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Original Article

A numerical scheme for the generalized Burgers–Huxley equation

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Abstract In this article, a numerical solution of generalized Burgers–Huxley (gBH) equation is approximated by using a new scheme: *modified cubic B-spline differential quadrature method* (MCB-DQM). The scheme is based on *differential quadrature method* in which the weighting coefficients are obtained by using modified cubic B-splines as a set of basis functions. This scheme reduces the equation into a system of first-order *ordinary differential equation* (ODE) which is solved by adopting SSP-RK43 scheme. Further, it is shown that the proposed scheme is stable. The efficiency of the proposed method is illustrated by four numerical experiments, which confirm that obtained results are in good agreement with earlier studies. This scheme is an easy, economical and efficient technique for finding numerical solutions for various kinds of (non)linear physical models as compared to the earlier schemes.

1. Introduction

The (generalized) Burgers–Huxley equation describes a wide class of physical nonlinear phenomena, for instance, a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports [42]. It have finds its applications in many fields such as biology, metallurgy, chemistry, metallurgy, combustion, mathematics and engineering [20,42]. In this article, we concerned with the numerical solution of one dimensional gBH equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha u^\delta \frac{\partial u}{\partial x} + \beta u f_{\delta,\gamma}(u), x \in \Omega, t \geq 0, \tag{1.1}$$

with the initial condition: $u(x,0) = g(x), x \in \Omega$ and boundary conditions: $u(x,t) = \psi_x(t), x \in \partial\Omega, t > 0$, where $\Omega = (a,b)$ and $f_{\delta,\gamma}(u) = (1 - u^\delta)(u^\delta - \gamma)$ is a nonlinear reaction term.

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The coefficient $\beta \geq 0$ and α are reaction and advection coefficient, respectively, $0 < \gamma < 1$ and $\delta > 0$. *Huxley equation* (Eq. (1.1) with $\alpha = 0, \delta = 1$) was proposed to explain the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. Eq. (1.1) with $\alpha = 0$ correspond to the well known Fitz-hugh–Nagoma equation [14], and for $\beta = 0$ Eq. (1.1) becomes a modified Burgers equation [7].

In the recent years, many numerical techniques to approximate nonlinear time dependent partial differential equations have been designed, several technics have been designed to such type of differential equations, see [2,9,23,32,38–40,44,48] and references therein. gBH equation have been studied theoretically/numerically by adopting various techniques. The solitary wave solutions of gBH equation are obtained by Wang et al. [47] using nonlinear transformation. The kink wave solution of gBH Eq. (1.1) as presented in [47] is given by

$$u(x,t) = \left[\frac{\gamma}{2}\{1 + \tanh(k(x - ct))\}\right]^{\frac{1}{\delta}}, \tag{1.2}$$

where the parameters c and k:

$$c = \frac{\alpha\gamma}{\delta + 1} - \frac{(1 + \delta - \gamma)\left[-\alpha + \sqrt{\alpha^2 + 4\beta(\delta + 1)}\right]}{2(\delta + 1)} \quad \text{and}$$

$$k = \frac{\gamma\delta\left[-\alpha + \sqrt{\alpha^2 + 4\beta(\delta + 1)}\right]}{4(\delta + 1)}$$

are the velocity and the wave number respectively.

The solitary wave solutions of the BH equation are obtained by Wazwaz [48] using the tanh-coth method, by Ismail et al. [22] and Hashim et al. [18,19] using Adomian decomposition method (ADM), by Bataineh et al. [4] and Molabahrami and Khami [35] using the homotopy analysis method. The travelling wave solutions for gBH equation were derived by Efimova and Kudryashov [11] using Hope-Cole transformation, by Batiha et al. [3] using variational iteration method (VIM) and by Gao and Zhao [15] using He's Exp-function method. The travelling wave analysis for BH equation have been reported by Griffiths and Schiesser [16]. A class of travelling solitary wave solutions for the gBH equation are obtained by Deng [10].

A large number of techniques have been developed for the numerical simulation of nonlinear BH equation, for instance, spectral collocation method [9,25,26], new domain decomposition algorithm based on Chebyshev polynomials (DDAC) [28], high order finite difference schemes [41], Chebyshev spectral collocation with the domain decomposition [27], a fourth-order finite difference scheme (FDS4) [8], spectral method (SM) [21,23], nodal Galerkin (Gauss Chebyshev Galerkin (GCG), El-Gendi Chebyshev Galerkin (ECG) and El-Gendi Legendre Galerkin (ELG)) methods [29], differential quadrature method (DQM) [32], optimal Homotopy asymptotic method (OHAM) [36], Homotopy analysis method [5], a monotone finite difference scheme [13], B-spline collocation method [33]. Dehghan et al. [12] derived new methods based on the interpolation scaling functions and the mixed collocation difference schemes for the solution of gBH equation. Gupta and Kadalbajoo [17] constructed a monotone finite difference operator for the singularly perturbed Burgers–Huxley equation, it is a natural development of monotone ϵ-convergent schemes for linear boundary value problems with exponential boundary layer. Zhou and Cheng [49] developed a linearly semi-implicit compact scheme for the BH equation with the help of time-splitting method. Recently, Mohanty et al. [34] developed a new two-level implicit operator compact method with accuracy of order two in time and four in space for the numerical simulation of time dependent BH equation.

DQM [6] has been widely used for numerical simulation of a number of (non)linear physical problems. In DQM the weighting coefficients are evaluated using various test functions: *spline functions*, *Lagrange interpolation polynomials, cubic B-splines, modified cubic B-splines and sinc function*, see [1,30,31,37,43,45] and references therein. This article present a numerical solution of gBH Eq. (1.1) approximated by using a new scheme: MCB-DQM [1]. The scheme is based on DQM where modified cubic B-splines are used as basis functions. On implementing DQM, the gBH equation is reduced into a system of first-order ODEs. Keeing stability criteria in mind, SSP-RK43 [46] scheme is used to solve the resulting system of ODEs. The proposed results are computed without using any transformation and linearization process. The efficiency of the proposed method is confirmed by four test problems.

This paper is organized as follows. In Section 2, the description of the modified cubic B-spline differential quadrature method is given. In Section 2.2, procedure for implementation of method is described. Four test problems are illustrated to establish the applicability and accuracy of the proposed method in Section 3. Section 4 concludes the article.

2. Description of modified cubic B-spline DQM method

The modified cubic B-spline differential quadrature method is the differential quadrature method (DQM is approximation to the derivatives of a function using the weighted sum of the functional values at certain discrete points) in which the weighting coefficients are obtained by using modified cubic B-spline functions as a set of basis functions. Since the weighting coefficients are dependent on the spatial grid spacing only, one can assume N grid points on the real axis distributed uniformly, that is, $a = x_1 < x_2, \ldots, x_{N-1} < x_N = b$ with $x_{i+1} - x_i = h$. The solution $u(x,t)$ at any time on the knot x_i is $u(x_i,t)$ for $i = 1, \ldots, N$. The approximate value of first and second order spatial derivatives are given by

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_i} = \sum_{j=1}^{N} a_{ij} u(x_j,t), \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_i} = \sum_{j=1}^{N} b_{ij} u(x_j,t), \quad i = 1, \ldots, N \tag{2.1}$$

where a_{ij} and b_{ij} are weighting coefficients of the first and second order derivatives with respect to x, respectively [6].

The cubic B-spline basis functions at the knots are defined as follows

$$\varphi_j(x) = \frac{1}{h^3}\begin{cases} (x - x_{j-2})^3 & x \in [x_{j-2}, x_{j-1}) \\ (x - x_{j-2})^3 - 4(x - x_{j-1})^3 & x \in [x_{j-1}, x_j) \\ (x_{j+2} - x)^3 - 4(x_{j+1} - x)^3 & x \in [x_j, x_{j+1}) \\ (x_{j+2} - x)^3 & x \in [x_{j+1}, x_{j+2}) \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

where $\{\varphi_0, \varphi_1, \ldots, \varphi_N, \varphi_{N+1}\}$ forms a basis over the region $[a, b]$.

Lemma 2.1. *The numerical values of φ_i and its derivatives φ_i', φ_i'' at jth nodal point are evaluated as*

$$\varphi_i(x_j) = \begin{cases} 4, & if\, i-j=0 \\ 1, & if\, i-j=\pm 1, \\ 0, & else \end{cases} \quad \varphi_i'(x_j) = \begin{cases} \pm 3/h, & i-j=\pm 1 \\ 0, & else \end{cases},$$

$$\text{and} \quad \varphi_i''(x_j) = \begin{cases} -12/h^2, & i-j=0 \\ 6/h^2, & i-j=\pm 1 \\ 0 & else. \end{cases}$$

The cubic B-spline basis functions are modified to result a diagonally dominant matrix system of equations. The modified cubic B-spline basis functions at the knots are defined as follows:

$$\left.\begin{array}{l} \phi_1(x) = \varphi_1(x) + 2\varphi_0(x);\ \phi_2(x) = \varphi_2(x) - \varphi_0(x) \\ \phi_j(x) = \varphi_j(x) \text{ for } j = 3, \dots, N-2 \\ \phi_{N-1}(x) = \varphi_{N-1}(x) - \varphi_{N+1}(x);\ \phi_N(x) = \varphi_N(x) + 2\varphi_{N+1}(x) \end{array}\right\} \tag{2.3}$$

where $\{\phi_1, \phi_2, \dots, \phi_N\}$ forms a basis over the region $[a, b]$.

2.1. *Computation of the weighting coefficients* $\mathrm{a_{ij}}$ *and* $\mathrm{b_{ij}}$

The first order derivative approximation at the grid point x_i, $i = 1, \dots, N$ is

$$\phi_k'(x_i) = \sum_{j=1}^{N} a_{ij}\phi_k(x_j), \quad k = 1, \dots, N, \tag{2.4}$$

From Lemma 2.1 and Eqs. (2.3) and (2.4) is reduced into a tridiagonal system of equations as

$$A\vec{a}[i] = \vec{H}[i], \text{ for } i = 1, \dots, N, \tag{2.5}$$

where A is the coefficients matrix given by

$$A = \begin{bmatrix} 6 & 1 & & & & & \\ 0 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 0 \\ & & & & & 1 & 6 \end{bmatrix},$$

$\vec{a}[i]$ denotes the weighting coefficient vector corresponding to grid point x_i, that is, $\vec{a}[i] = [a_{i1}, a_{i2} \dots a_{iN}]^T$, and the coefficient vector $\vec{H}[i] = [h_{i1}, h_{i2} \dots h_{iN}]^T$ corresponding to x_i, $i = 1, 2, \dots, N$ are evaluated as $\vec{H}[1] = [-6/h, 6/h, 0, \dots, 0]^T$, $\vec{H}[2] = [-3/h, 0, 3/h, 0, \dots, 0]^T$, $\dots$, $\vec{H}[N-1] = [0, \dots, 0, -3/h, 0, 3/h]^T$ and $\vec{H}[N] = [0, \dots, 0, -6/h, 6/h]^T$.

Now, we apply the well known "Thomas algorithm" to solve the resulting tridiagonal system of equations which provides the vector $\vec{a}[i]$, that is, the weighting coefficients $a_{i1}, a_{i2}, \dots, a_{iN}$, for $i = 1, \dots N$. Using the coefficients a_{ij}, the weighting coefficient b_{ij} for $i = 1, 2, \dots, N;\ j = 1, 2, \dots, N$ is evaluated as follows [45]

$$b_{ij} = 2a_{ij}\left(a_{ij} - \frac{1}{x_i - x_j}\right), \text{ for } i \neq j, \text{ and } b_{ii} = -\sum_{i=1, i\neq j}^{N} b_{ij}.$$

2.2. *Implementation of method for gBH equation and stability analysis*

On substituting the first and second order approximation of the spatial derivatives, obtained by using MCB-DQM, gBH Eq. (1.1) can be rewritten as

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{N} b_{ij}u_j - \alpha u_i^\delta \sum_{j=1}^{N} a_{ij}u_j + \beta u_i f_{\delta,\gamma}(u_i), \tag{2.6}$$

where $u(x_i, t) = u_i$. After implementing the boundary conditions: $u(a, t) = \psi_a(t)$ and $u(b, t) = \psi_b(t)$, Eq. (2.6) with initial conation can be re-written as

$$\begin{cases} \frac{\partial u_i}{\partial t} = \sum_{j=2}^{N-1} b_{ij}u_j - \alpha u_i^\delta \sum_{j=2}^{N-1} a_{ij}u_j + G_i, \\ u(x_i, t=0) = u_i^0, \quad i = 2, 3, \dots N-1, \end{cases} \tag{2.7}$$

where $G_i = b_{i1}u_1 + b_{iN}u_N - \alpha u_i^\delta(a_{i1}u_1 + a_{iN}u_N) + \beta u_i f_{\delta,\gamma}(u_i)$. This equation is the resulting system of first order ordinary differential equations. In matrix form the above system of ordinary differential equations is written as

$$\begin{cases} \frac{\partial \vec{U}}{\partial t} = M\vec{U} + G, \\ U(t=0) = U_0, \end{cases} \tag{2.8}$$

where $\vec{U} = (u_2, u_3, \dots, u_{N-1})^T$, $M_{ij} = b_{ij} - \alpha_i a_{ij}$, $\alpha_i = \alpha u_i^\delta$. Keeping in mind the memory allocation, computational cost, accuracy and stability criteria SSP-RK43 scheme [46] as mentioned below, is preferred to solve the resulting system.

$$u^{(1)} = u^m + \frac{\Delta t}{2}L(u^m);\ u^{(2)} = u^{(1)} + \frac{\Delta t}{2}L(u^{(1)});\ u^{(3)} = \frac{2}{3}u^m + \frac{u^{(2)}}{3} + \frac{\Delta t}{6}L(u^{(2)}) \text{and } u^{m+1} = u^{(3)} + \frac{\Delta t}{2}L(u^{(3)}).$$

Moreover, the stability of the proposed scheme is directly depends upon the stability of the system of Eq. (2.8). Noticed that if the system of ordinary differential Eq. (2.8) is unstable, then the numeric scheme for temporal discretization may not be generate converged solution. The stability of (2.8) is depends on the eigen values of the coefficient matrix M [24,31]. Hence, to show the stability of the exact solution $\{U\}$ of (1.1), it is sufficient to show that the real part, $Re(\lambda_i)$ of every eigenvalue λ_i of the matrix M is non-positive, i.e., $Re(\lambda_i) \leq 0$ for all $i = 1, 2, \dots n$, for more details, see [24].

In Fig. 1, the eigenvalues of the matrices A and B are depicted by taking grid points 10, 20, 30 and 40. It shows that the computed eigenvalues of B are either zero or negative, and that of A are imaginary with real part zero. It confirms that the eigenvalues of M are either negative reals or complex with negative real part, and hence the proposed scheme for gBH equation is stable.

3. Numerical experiments and discussion

In this section the accuracy and the efficiency of MCB-DQM, for numeric solutions of generalized gBH Eq. (1.1) is analysed by evaluating the discrete absolute errors and maximum error L_∞ norms:

$$E_{abs}(x_i) = \left|u_i^{exact} - u_i^*\right| \text{ and } L_\infty = \max_{i=1}^{N}\left|u_i^{exact} - u_i^*\right|,$$

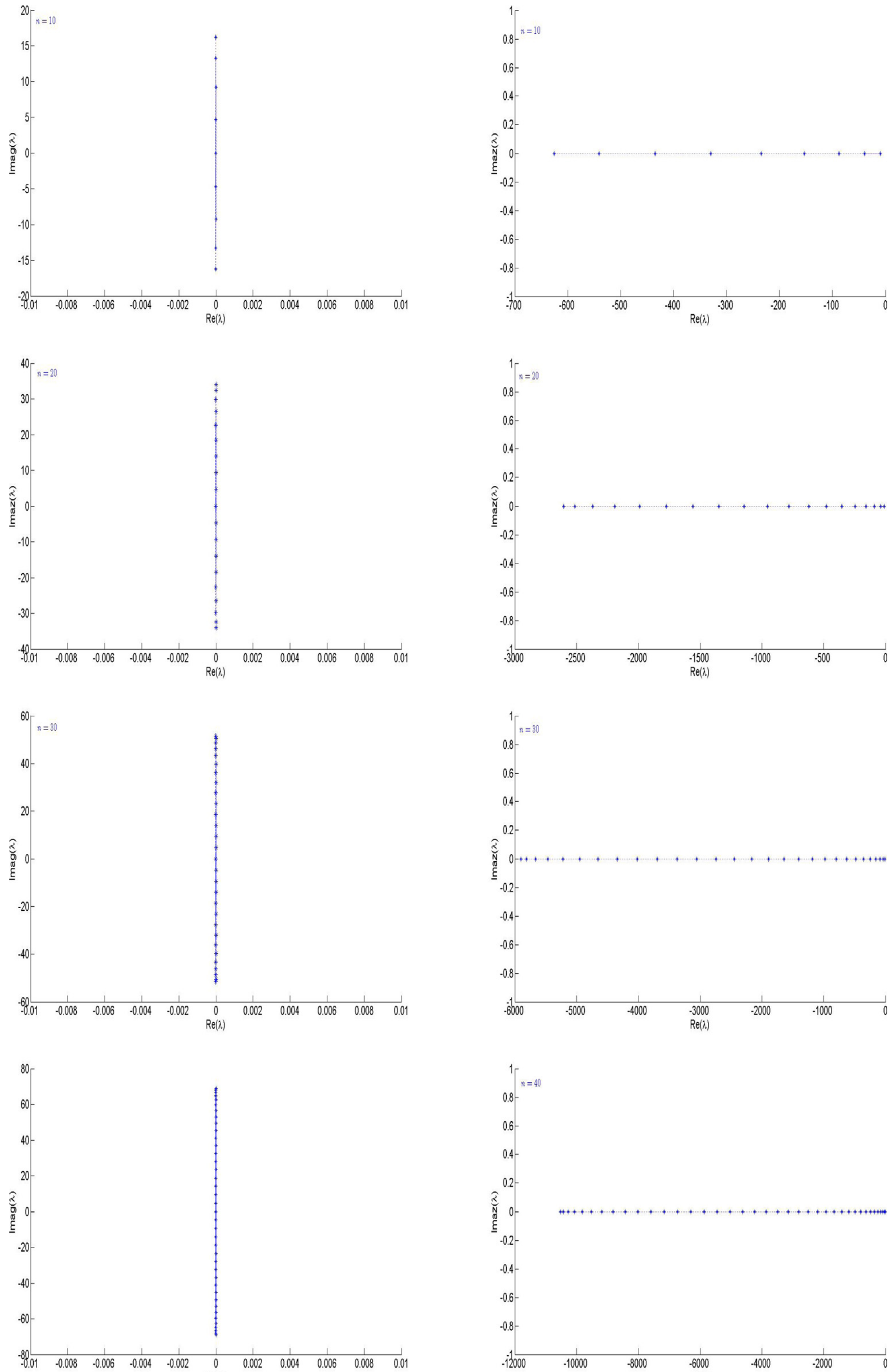

Fig. 1 Eigenvalues of A (left) and B (right) for different values of grid points.

where u_i^* represent the numerical solution at node i. The numerical solutions are obtained by MCB-DQM taking spatial space length $h = 0.1$ and time-step $\triangle t = 0.001$.

In the following problems the numerical results of gBH Eq. (1.1) with the initial and boundary conditions extracted from the exact solution (1.2) are compared with the results using various schemes, in literature.

Problem 1. The numerical solution of the gBH Eq. (1.1) is obtained for $\alpha = \beta = \delta = 1$ and $\gamma = 0.001$ at different time levels.

Table 1 Comparison of E_{abs} in MCB-DQM solutions of Problem 1 at some grid points.

Schemes/t	0.1			1.0		
x	0.1	0.5	0.9	0.1	0.5	0.9
MCB-DQM	1.1118E−08	2.8706E−08	1.1119E−08	1.6683E−08	4.6658E−08	1.6685E−08
FDS4[8]	6.3953E−09	3.9956E−08	7.6633E−08	3.2922E−07	3.7922E−07	4.2922E−07
ADM[22]	3.8743E−07	3.8746E−07	3.8749E−07	3.8750E−06	3.8753E−06	3.8756E−06
ADM[18]	3.7481E−08	3.7481E−08	3.7481E−08	3.7481E−07	3.7481E−07	3.7481E−07
VIM [3]	3.7481E−08	1.3748E−08	3.7481E−08	3.7481E−07	3.7481E−07	3.7481E−07
OHAM[36]	3.7481E−08	3.7481E−08	3.7481E−08	3.7481E−07	3.7481E−07	3.7481E−07

Table 2 $\mathbb{L}_\infty$ errors in the MCB-DQM solutions of Problem 1 at different time levels.

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0
MCB-DQM	2.870E−08	3.996E−08	4.416E−08	4.573E−08	4.631E−08	4.653E−08	4.661E−08	4.664E−08	4.665E−08
GCG[29]		3.913E−08							1.954E−07
ECG[29]		3.659E−08							1.879E−07
DQM [32]	1.737E−06	2.598E−06	3.460E−06	4.324E−06	5.189E−06	6.055E−06	6.922E−06	7.791E−06	9.531E−06
SM [23]	6.695E−08	1.093E−07	1.516E−07	1.938E−07	2.361E−07	2.784E−07	3.207E−07	3.630E−07	4.475E−07
ELG[29]		3.721E−08							1.870E−07
DDAC[28]		4.014E−08							4.685E−08

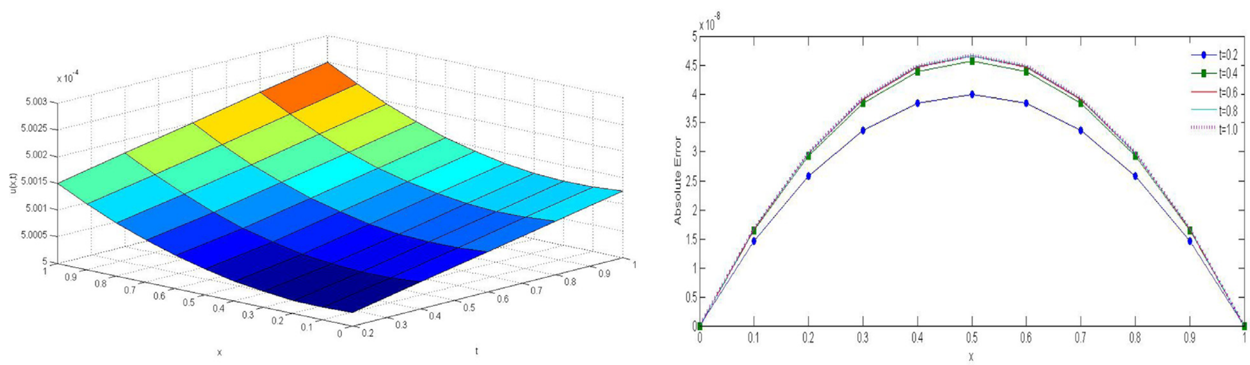

Fig. 2 Traveling wave solutions (left) and absolute errors (right) of Problem 1 at different time-levels $t \le 1$.

The comparison of the absolute errors for different time levels at different node points is presented in Table 1. The computed L_∞ errors for different time levels $t \le 1$ are compared with some earlier schemes in Table 2. It is evident from Tables 1 and 2 that we are getting better results than the results obtain in [3,8,18,22,23,28,29,32,36], and approaching towards the exact solution. The traveling wave behavior of the solution and the obtained absolute errors are shown graphically in Fig. 2.

Problem 2. The numerical solution of the gBH Eq. (1.1) is obtained for $\alpha = 0.02, \beta = \delta = 1$ and $\gamma = 0.001$. The computed L_∞ errors at different time levels $t \le 1$ are compared with SM [23], and reported in Table 3. The absolute errors are shown graphically in Fig. 3, for domain [0, 1], and Fig. 3 shows the physical behavior (left) and the absolute errors (right) of the numerical solution, for large domain $\Omega = [-10^4, 10^4]$ by taking $N = 301$.

Problem 3. The numerical solution of the gBH Eq. (1.1) is obtained for $\alpha = 0, \beta = 1, \gamma = 0.001$ taking different values of δ at different time levels $t \le 1$. The absolute errors at some selected node points are compared with the errors in some earlier schemes, and reported in Tables 4, 5 and 6 for $\delta = 1, 2$ and 3, respectively. It is evident from Tables 4–6 that our results are better than the results obtain in [8,22,36], and comparable with the three schemes presented in [29]. The obtained absolute errors are shown graphically in Fig. 4 for $\delta = 1$ (left) and $\delta = 3$ (right). The physical behavior (left) and the absolute errors (right) for large domain $\Omega = [-10^4, 10^4]$ are depicted in Fig. 5 by taking $N = 301$.

Problem 4. The numerical solution of the gBH Eq. (1.1) is obtained at different time levels $t \le 1$ with parameters $\alpha = 5, \delta = 1$ with different values of $\beta = 1, 10, 100$ and $\gamma = 10^{-3}, 10^{-4}, 10^{-5}$. The maximum error, L_∞ obtained for different time levels $t = 0.3$ and $t = 0.9$ is presented in Tables 7 and 8, respectively. It is evident that we are getting comparable results with the solutions obtained by analytical approximate solution [8,28,29]. Fig. 6 depicts the physical behavior of the numerical solution at different time levels $t \le 1$ for the parameters $\gamma = 0.001$ with $\beta = 1$(right)

Table 3 Comparison of L_∞ errors in the MCB-DQM solutions of Problem 2 at different levels $t \le 1$.

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0
MCB-DQM	3.80E−08	5.29E−08	5.84E−08	6.05E−08	6.13E−08	6.16E−08	6.17E−08	6.17E−08	6.17E−08
SM[23]	7.29E−08	1.21E−07	1.69E−07	2.17E−07	2.65E−07	3.13E−07	3.61E−07	4.09E−07	5.05E−07

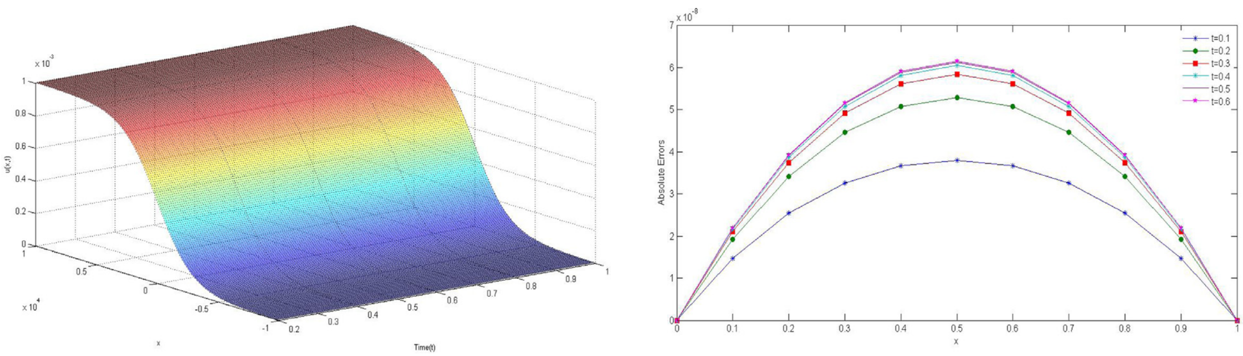

Fig. 3 Physical behavior (left) and absolute errors (right) of Problem 2 at different time-levels $t \le 1$.

Table 4 Comparison of E_{abs} errors in MCB-DQM solutions of Problem 3 for $\alpha = 0$, $\beta = 1$ and $\gamma = 0.001$ at $\delta = 1$ at $t \le 1$.

t	x	MCB-DQM	FDS4 [8]	ADM [22]	OHAM [36]	GCG[29]	ECG[29]	ELG[29]
0.05	0.1	1.0044E−08	2.4988E−08	1.8747E−07	2.4988E−08	1.0698E−08	1.0683E−08	9.2752E−09
	0.5	2.3047E−08	2.4988E−08	1.8749E−07	2.4988E−08	9.2595E−09	9.2595E−09	9.2595E−09
	0.9	1.0044E−08	2.4987E−08	1.8751E−07	2.4988E−08	7.8921E−09	7.8701E−09	9.2845E−09
0.10	0.1	1.4790E−08	4.9975E−08	3.7493E−07	4.9975E−08	2.3188E−08	2.3188E−08	2.3173E−08
	0.5	3.8252E−08	4.9975E−08	3.7498E−07	4.9975E−08	2.1749E−08	2.1748E−08	2.1749E−08
	0.9	1.4790E−08	4.9975E−08	3.7502E−07	4.9975E−08	2.0382E−08	2.0381E−08	2.0360E−08
1.00	0.1	2.2205E−08	4.9975E−07	3.7500E−06	4.9975E−07	2.4872E−07	2.4870E−07	2.4729E−07
	0.5	6.2169E−08	4.9975E−07	3.7504E−06	4.9975E−07	2.4728E−07	2.4728E−07	2.4728E−07
	0.9	2.2205E−08	4.9975E−07	3.7509E−06	4.9975E−07	2.4591E−07	2.4585E−07	2.4530E−07

Table 5 Comparison of E_{abs} errors in MCB-DQM solutions of Problem 3 for $\alpha = 0$, $\beta = 1$, $\gamma = 0.001$ and $\delta = 2$ at $t \le 1$.

t	x	MCB-DQM	FDS4[8]	ADM[22]	GCG[29]	ECG[29]	ELG[29]
0.05	0.1	4.4924E−07	1.1176E−06	5.5890E−07	4.8110E−07	4.5567E−07	4.2020E−07
	0.5	1.0307E−06	1.1175E−06	5.5884E−07	3.9966E−07	3.9966E−07	3.9966E−07
	0.9	4.4917E−07	1.1174E−06	5.5877E−07	3.9240E−07	3.5649E−07	4.3297E−07
0.10	0.1	6.6147E−07	2.2353E−06	1.1178E−06	1.0397E−06	1.0142E−06	9.7883E−07
	0.5	1.7107E−06	2.2350E−06	1.1177E−06	9.5823E−07	9.5822E−07	9.5823E−07
	0.9	6.6139E−07	2.2347E−06	1.1175E−06	9.5091E−07	9.1499E−07	9.9147E−07
1.00	0.1	9.9267E−07	2.2353E−05	1.1175E−05	1.1021E−05	1.1105E−05	1.1008E−05
	0.5	2.7793E−06	2.2350E−05	1.0074E−05	1.1057E−05	1.1056E−05	1.1057E−05
	0.9	9.9260E−07	2.2347E−05	1.1173E−05	1.0841E−05	1.0995E−05	1.0955E−05

Table 6 Comparison of E_{abs} errors in MCB-DQM solutions of Problem 3 for $\alpha = 0$, $\beta = 1$, $\gamma = 0.001$ and $\delta = 3$ at $t \le 1$.

t	0.05			0.10		
x	MCB-DQM	FDS4[8]	ADM[22]	MCB-DQM	FDS4[8]	ADM[22]
0.10	1.5946E−06	3.9673E−06	1.9841E−06	2.3479E−06	7.9346E−06	3.96811E−06
0.50	3.6584E−06	3.9665E−06	1.9837E−06	6.0721E−06	7.9330E−06	3.96731E−06
0.90	1.5942E−06	3.9657E−06	1.9833E−06	2.3475E−06	7.9314E−06	3.96652E−06
	t=1.00					
0.10	3.5221E−06	7.93462E−05	3.96632E−05			
0.50	9.8610E−06	7.93303E−05	3.96553E−05			
0.90	3.5217E−06	7.93143E−05	3.96473E−05			

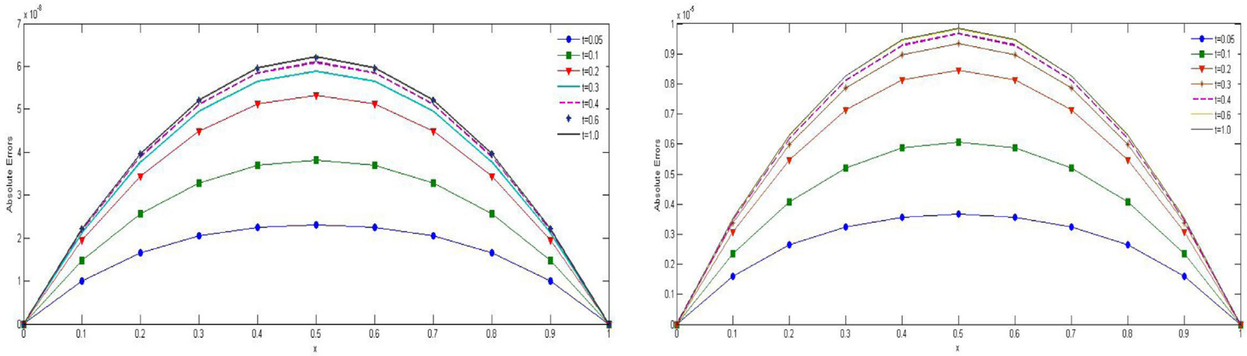

Fig. 4 Absolute errors for $\delta = 1$ (left) and $\delta = 3$ (right) of Problem 3 at different time-levels $t \leq 1$.

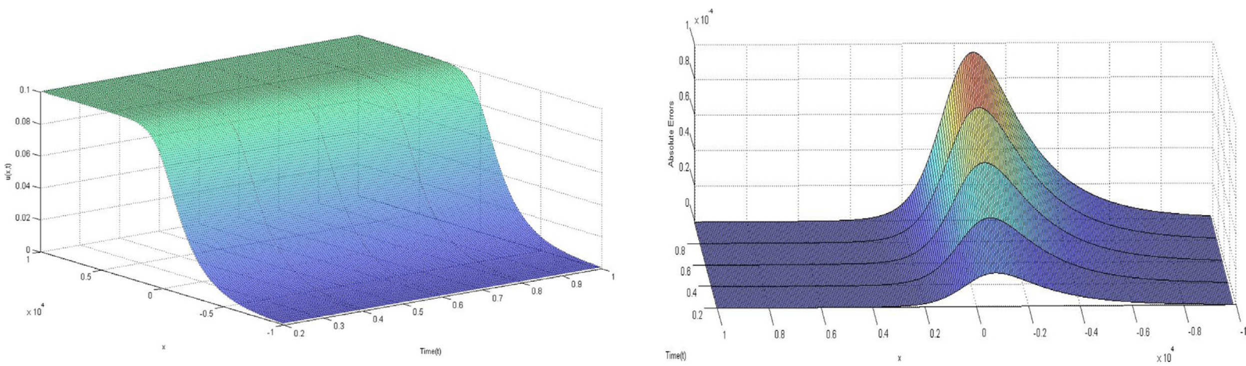

Fig. 5 Physical behavior (left) and absolute errors (right) of Problem 3 at different time-levels $t \leq 1$ for $\delta = 3$.

Table 7 Comparison of L_∞ errors in MCB-DQM solutions of Problem 4 with parameters $\alpha = 5, \delta = 1, \beta = 1, 10, 100$ and $\gamma = 10^{-3}, 10^{-4}, 10^{-5}$ at $t = 0.3$.

γ	β	MCB-DQM	FDS4[8]	DDAC[28]	GCG[29]	ECG[29]	ELG[29]
10^{-3}	1	3.1513E−08	3.1570E−08	3.1616E−08	5.6588E−09	5.6487E−09	5.5590E−09
	10	3.9583E−07	3.9684E−07	3.9742E−07	4.0136E−08	3.9438E−08	3.5810E−08
	100	5.0132E−06	5.0291E−06	5.0365E−06	1.5500E−06	1.5469E−06	1.5425E−06
10^{-4}	1	3.1527E−10	3.1584E−10	3.1630E−10	5.6218E−11	5.6246E−11	5.2789E−11
	10	3.9601E−09	3.9702E−09	3.9760E−09	3.9544E−10	3.9474E−10	3.7009E−10
	100	5.0156E−08	5.0316E−08	5.0389E−08	1.5484E−08	1.5480E−08	1.5369E−08
10^{-5}	1	3.1529E−12	3.3410E−12	3.1632E−12	5.6219E−13	5.6222E−13	5.2509E−13
	10	3.9602E−11	3.9704E−11	3.9762E−11	3.9485E−12	3.9478E−12	3.6859E−12
	100	5.0158E−10	5.0318E−10	5.0392E−10	1.5482E−10	1.5481E−10	1.5363E−10

Table 8 Comparison of L_∞ errors in MCB-DQM solutions of Problem 4 with parameters $\alpha = 5, \delta = 1, \beta = 1, 10, 100$ and $\gamma = 10^{-3}, 10^{-4}, 10^{-5}$ at $t = 0.9$.

β	$\gamma = 10^{-3}$			$\gamma = 10^{-4}$		
	MCB-DQM	FDS4[8]	DDAC[28]	MCB-DQM	FDS4[8]	DDAC[28]
1	3.3294E−08	3.3393E−08	3.3394E−08	3.3309E−10	3.3408E−10	3.3409E−10
10	4.1819E−07	4.1976E−07	4.1977E−07	4.1838E−09	4.1995E−09	4.1996E−09
100	5.2933E−06	5.3165E−06	5.3166E−06	5.2989E−08	5.3221E−08	5.3223E−08
	$\gamma = 10^{-5}$					
1	3.3310E−12	3.3410E−12	3.3411E−12			
10	4.1840E−11	4.1997E−11	4.1998E−11			
100	5.2992E−10	5.3224E−10	5.3225E−10			

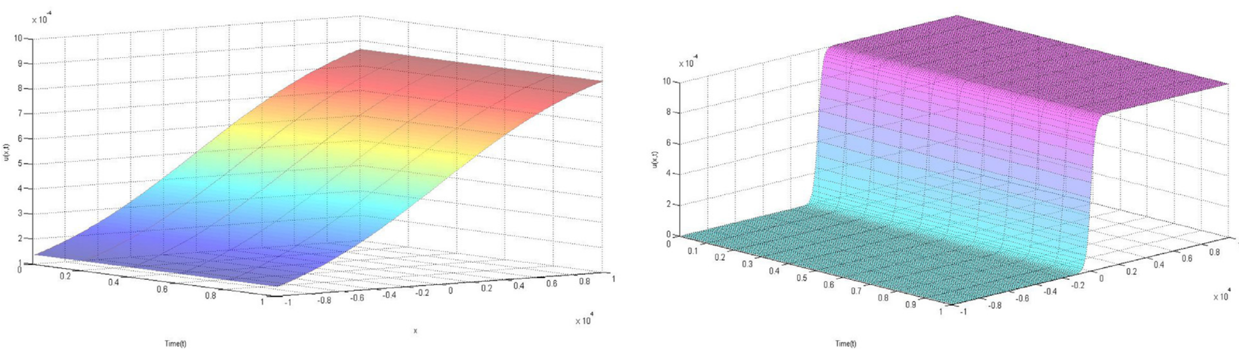

Fig. 6 Physical behavior at $\beta = 1$ (left) and $\beta = 100$ (right) of Problem 4 at different time-levels $t \leq 1$.

and $\beta = 100$ (left) for large domain $\Omega = [-10^4, 10^4]$ taking $N = 301$.

4. Conclusion

In this article, the efficiency of the recent numerical scheme MCB-DQM is demonstrated for the gBH equation. It is shown that the proposed scheme is stable. The scheme is tested on four test problems. The computed results are compared with the exact solutions in terms of L_∞ errors and absolute errors. The computed results are also compared with the results given by earlier schemes. It is evident from the numerical experiments that the solutions obtained by MCB-DQM are in good agreement with the exact solutions. The present scheme is an easy, economical and efficient technique for finding numerical solutions for various kinds of (non)linear physical models, and coupled systems of partial differential equations.

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