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Original Article

# Unsteady boundary layer flow of a nanofluid over a stretching/shrinking sheet with a convective boundary condition

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Stretching/shrinking sheet;
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Nanofluid

**Abstract** The unsteady boundary layer flow of a nanofluid past a stretching/shrinking sheet with a convective surface boundary condition is studied. The effects of the unsteadiness parameter, stretching/shrinking parameter, convective parameter, Brownian motion parameter and thermophoresis parameter on the local Nusselt number are investigated. Numerical solutions to the governing equations are obtained using a shooting method. The results for the local Nusselt number are presented for different values of the governing parameters. The local Nusselt number decreases as the stretching/shrinking parameter increases. The local Nusselt number is consistently higher for higher values of the convective parameter but lower for higher values of the unsteadiness parameter, Brownian motion parameter and thermophoresis parameter.

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## 1. Introduction

The term “nanofluid” which was first used by Choi and Eastman [1] refers to the dispersions of nanometer-sized particles in a base fluid such as water, ethylene glycol and propylene glycol, to increase their thermal conductivities. Nanofluids have attracted much attention as a new generation of coolants for various industrial and automotive applications. As a result,

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many papers on nanofluids have been published, such as the papers by Xuan and Li [2], Xuan and Roetzel [3], Eastman et al. [4], Tiwari and Das [5] and Buongiorno [6]. In his paper, Buongiorno [6] developed an analytical model for convective transport in nanofluids which takes into account the Brownian diffusion and thermophoresis effects. Buongiorno model was used in many recent papers, e.g. Neild and Kuznetsov [7–9] and Bachok et al. [10, 11], among others.

The boundary layer flow over a stretching sheet is important in applications such as extrusion, wire drawing, metal spinning, hot rolling, etc. [12]. The flow over a stretching sheet was first studied by Crane [13] who presented an exact analytical solution for the steady two-dimensional flow over a stretching plate in a quiescent fluid. However, recently, the study on the flow over a shrinking sheet has garnered considerable attention. Miklavčič and Wang [14] initiated the study of flow over a shrinking sheet. They found that the vorticity is not confined within a boundary layer and a steady flow cannot exist without exerting adequate suction at the boundary. Ever since, numerous studies emerge, investigating different aspects of this problem such as those studied by Wang [15], Fang [16], and Zaimi and Ishak [17], to name just a few.

In the boundary layer flow and heat transfer analysis, it is customary for the flow to be assumed as steady. Nevertheless, in many engineering applications, unsteadiness becomes an integral part of the problem where the flow becomes time-dependent [11, 18, 19]. Thus, motivated by this, we extend the study of Bachok et al. [11] to the case of convective surface boundary condition. For a long time, constant surface temperature and heat flux are customarily used. However, there are times when heat transfer at the surface relies on the surface temperature, as what mostly occurs in heat exchangers. In this situation, convective boundary condition is used to replace the condition of prescribed surface temperature. Aziz [20] employed the convective boundary condition in his research to study the heat transfer characteristics for the Blasius flow. Ishak [21] introduced the effects of suction and injection at the boundary. Makinde and Aziz [22] investigated the boundary layer flow of a nanofluid past a stretching sheet with a convective surface boundary condition. The dependency of the local Nusselt number on five parameters, namely the stretching/shrinking, unsteadiness, convective, Brownian motion and thermophoresis parameters is the main focus of the present investigation. Numerical solutions are presented graphically and in tabular forms to show the effects of these parameters on the local Nusselt number.

## 2. Mathematical formulation

Consider an unsteady, two-dimensional $(x,y)$ boundary layer flow of a viscous and incompressible fluid over a stretching/shrinking sheet immersed in a nanofluid. It is assumed that at time $t=0$, the velocity of the sheet is $U_w(x,t)=0$. The unsteadiness in the flow field is caused by the time-dependent velocity of the stretching sheet, which is given by $U_w = Ax/t$ where $A>0$, $t>0$ [11, 23–25]. It is also assumed that the constant mass flux velocity is $v_0(x,t)$ with $v_0(x,t)<0$ for suction and $v_0(x,t)>0$ for injection or withdrawal of the fluid. The nanofluid is confined to $y>0$, where $y$ is the coordinate measured normal to the stretching/shrinking surface as shown in Fig. 1. It is further assumed that the bottom surface of the sheet is heated by convection from a hot fluid at temperature $T_f$ which provides a heat transfer coefficient $h$. The surface temperature $T_w$ is the result of a convective heating process characterized by the hot fluid.

The governing equations for the steady conservation of mass, momentum, thermal energy and nanoparticle volume fraction equations can be written as [14]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho_f}\frac{\partial P}{\partial x}+\omega\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho_f}\frac{\partial P}{\partial y}+\omega\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}&=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)\\ &+\tau\left\{D_B\left(\frac{\partial \varphi}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial \varphi}{\partial y}\frac{\partial T}{\partial y}\right)+\frac{D_T}{T_\infty}\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right]\right\}\end{aligned} \tag{4}$$

$$\frac{\partial \varphi}{\partial t}+u\frac{\partial \varphi}{\partial x}+v\frac{\partial \varphi}{\partial y}=D_B\left(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{5}$$

where $u$ and $v$ are the velocity components along the $x$- and $y$-axis, respectively, $P$ is the fluid pressure, $T$ is the fluid temperature, $\alpha$ is the thermal diffusivity, $\omega$ is the kinematic viscosity, $D_B$ is the Brownian diffusion coefficient, $D_T$ is the thermophoresis diffusion coefficient and $\varphi$ is the nanoparticle volume fraction. Furthermore, $\tau=(\rho c)_p/(\rho c)_f$ is the ratio of the effective heat capacity of the particles to that of the fluid with $\rho$ and $c$ being the density and the specific heat at constant pressure, respectively. The subscript $\infty$ represents the values at large values of $y$ (outside the boundary layer). Details of the derivation of Eqs. (4) and (5) are given in the papers by Buongiorno [6] and Nield and Kuznetsov [8].

Eqs. (1)–(5) are subjected to the following boundary conditions [11,20]:

$$\begin{aligned}&t=0:\ v(x,y,t)=0,\ u(x,y,t)=0,\ T(x,y,t)=T_w,\\ &\qquad \varphi(x,y,t)=\varphi_w\\ &t>0:\ \begin{cases} v(x,t)=v_0(x,t),\ u(x,t)=\sigma U_w(x,t),\\ -k\frac{\partial T}{\partial y}=h\left(T_f-T_w\right),\quad \varphi(x,t)=\varphi_w \quad \text{at} \quad y=0\end{cases}\end{aligned} \tag{6}$$

$$\begin{aligned}&u(x,y,t)\to 0,\quad v(x,y,t)\to 0,\quad T(x,y,t)\to T_\infty,\\ &\varphi(x,y,t)\to\varphi_\infty \quad \text{as} \quad y\to\infty\end{aligned} \tag{7}$$

where $\sigma$ is the stretching/shrinking velocity with $\sigma>0$ for a stretching sheet and $\sigma<0$ for a shrinking sheet and $k$ is the thermal conductivity of the base fluid. The subscript w denotes the values at the solid surface. The governing Eqs. (1)–(5) subjected to the boundary conditions (6) and (7) can be expressed in a simpler form by introducing the following transformation:

$$\begin{aligned}&\psi=Ax(\omega/t)^{1/2}f(\eta),\quad \eta=(\omega t)^{-1/2}y,\\ &\theta(\eta)=\frac{T-T_\infty}{T_f-T_\infty},\quad \beta(\eta)=\frac{\varphi-\varphi_\infty}{\varphi_w-\varphi_\infty}\end{aligned} \tag{8}$$

where $\eta$ is the similarity variable and $\psi$ is the stream function defined as $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$, which identically satisfies Eq. (1). By employing the boundary layer approximations

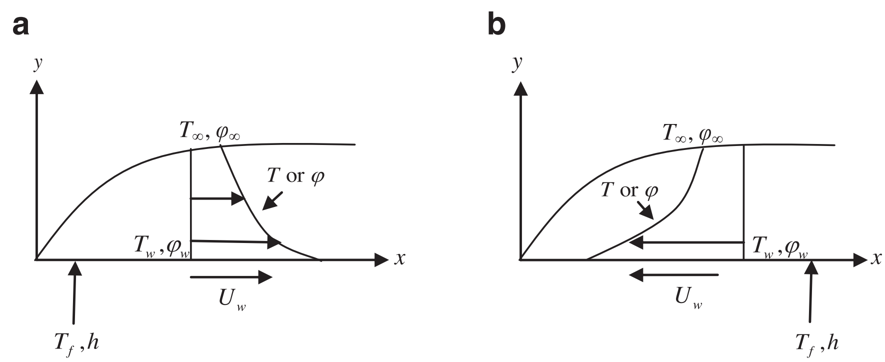


**Fig. 1** Geometry of the problem for (a) stretching and (b) shrinking sheets.

and the similarity variables (8), Eqs. (2)–(5) reduce to the following nonlinear ordinary differential equations:

$$f''' + A\left(ff'' - f'^2\right) + f' + \frac{\eta}{2}f'' = 0 \tag{9}$$

$$\frac{1}{Pr}\theta'' + \left(Af + \frac{\eta}{2}\right)\theta' + Nb\beta'\theta' + Nt\theta'^2 = 0 \tag{10}$$

$$\beta'' + \frac{Nt}{Nb}\theta'' + Le\left(Af + \frac{\eta}{2}\right)\beta' = 0 \tag{11}$$

and the boundary conditions (6) and (7) become

$$f(0) = S, \quad f'(0) = \sigma, \quad \theta'(0) = -\gamma[1 - \theta(0)], \quad \beta(0) = 1 \tag{12}$$

$$f' = 0, \quad \theta = 0, \quad \beta = 0 \text{ as } \eta \to \infty \tag{13}$$

where primes denote differentiation with respect to $\eta$. Further, $Pr$ is the Prandtl number, $Nb$ is the Brownian motion parameter, $Nt$ is the thermophoresis parameter, $Le$ is the Lewis number, $S$ is the mass flux parameter with $S > 0$ for suction and $S < 0$ for injection and $\gamma$ is the Biot number (convective parameter), which are defined as

$$Pr = \frac{\omega}{\alpha}, \quad Nb = \frac{\tau D_B(\varphi_w - \varphi_\infty)}{\omega}, \quad Nt = \frac{\tau D_T(T_f - T_\infty)}{\omega T_\infty}$$
$$Le = \frac{\omega}{D_B}, \quad S = -\frac{v_0(x,t)}{A\sqrt{\omega/t}}, \quad \gamma = c\sqrt{\omega}/k \tag{14}$$

where we take $h = c/\sqrt{t}$, to obtain similarity solution. When $Nb = Nt = 0$, the present problem reduces to a regular viscous fluid, and the nanoparticle volume fraction Eq. (11) becomes ill-posed and is of no physical significance. Furthermore, Eq. (9) for $A = 1$ becomes identical with Eq. (6) found by Fang et al. [19] for the unsteady two-dimensional shrinking sheet when their unsteadiness parameter $\beta = -1$.

The physical quantities of interest are the skin friction coefficient $C_f$ and the local Nusselt number $Nu_x$ which are defined as

$$C_f = \frac{\tau_w}{\rho U_w^2}, \quad Nu_x = \frac{x q_w}{k(T_f - T_\infty)} \tag{15}$$

where $\tau_w$ and $q_w$ are the surface shear stress and heat flux, respectively, which are given by [20]

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{16}$$

Using the similarity variables (8), we obtain

$$C_f Re_x^{1/2} = A^{-1/2} f''(0), \quad Nu_x Re_x^{-1/2} = -A^{-1/2}\theta'(0) \tag{17}$$

where $Re_x = U_w x/\omega$ is the local Reynolds number.

**Table 1** Values of $\sigma_c$ for different $A$ and $S$.

| $A$ | $S$ | $\sigma_c$ |
|---|---|---|
| 1.0 | 2 | −0.925998 |
| 1.1 | | −1.027556 |
| 1.2 | | −1.129323 |
| 1.0 | 3 | −2.152694 |
| | 4 | −3.888846 |

## 3. Results and discussions

The set of ordinary differential Eqs. (9)–(11) with the boundary conditions (12) and (13) were solved numerically using a shooting method. In this method, the dual solutions are obtained by setting different initial guesses for the values of $f''(0)$, $-\theta'(0)$ and $-\beta'(0)$, where all profiles satisfy the far field boundary conditions (13) asymptotically but with different boundary layer thicknesses. The problem for a regular (viscous) fluid involves five parameters: Prandtl number, stretching/shrinking, suction/injection, unsteadiness and convective parameters. The asymptotic boundary conditions (13) at $\eta = \infty$ are replaced by $\eta = 15$ as customary in the boundary layer analysis. This choice is adequate for the velocity and temperature profiles to reach the far field boundary conditions asymptotically.

Variation of the skin friction coefficients and local Nusselt number ($C_f Re_x^{1/2}$ and $Nu_x Re_x^{-1/2}$ respectively) with $\sigma$ for S = 2, $Le = 2$, $Nt = Nb = 0.5$, $Pr = 6.8$ and different values of the acceleration parameter $A$ and the convective parameter $\gamma$ are shown in Figs. 2–4. As can be seen in these figures, there are more than one solution obtained for a fixed value of $\sigma$. When $\sigma$ is equal to a certain value $\sigma = \sigma_c$ where $\sigma_c$ (<0) is the critical value of $\sigma$, there is only one solution, and when $\sigma < \sigma_c$, there is no solution. Based on our computations, the values of $\sigma_c$ are computed in Table 1. Furthermore, the values of $\sigma_c$ for $S = 3$ and $S = 4$ are also included in Table 1 for future reference. It is noted that the unsteadiness and mass suction parameters widen the range of $\sigma$ for which the solution exists. However, it is also seen that the values of $\sigma_c$ remain constant for different values of $\gamma$. This is clear from Eqs. (9)–(13) where the thermal field does not affect the flow field.

From Figs. 2– 4, the skin friction coefficient and the local Nusselt number (surface heat transfer rate) change with the variations of $\sigma$, $A$ and $\gamma$. The skin friction coefficient and the local Nusselt number generally decrease as $\sigma$ increases. The local Nusselt number is also consistently higher for a nanofluid

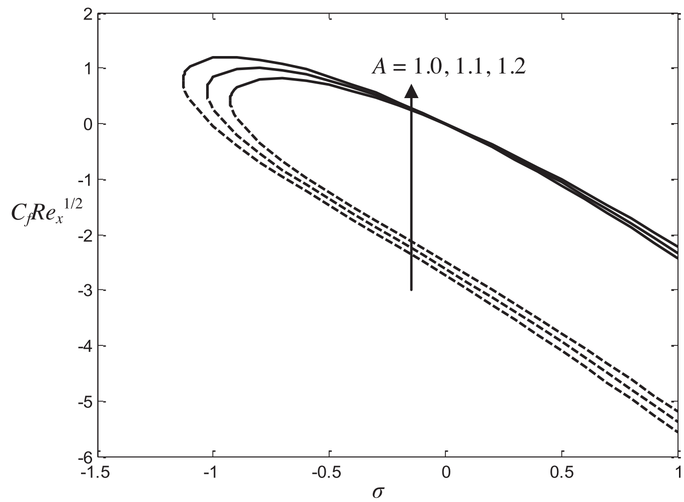


**Fig. 2** Variation of the skin friction coefficient with $\sigma$ for different values of $A$ when $S=2$.

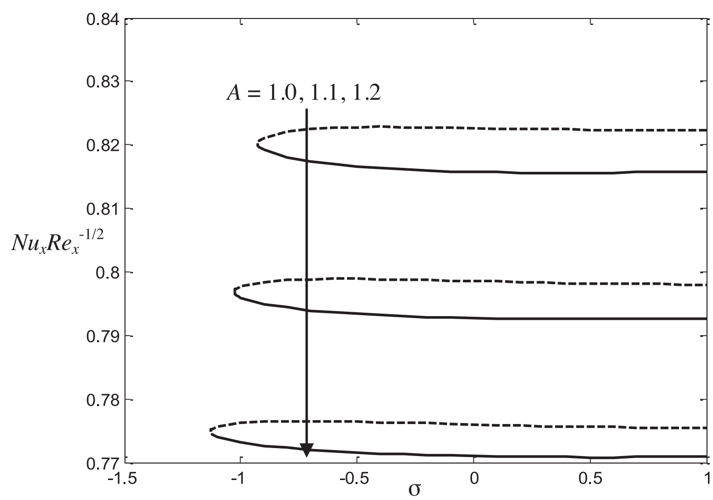


**Fig. 3** Variation of the local Nusselt number with $\sigma$ for different values of $A$ when $S=2$, $Pr=6.8$, $\gamma=1$, $Le=2$, $Nb=Nt=0.5$.

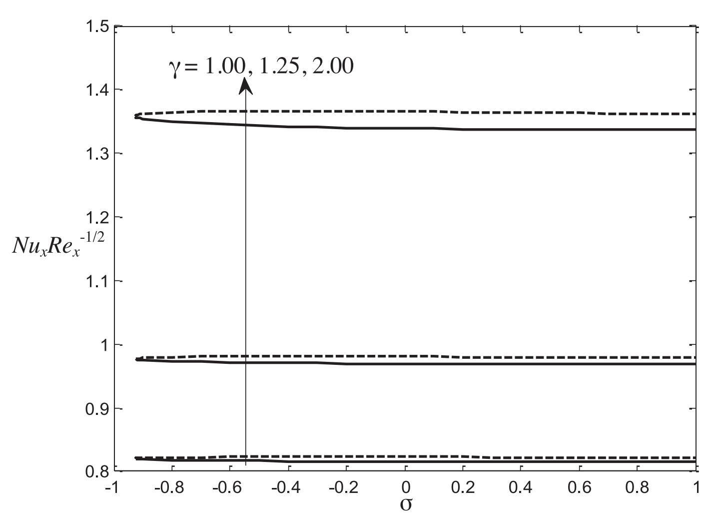


**Fig. 4** Variation of the local Nusselt number with $\sigma$ for different values of $\gamma$ when $A=1$, $Pr=6.8$, $S=2$, $Le=2$, $Nb=Nt=0.5$.

**Table 2** Variations of the local Nusselt number $Nu_x Re_x^{-1/2}$ for different values of $Nb$ and $Nt$ when $Pr = 6.8$, $S = 2$, $\sigma = -0.1$, $\gamma = 0.1$, $Le = 2$ and $A = 1$.

| $Nb$ | $Nt$ | $Nu_x Re_x^{-1/2}$ |
|---|---|---|
| 0.1 | 0.2 | 0.099150 |
| 0.3 | | 0.098751 |
| 0.5 | | 0.098012 |
| 0.2 | 0.1 | 0.098980 |
| | 0.3 | 0.098978 |
| | 0.5 | 0.098976 |

with higher values of $\gamma$. This phenomenon can be explained by expression (14) where $\gamma$ is directly proportional to the heat transfer coefficient $h$ where it is implied that the former is inversely proportional to thermal resistance. Hence, the thermal resistance becomes weaker as the convective parameter intensifies which results in higher surface heat transfer rate.

In this study, two parameters are added, namely the Brownian motion $Nb$ and thermophoresis parameters $Nt$. The effects of these parameters on the local Nusselt number are portrayed in Table 2. The local Nusselt number is consistently lower for higher values of $Nb$ and Nt. This observation agrees with the findings of Nield and Kuznetsov [7– 9]. The local Nusselt number decreases due to the thickening of thermal boundary layer as the Brownian motion parameter and thermophoresis parameter increase.

To support the validity of the numerical results obtained, velocity profiles are shown in Fig. 5 at different values of $A$. These profiles satisfy the far field boundary conditions (13), as well as supporting the existence of the dual solutions shown in Figs. 2– 4. For a similar problem where dual solutions exist, Merkin [26], Weidman et al. [27] and Postelnicu and Pop [28] have shown that the first solution is linearly stable, while the second solution is not. Thus for the present problem, it is expected that only the first solution is stable and physically realizable.

## 4. Conclusions

The unsteady boundary layer flow of a nanofluid past a stretching/shrinking sheet with a convective boundary condition was studied. The effects of the acceleration parameter, stretching/shrinking parameter, convective parameter, Brownian motion parameter and thermophoresis parameter on the local Nusselt number were studied. Numerical solutions to the governing equations were obtained using a shooting method. The results for the local Nusselt number are presented for different values of the governing parameters. The local Nusselt number decreases as the stretching/shrinking parameter increases. It is also consistently higher for higher values of the convective parameter but lower for higher values of the unsteadiness parameter, Brownian motion parameter and thermophoresis parameter. The results also indicate the existence of dual solutions for both stretching and shrinking cases.

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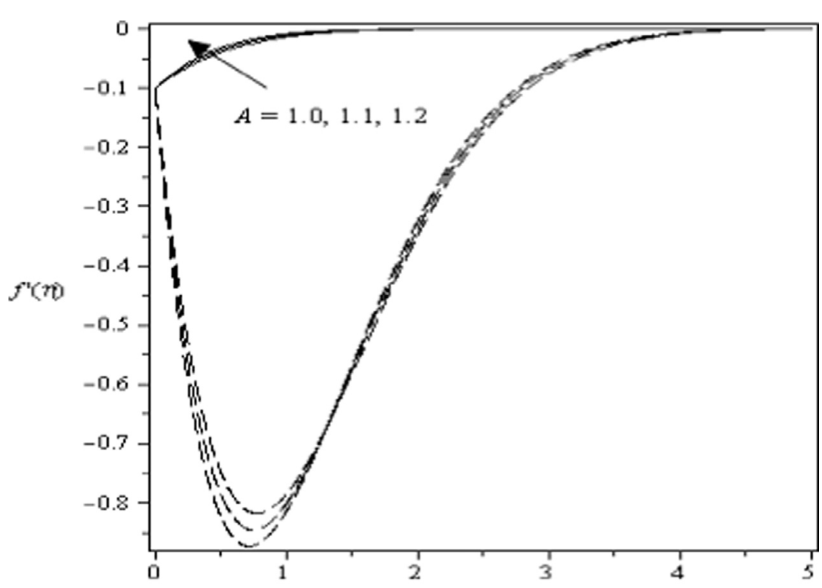


**Fig. 5** Velocity profiles for different values of $A$ when $S = 2$ and $\sigma = -0.1$.

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