbf{r})=0 \quad (10)$$

B_g^2 is the geometric buckling appropriate for the reactor geometry (Table 3.3, page (60), [18]).

Using Eq. (10) in Eqs. (7) and (8) leads to the fractional modified point kinetics equations with multi-group delayed neutrons as

$$\tau^\kappa\frac{d^\kappa}{dt^\kappa}\left[\frac{dp(t)}{dt}-\left(\frac{\rho}{\Lambda}-\mu+\alpha\right)p(t)-\sum_{i=1}^{I}\lambda_ic_i(t)\right] +\frac{dp(t)}{dt}-\left(\frac{\rho}{\Lambda}-\mu\right)p(t)-\sum_{i=1}^{I}\lambda_ic_i(t)=0 \quad (11)$$

$$\frac{dc_i(t)}{dt}=\mu_ip(t)-\lambda_ic_i(t), \quad i=1,2,3,\ldots,I \quad (12)$$

where $p(t)$ is the neutron density, $\rho=\frac{\nu\Sigma_f-\Sigma_a(1+L^2B_g^2)}{\nu\Sigma_f}$ is the time dependent reactivity, $L^2=\frac{D}{\Sigma_a}$ is the diffusion length, $\Lambda=\frac{1}{v\nu\Sigma_f}$ is the prompt neutron generation time, $\alpha=vDB_g^2$, $\mu_i=\frac{\beta_i}{\Lambda}$ and the initial conditions of this differential equations are $p(0)=p_0$, $c_i(0)=\frac{\mu_i}{\lambda_i}p_0$, $i=1,2,3,\ldots,I$.

The fractional calculus involves different definitions of the fractional operator as well as the Grünwald–Letnikov derivative, Riemann–Liouville fractional derivative, and Caputo derivative. Let us introduce the definition of the fractional derivatives as:

1. The Grünwald–Letnikov definition of fractional derivatives is defined as

$$ {}^{GL}D^{\kappa} f(t) = \lim_{h\to 0} h^{-\kappa} \sum_{m=0}^{n} \gamma_m^{\kappa} f(t-mh) \tag{13} $$

where $\gamma_m^{\kappa} = (-1)^m \binom{\kappa}{m}$, $\gamma_0^{\kappa} = 1$ and $\gamma_m^{\kappa} = \left(1 - \frac{\kappa+1}{m}\right)\gamma_{m-1}^{\kappa}, m = 1, 2, \ldots$.

2. The Riemann–Liouville definition of fractional derivatives is defined as

$$ {}^{RL}D^{\kappa} f(t) = \frac{1}{\Gamma(m-\kappa)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{f(\xi)}{(t-\xi)^{\kappa+1-m}} d\xi, \quad m-1 < \kappa < m \tag{14} $$

where m is a positive integer and $\Gamma(m-\kappa)$ is the gamma function whose argument is $(m-\kappa)$.

3. The Caputós derivative is defined as

$$ {}^{C}D^{\kappa} f(t) = \frac{1}{\Gamma(m-\kappa)} \int_0^t \frac{f^{(m)}(\xi)}{(t-\xi)^{\kappa+1-m}} d\xi, m-1 < \kappa < m \tag{15} $$

where $f^{(m)}(\xi) = \frac{\partial^m f(\xi)}{\partial \xi^m}$

3. Analytical solution of the FMPKE

The solution for the fractional modified point kinetics equations (FMPKE) can be obtained in several ways, in this paper an analytical solution of the FMPKE is proposed. Let us assume that

$$ q(t) = \frac{dp(t)}{dt} - \left(\frac{\rho}{\Lambda} - \mu + \alpha\right) p(t) - \sum_{i=1}^{I} \lambda_i c_i(t) \tag{16} $$

with initial value $q(0) = -\alpha p_0$.

Substituting into Eq. (11), yields

$$ \frac{dp(t)}{dt} = \left(\frac{\rho(t)}{\Lambda} - \mu\right) p(t) + \sum_{i=1}^{I} \lambda_i c_i(t) - \tau^{\kappa} \frac{d^{\kappa} q(t)}{dt^{\kappa}} \tag{17} $$

Let us rewrite Eqs. (17) and (12) in matrix form as

$$ \frac{d}{dt}|\Psi(t)\rangle = \mathbf{N}|\Psi(t)\rangle - \tau^{\kappa} \frac{d^{\kappa} q(t)}{dt^{\kappa}} |1\rangle \tag{18} $$

where,

$$ |\Psi(t)\rangle = \begin{pmatrix} p(t) \\ c_1(t) \\ c_2(t) \\ \vdots \\ c_I(t) \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} \frac{\rho}{\Lambda} - \mu & \lambda_1 & \lambda_2 & \cdots & \lambda_I \\ \mu_1 & -\lambda_1 & 0 & \cdots & 0 \\ \mu_2 & 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_I & 0 & 0 & \cdots & -\lambda_I \end{pmatrix}, $$

$$ |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. $$

The vector $|\Psi(t)\rangle$ can be represented by a power series of the relaxation time τ^{κ} :

$$ |\Psi(t)\rangle = |\psi_0(t)\rangle + \tau^{\kappa}|\psi_1(t)\rangle + \cdots \tag{19} $$

since $\tau \le 10^{-4}$, then we can neglect the terms of order two $O(\tau^{2\kappa})$.

Substituting into Eq. (18) we get

$$ \frac{d}{dt}|\psi_0(t)\rangle + \tau^{\kappa}\frac{d}{dt}|\psi_1(t)\rangle = \mathbf{N}|\psi_0(t)\rangle + \tau^{\kappa}\mathbf{N}|\psi_1(t)\rangle - \tau^{\kappa}\frac{d^{\kappa}q(t)}{dt^{\kappa}}|1\rangle + \cdots \tag{20} $$

3.1. Zero order of relaxation time

The zero order of relaxation time, τ^0, is

$$ \frac{d}{dt}|\psi_0(t)\rangle = \mathbf{N}|\psi_0(t)\rangle \tag{21} $$

This equation represents the point kinetics model which can be derived using Fick's approximation. The analytical solution of this system is reported in most reactor analysis articles. For completeness, it is recalled briefly that, with this assumption, the coefficient matrix $\mathbf{N}$ is independent on time and all eigenvalues ω_k are real which can be determined as:

$$ det(\mathbf{N} - \omega\mathbf{I}) = 0 \;\Rightarrow\; \left(\omega - \frac{\rho}{\Lambda}\right) + \sum_{i=1}^{I} \frac{\omega\mu_i}{\omega + \lambda_i} = 0 \tag{22} $$

This equation is called *inhour equation*. The eigenvectors $|\mathbf{U}_j\rangle$ corresponding to different eigenvalues ω_j are orthogonal i.e. $\langle \mathbf{V}_l|\mathbf{U}_j\rangle = \delta_{l,j} = \begin{cases} 1, & l=j; \\ 0, & l\neq j. \end{cases}$ and $\sum_{j=0}^{I} |\mathbf{U}_j\rangle\langle \mathbf{V}_j| = \mathbf{I}$.

According to the references [20–24], the analytical formula of the eigenvectors $|\mathbf{U}_j\rangle$ of coefficient matrix $\mathbf{N}$ can be determined from $(\mathbf{N} - \omega_j\mathbf{I})|\mathbf{U}_j\rangle = 0$ as

$$ |\mathbf{U}_j\rangle = \sigma_j \begin{pmatrix} 1 \\ \frac{\mu_1}{(\omega_j + \lambda_1)} \\ \frac{\mu_2}{(\omega_j + \lambda_2)} \\ \vdots \\ \frac{\mu_I}{(\omega_j + \lambda_I)} \end{pmatrix}, \tag{23} $$

and the eigenvectors $\langle \mathbf{V}_j|$ of the adjoint matrix, the transport of coefficient matrix, $\mathbf{N}^T = \mathbf{N}^{\dagger}$ can be determined by solve the equation $[(\mathbf{N} - \omega_j\mathbf{I})|\mathbf{V}_j\rangle]^{\dagger} = 0$ or $\langle \mathbf{V}_j|(\mathbf{N}^T - \omega_j\mathbf{I}) = 0$ as

$$ \langle \mathbf{V}_j| = \sigma_j \begin{pmatrix} 1 & \frac{\lambda_1}{(\omega_j + \lambda_1)} & \frac{\lambda_2}{(\omega_j + \lambda_2)} & \cdots & \frac{\lambda_I}{(\omega_j + \lambda_I)} \end{pmatrix} \tag{24} $$

By the normalization condition $\langle \mathbf{V}_j|\mathbf{U}_j\rangle = 1$ then $\sigma_j = (1 + \sum_{i=1}^{I} \frac{\mu_i\lambda_i}{(\omega_j+\lambda_i)^2})^{-\frac{1}{2}}$, $\forall j$.

The exact analytical solution of Eq. (21) is

$$|\psi_0(t)\rangle = \exp(\mathbf{N}t)|\psi_0(0)\rangle = \sum_{j=0}^{I} e^{\omega_j t}|\mathbf{U}_j\rangle\langle\mathbf{V}_j|\psi_0(0)\rangle \tag{25}$$

where $|\psi_0(0)\rangle$ is the initial state vector of reactor.

Eq. (25) represents the exact solution of the point kinetics model or zero order of relaxation time. Generally, in the point kinetics model, the reactivity ρ is a time dependent. For this case, we have an n time intervals, $t_{m+1} - t_m = h$ and $t = t_n = nh$. Through the time interval $[t_m, t_{m+1}]$, the reactivity is considered constant. Then, the solution of the point kinetics model with time dependent reactivity takes the form

$$|\psi_0(t_{m+1})\rangle = \exp(\mathbf{N}h)|\psi_0(t_m)\rangle = \sum_{j=0}^{I} e^{\omega_j h}|\mathbf{U}_j\rangle\langle\mathbf{V}_j|\psi_0(t_m)\rangle \tag{26}$$

3.2. First order of relaxation time

The first order of relaxation time, τ^κ, is

$$\frac{d}{dt}|\psi_1(t)\rangle = \mathbf{N}|\psi_1(t)\rangle - \frac{d^\kappa q(t)}{dt^\kappa}|1\rangle \tag{27}$$

The final term can be calculated from Eqs. (16) and (21) as

$$\frac{d^\kappa q(t)}{dt^\kappa} = -\alpha\frac{d^\kappa p(t)}{dt^\kappa} \tag{28}$$

Substituting from Eq. (25) yields

$$\frac{d^\kappa p(t)}{dt^\kappa} = \frac{d^\kappa}{dt^\kappa}\sum_{j=0}^{I} e^{\omega_j t}\sigma_j\langle\mathbf{V}_j|\psi_0(0)\rangle \tag{29}$$

Let us use the Riemann–Liouville fractional derivatives [19]:

$$\frac{d^\kappa p(t)}{dt^\kappa} = \sum_{j=0}^{I} t^{-\kappa}E_{1,1-\kappa}(\omega_j t)\sigma_j\langle\mathbf{V}_j|\psi_0(0)\rangle \tag{30}$$

where $E_{1,1-\kappa}(\omega_j t)$ is the Mittag-Leffler function which is defined as:

$$E_{a,b}(z) = \sum_{l=0}^{\infty}\frac{z^l}{\Gamma(al+b)}, a, b > 0. \tag{31}$$

The general analytical solution of Eq. (27) takes the form:

$$|\psi_1(t)\rangle = \alpha\exp(\mathbf{N}t)\int_0^t \exp(-\mathbf{N}\xi)\frac{d^\kappa p(\xi)}{d\xi^\kappa}d\xi|1\rangle \tag{32}$$

where $|\psi_1(0)\rangle = 0$.

To calculate the integration, substituting from Eq. (34) we have:

$$I = \int_0^t \exp(-\mathbf{N}\xi)\frac{d^\kappa p(\xi)}{d\xi^\kappa}d\xi = \sum_{j=0}^{I}\sigma_j I_j\langle\mathbf{V}_j|\psi_0(0)\rangle \tag{33}$$

where

$$\begin{aligned} I_j &= \int_0^t \exp(-\mathbf{N}\xi)\xi^{-\kappa}E_{1,1-\kappa}(\omega_j\xi)d\xi \\ &= \sum_{i=0}^{I}\int_0^t e^{-\omega_i\xi}\xi^{-\kappa}E_{1,1-\kappa}(\omega_j\xi)d\xi|\mathbf{U}_i\rangle\langle\mathbf{V}_i| = \sum_{i=0}^{I} I_{j,i}|\mathbf{U}_i\rangle\langle\mathbf{V}_i| \end{aligned} \tag{34}$$

Substituting from Eq. (31) yields:

$$\begin{aligned} I_{j,i} &= \int_0^t e^{-\omega_i\xi}\xi^{-\kappa}E_{1,1-\kappa}(\omega_j\xi)d\xi = \sum_{l=0}^{\infty}\int_0^t \frac{e^{-\omega_i\xi}\xi^{-\kappa}(\omega_j\xi)^l}{\Gamma(l-\kappa+1)}d\xi \\ &= \sum_{l=0}^{\infty}\omega_j^l\int_0^t\frac{e^{-\omega_i\xi}\xi^{l-\kappa}}{\Gamma(l-\kappa+1)}d\xi = \sum_{l=0}^{\infty}\frac{\omega_j^l\gamma(l-\kappa+1,\omega_i t)}{\omega_i^{l-\kappa+1}\Gamma(l-\kappa+1)} \end{aligned} \tag{35}$$

where $\gamma(l-\kappa+1,\omega_i t)$ is lower incomplete gamma function which is defined as:

$$\gamma(s,z) = z^s\Gamma(s)e^{-z}\sum_{l=0}^{\infty}\frac{z^l}{\Gamma(s+l+1)} \tag{36}$$

Substituting from Eqs. (35) and (34) into Eq. (33) we have:

$$I = \sum_{j=0}^{I}\sigma_j\langle\mathbf{V}_j|\psi_0(0)\rangle\sum_{i=0}^{I}\sum_{l=0}^{\infty}\frac{\omega_j^l\gamma(l-\kappa+1,\omega_i t)}{\omega_i^{l-\kappa+1}\Gamma(l-\kappa+1)}|\mathbf{U}_i\rangle\langle\mathbf{V}_i| \tag{37}$$

Substituting into Eq. (32) yields:

$$\begin{aligned} |\psi_1(t)\rangle = \alpha&\sum_{i=0}^{I}\sigma_i e^{\omega_i t}\sum_{j=0}^{I}\sigma_j\sum_{l=0}^{\infty} \\ &\times\frac{\omega_j^l\gamma(l-\kappa+1,\omega_i t)}{\omega_i^{l-\kappa+1}\Gamma(l-\kappa+1)}|\mathbf{U}_i\rangle\langle\mathbf{V}_j|\psi_0(0)\rangle \end{aligned} \tag{38}$$

Let us divide the time t to n intervals as $t_{m+1} - t_m = h$ and $t = t_n = nh$. Through the small time interval $[t_m, t_{m+1}]$, the Eq. (38) becomes:

$$\begin{aligned} |\psi_1(t_{m+1})\rangle \simeq \alpha&\sum_{i=0}^{I}\sigma_i e^{\omega_i h}\sum_{j=0}^{I}\sigma_j\sum_{l=0}^{L} \\ &\times\frac{\omega_j^l\gamma(l-\kappa+1,\omega_i h)}{\omega_i^{l-\kappa+1}\Gamma(l-\kappa+1)}|\mathbf{U}_i\rangle\langle\mathbf{V}_j|\psi_0(t_m)\rangle \end{aligned} \tag{39}$$

Substituting from Eqs. (26) and (39) into Eq. (19) we get:

$$\begin{aligned} |\Psi(t_{m+1})\rangle \simeq &\sum_{i=0}^{I}e^{\omega_i h}|\mathbf{U}_i\rangle\langle\mathbf{V}_i|\psi_0(t_m)\rangle + \tau^k\alpha\sum_{i=0}^{I}\sigma_i e^{\omega_i h}\sum_{j=0}^{I}\sigma_j\sum_{l=0}^{L} \\ &\times\frac{\omega_j^l\gamma(l-\kappa+1,\omega_i h)}{\omega_i^{l-\kappa+1}\Gamma(l-\kappa+1)}|\mathbf{U}_i\rangle\langle\mathbf{V}_j|\psi_0(t_m)\rangle \end{aligned}$$

4. Results and discussion

The behavior of the fractional modified point kinetics equations with multi-group of delayed neutrons are presented for pressurized water reactor (PWR) [14,18]. The delayed neutron parameters of this reactor are: $\tau = 10^{-4}(s)$, $\lambda_1 = 0.0127$, $\lambda_2 = 0.0317$, $\lambda_3 = 0.115$, $\lambda_4 = 0.311$, $\lambda_5 = 1.4$, $\lambda_6 = 3.87\ (s^{-1})$, $\beta_1 = 0.000266$, $\beta_2 = 0.001491$, $\beta_3 = 0.001316$, $\beta_4 = 0.002849$, $\beta_5 = 0.000896$, $\beta_6 = 0.000182$, $\beta = 0.007$, $\Lambda = 2.0 \times 10^{-5}$ (s) and $\alpha = 2220.0\ (s^{-1})$. The initial conditions are $p(0) = p_0 = 1.0$, $c_i(0) = \frac{\beta_i p_0}{\Lambda\lambda_i}$, and $\frac{dp(t)}{dt}|_{t=0} = \frac{dc_i(t)}{dt}|_{t=0} = 0$, $i = 1, 2, 3, \ldots, 6$. The neutron density is calculated for three types of reactivity: step, ramp and sinusoidal reactivities.

Figs. (1)–(4) show the neutron density for a step reactivity $\rho = -0.007 = -1(\$)$, $\rho = -0.003 = -0.429(\$)$, $\rho = 0.003 = 0.429(\$)$ and $\rho = 0.007 = 1(\$)$ respectively. As shown in these

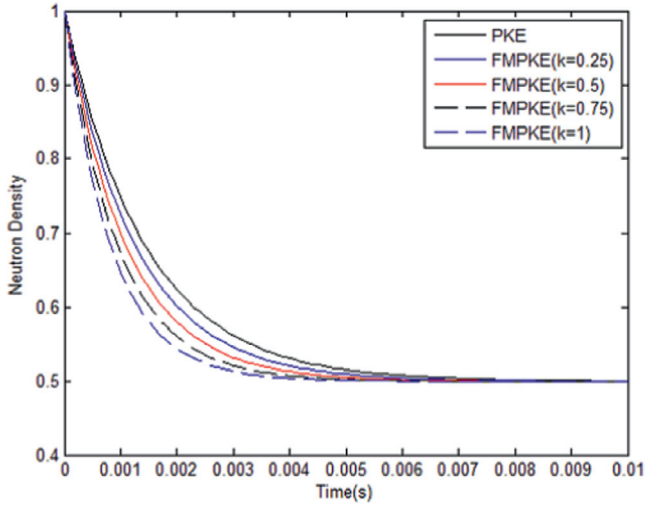

Fig. 1 Neutron density for negative step reactivity $\rho = -0.007$.

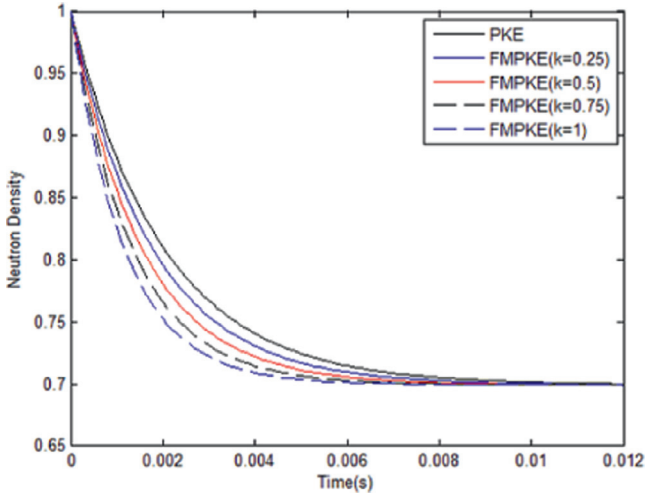

Fig. 2 Neutron density for negative step reactivity $\rho = -0.003$.

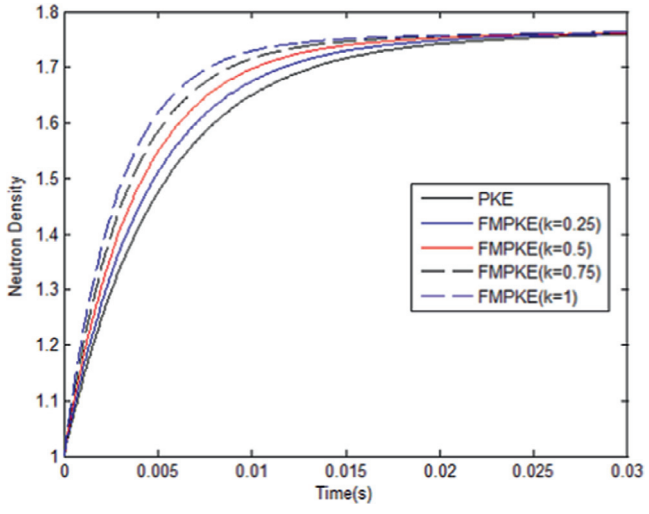

Fig. 3 Neutron density for positive step reactivity $\rho = +0.003$.

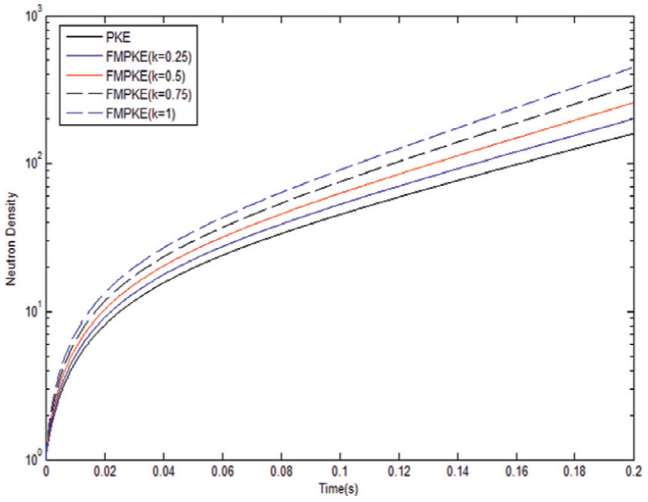

Fig. 4 Neutron density for positive step reactivity $\rho = +0.007$.

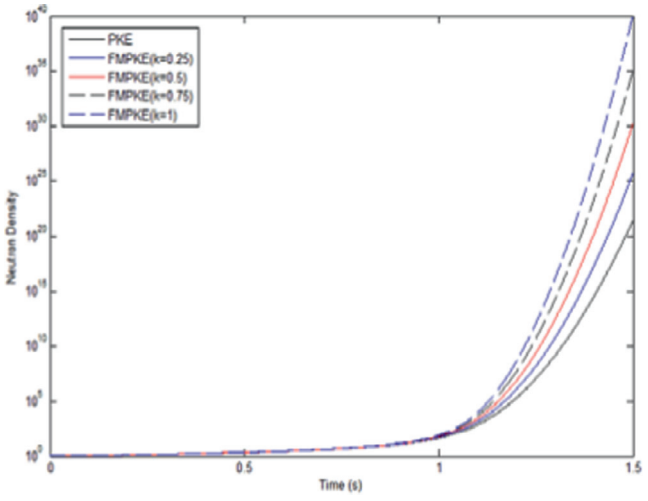

Fig. 5 Neutron density for ramp reactivity $\rho = \beta t$.

figures, the results of fractional modified point kinetics equations (FMPKE) with fractional order $\kappa = 0.25, 0.5, 0.75$ and 1.0 are compared with the results of the point kinetics equations (PKE). The important fact in the presented Figs. (1)–(3), is that there is a relaxation in the neutron density of the FMPKE compared with the PKE at the beginning. Figs. (1) and (2) shows for negative reactivity the FMPKE presents sub diffusive effects, i.e., greater resistance to movement of the neutron respect to PKE due to the decreasing of the neutron density with increasing the fractional order, in addition to the negative term including the relaxation time and the fraction rate of change of the neutron density. On the other side, when the positive reactivity is applied, apparently showing a super diffusive behavior due to the increasing of the neutron density with time for PKE model, in addition to the relaxation time and the fraction rate of change for FMPKE model as show in Figs. (3) and (4). At a large positive reactivity, Fig. (4) shows a good representation for the relaxation in the neutron density of the FMPKE and PKE, which confirm that the FMPKE is the best representation for supercritical reactor.

The effects of the ramp, $\rho = \beta t = 1.0(\$/s)$, and sinusoidal reactivity, $\rho = \beta sin\left(\frac{2\pi}{5}t\right)$, on the FMPKE are also included and shown in Figs. (5) and (6), respectively. The effect of the

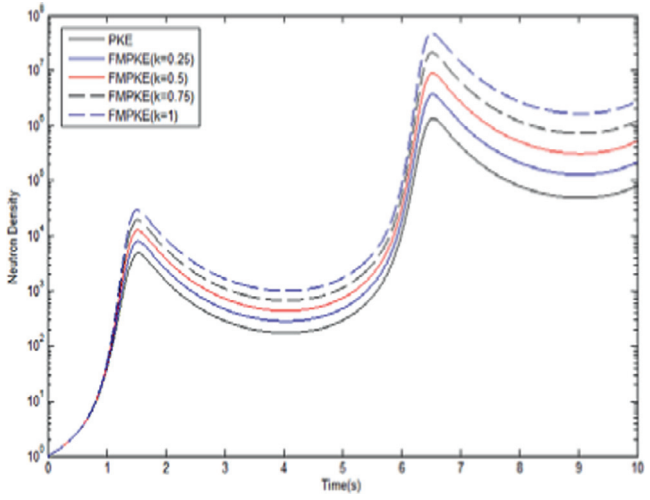

Fig. 6 Neutron density for sinusoidal reactivity $\rho = \beta sin(\frac{2\pi t}{5})$.

relaxation time on the neutron density for time varying reactivity using FMPKE is also satisfied and clear.

5. Conclusion

Fractional neutron point kinetic model for the FMPKE has been analyzed for the dynamic behavior of the neutron motion in which the relaxation time associated with a variation in the neutron flux involves a fractional order acting as exponent of the relaxation time, to obtain the best operation of a nuclear reactor dynamics. The numerical stability of the results for neutron dynamic behavior for subcritical reactivity, supercritical step and time varying reactivity and for different values of fractional order are shown and compared with the classic neutron point kinetic equations (PKE). The fractional model retains the main dynamic characteristics of the neutron motion in which the relaxation time associated with a rapid variation in the neutron flux contains a fractional order, acting as exponent of the relaxation time, to obtain the best representation of a nuclear reactor dynamics.

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