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Original Article

Reconstructing past fractional record values

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Abstract In this paper, reconstructing past fractional upper (lower) records from any absolutely continuous distribution is proposed. For this purpose, two pivotal quantities are given and their exact distributions are derived. More detailed results, including the case of unknown parameters, are given for the exponential and Fréchet distributions. Moreover, the exact mean square reconstructor errors are obtained and some comparisons between the pivotal quantities are performed. To explore the efficiency of the obtained results, a simulation study is conducted and two real data sets are analyzed.

2010 MATHEMATICAL SUBJECT CLASSIFICATION: 62E15; 62G30; 62G32; 62M20; 62F25

1. Introduction

Let $\{Z_n, n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables (rv's) with cumulative distribution function (cdf) $F(z)$ and probability density function (pdf) $f(z)$. Furthermore, assume that $Z_{1:n}, Z_{2:n}, \ldots, Z_{n:n}$ denote the order statistics of the random sample $Z_1, Z_2, \ldots, Z_n$. The kth upper record times, $T_k(n)$, $n \geq 1$, of the sequence $\{Z_n, n \geq 1\}$ is defined for fixed $k \geq 1$, as $T_k(1) = 1$ and

$$T_k(n+1) = \min\{j > T_k(n) : Z_{j:j+k-1} > Z_{T_k(n):T_k(n)+k-1}\},\ n > 1,$$

and the kth upper record values as $X_n^{(k)} = Z_{T_k(n):T_k(n)+k-1}$, $n \geqslant 1$. Clearly, $X_1^{(k)} = Z_{1:k} = \min\{Z_1, \ldots, Z_k\}$. For $k = 1$ we have $X_n = Z_{n:n} = \max\{Z_1, \ldots, Z_n\}$ is the upper record value of a random sample of size n. (cf. Dziubdziela and Kopocinski [1]). The kth lower record times and the kth lower record values are defined similarly.

The first result of record values for iid observations was reported by Chandler [2]. Dziubdziela and Kopocinski [1] generalized the concept of record values to a more generalized nature and called them kth record values. The concept of order statistics process was originally introduced by Stigler [3], which may be considered as fractional order statistics for non-integer index. Jones [4] gave an alternative construction of Stigler's uniform fractional order statistics. Bieniek and Szynal [5] followed a similar method of fractional order statistics to introduce the

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fractional record values or the record-values process, which can be considered as a family of record values with non-integer or fractional indices.

Inference based on record values have been extensively studied by many authors, including Kaminsky and Nelson [6], Ahsanullah [7,8], Dunsmore [9], Nagaraja [10], Balakrishnan et al. [11], AL-Hussaini and Ahmed [12], Raqab and Balakrishnan [13], Barakat et al. [14] and Barakat et al. [15], among others. The problem of reconstructing missing records based on current available records is of special interest in a wide fields of applications. Recent works on the reconstruction problem include, Klimczak and Rychlik [16], Balakrishnan et al. [17], Khatib and Ahmadi [18], Khatib et al. [19] and Aly [20].

Fractional record values (or record values with fractional indices) concept was originally introduced by Bieniek and Szynal [5]. In the rest of this section, some basic concepts and results which will be needed in the sequel are given (for more details see Bieniek and Szynal [5]).

Let $k \in \mathbb{N}$ be fixed and $W^{(k)} = \left\{W_t^{(k)},\ t \geqslant 0\right\}$ be a stochastic process such that:

(i) $W_0^{(k)} = 0$ almost sure,
(ii) $W_t^{(k)}$ has independent increments,
(iii) for every $t > s \geq 0$, $W_t^{(k)} - W_s^{(k)}$ has gamma distribution with parameters $t - s$ and k, respectively.

Then $\left\{W_t^{(k)},\ t \geqslant 0\right\}$ is called the exponential kth upper record-values process. Moreover, the rv's $W_t^{(k)}$, $t > 0$, are said to be exponential fractional kth upper record -values. Furthermore, the stochastic process $X^{(k)} = \left\{X_t^{(k)},\ t \geqslant 0\right\}$, where

$$X_t^{(k)} = F^{-1}\left(1 - \exp[-W_t^{(k)}]\right), \quad t \geqslant 0,$$

is called the kth upper record-values process based on the cdf F and the rv's $X_t^{(k)}$, $t > 0$, are said to be fractional kth upper record values from F. Fractional kth lower record values from F are defined similarly (cf. Bieniek and Szynal [5]).

In what follows, we shall assume that F is an absolutely continuous cdf with pdf f. The pdf $f_t^{(k)}(x)$, of $X_t^{(k)}$ is given by

$$f_t^{(k)}(x) = \frac{k^t}{\Gamma(t)}[H(x)]^{t-1}[\bar{F}(x)]^{k-1} f(x), \ -\infty < x < \infty,\ t > 0,$$

and the joint pdf, $f_{t_r,t_s}^{(k)}(x_r, x_s)$ of $X_{t_r}^{(k)}$ and $X_{t_s}^{(k)}$, with $t_s > t_r$ can be written as

$$f_{t_r,t_s}^{(k)}(x_r, x_s) = \frac{k^{t_s}}{\Gamma(t_r)\Gamma(t_s - t_r)}[H(x_r)]^{t_r-1}\left[H(x_s) - H(x_r)\right]^{t_s-t_r-1} \times [\bar{F}(x_s)]^k h(x_r) h(x_s),\ -\infty < x_r < x_s < \infty,$$

where, $H(x) = -\ln[1 - F(x)]$. The pdf $g_t^{(k)}(y)$ of the fractional kth lower record values, $Y_t^{(k)}$, is

$$g_t^{(k)}(y) = \frac{k^t}{\Gamma(t)}[-\ln F(y)]^{t-1}[F(y)]^{k-1} f(y), -\infty < y < \infty,\ t > 0,$$

and the joint pdf of $Y_{t_r}^{(k)}$ and $Y_{t_s}^{(k)}$, $t_s > t_r \geqslant 0$ for $-\infty < y_s < y_r < \infty$, is given by

$$g_{t_r,t_s}^{(k)}(y_r, y_s) = \frac{k^{t_s}}{\Gamma(t_r)\Gamma(t_s - t_r)}[-\ln F(y_r)]^{t_r-1}\left[\ln F(y_r) - \ln F(y_s)\right]^{t_s-t_r-1}\left[F(y_s)\right]^k \frac{f(y_r)}{F(y_r)} \frac{f(y_s)}{F(y_s)}.$$

The rest of this paper is organized as follows: Section 2 contains the theoretical results for the fractional kth upper record values formulated in general form and more details for two-parameter exponential distribution whenever the parameters are known or unknown are derived. A similar results are obtained for fractional kth lower record values with some detailed results of Fréchet distribution in Section 3. A simulation study is presented in Section 4, while illustrative examples are given in Section 5.

2. Reconstruction of past fractional kth upper record values

In this section, a reconstruction method based on reconstructive pivotal quantities, that allows one to obtain statistical intervals, called reconstructive confidence intervals (RCI's), Furthermore, point reconstructors are given for past missing fractional kth upper record values from two-parameter exponential distribution.

Theorem 2.1. *Assume that $X_{t_1}^{(k)}, X_{t_2}^{(k)}, ..., X_{t_r}^{(k)}$ are missing fractional kth upper record values with fractional indices, $0 = t_0 < t_1 < t_2 < \cdots < t_n$, from an absolutely continuous population with cdf F, pdf f and that $X_{t_s}^{(k)}, ..., X_{t_n}^{(k)}$, $r < s < n$ are observed ones from the same population. Then, the pivotal quantity $P = \left(X_s^* - X_r^*\right)/X_s^*$, follows beta distribution with parameters $t_s - t_r$ and t_r, respectively. Furthermore, an observed $100(1-\delta)\%$ RCI for $X_{t_r}^{(k)}$, is (L_P, U) where,*

$$L_P = F^{-1}\left(1 - \left(\bar{F}(X_{t_s}^{(k)})\right)^{1-p_\delta}\right), \qquad U = X_{t_s}^{(k)},$$

p_δ can be obtained by solving the non linear equation $Pr(P \leqslant p_\delta) = 1 - \delta$, and $X_i^* = H(X_{t_i}^{(k)})$.

Theorem 2.2. *Under the conditions of Theorem 2.1, the cdf of the pivotal quantity, $Q = \dfrac{X_s^* - X_r^*}{X_n^* - X_s^*}$, is given by,*

$$F_Q(q) = I_{\frac{q}{1+q}}(t_s - t_r, t_n - t_s), \quad q \geqslant 0, \tag{2.1}$$

where, $I_z(a, b)$, denote the incomplete beta function defined by

$$I_z(a, b) = \frac{1}{B(a, b)} \int_0^z u^{a-1}(1-u)^{b-1} du, \quad \text{for} \quad 0 < z < 1,$$

and $B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1} du$, is the beta function. Moreover, an observed $100(1-\delta)\%$ RCI for $X_{t_r}^{(k)}$, is (L_Q, U) where,

$$L_Q = F^{-1}\left(1 - \bar{F}^{1+q_\delta}\left(X_{t_s}^{(k)}\right)\bar{F}^{-q_\delta}\left(X_{t_n}^{(k)}\right)\right), \qquad U = X_{t_s}^{(k)},$$

and q_δ satisfies the non linear equation $F_Q(q_\delta) = 1 - \delta$.

2.1. Two-parameter exponential distribution with known parameters

Corollary 2.1. *Under the conditions of Theorem 2.1, if the population follows two-parameter exponential distribution, $Exp(\mu, \beta)$,* *where the location parameter* μ *and the scale parameter* β *are assumed to be known. Then an observed and expected limits of a* $100(1-\delta)\%$ *RCI for* $X_{t_r}^{(k)}$*, based on the pivotal quantity* P*, respectively, are* (L_P, U) *and* $(E[L_P], E[U])$ *where,*

$$L_P = (1-p_\delta)X^{(k)}(t_s) + p_\delta \mu,\ U = X_{t_s}^{(k)} \text{ and}$$
$$E[L_P] = \beta(1-p_\delta)t_s/k + \mu,\ E[U] = \beta t_s/k + \mu.$$

Moreover, an unbiased point reconstructor for $X_{t_r}^{(k)}$*, based on* P *and its mean square reconstructor error (MSRE), respectively, are given by*

$$\tilde{X}_{t_r}^{(k)} = \frac{t_r}{t_s}(X_{t_s}^{(k)} - \mu) + \mu, \text{ and}$$
$$\mathrm{MSRE}\left(\tilde{X}_{t_r}^{(k)}\right) = E\left[\tilde{X}_{t_r}^{(k)} - X_{t_r}^{(k)}\right]^2 = \frac{\beta^2 t_r}{k^2 t_s}(t_s - t_r). \tag{2.2}$$

Corollary 2.2. *With the same conditions of Corollary 2.1, an observed and expected lower limits of a* $100(1-\delta)\%$ *RCI of* $X_{t_r}^{(k)}$*, based on the pivotal quantity* Q*,* (L_Q, U)*, are*

$$L_Q = X_{t_s}^{(k)} - q_\delta(X_{t_n}^{(k)} - X_{t_s}^{(k)}) \quad \text{and}$$
$$E[L_Q] = \frac{\beta}{k}(t_s - q_\delta(t_n - t_s)) + \mu.$$

Furthermore, an unbiased point reconstructor for $X_{t_r}^{(k)}$*, based on* Q *and its MSRE, respectively, are*

$$\hat{X}_{t_r}^{(k)} = \frac{(t_n - t_r)X_{t_s}^{(k)} - (t_s - t_r)X_{t_n}^{(k)}}{t_n - t_s} \quad \text{and}$$
$$\mathrm{MSRE}\left(\hat{X}_{t_r}^{(k)}\right) = \frac{\beta^2(t_n - t_r)(t_s - t_r)}{k^2(t_n - t_s)}. \tag{2.3}$$

2.2. *Two-parameter exponential distribution with unknown parameters*

The main task of this subsection, is to apply our method for the two-parameter exponential distribution, whenever the distribution parameters are unknown. The likelihood function of the fractional kth upper record values, $X_{t_s}^{(k)}, \ldots, X_{t_n}^{(k)}$, can be obtained by using (2.4) of Bieniek and Szynal [5], that is,

$$L(\mathbf{x};\Theta) = f(x_s, \ldots, x_n;\Theta) = C(H(x_s;\Theta))^{t_{s-1}}(1 - F(x_n;\Theta))^k$$
$$\times \prod_{i=s}^{n}\left([H(x_i) - H(x_{i-1})]^{t_i - t_{i-1} - 1} h(x_i;\Theta)\right), \tag{2.4}$$

where, $x_s < x_{s+1} < \ldots < x_n$, with $s < n$, are observed fractional kth upper record values., Θ is an unknown vector of parameters and $C = k^{t_n}\left[\Gamma(t_{s-1}+1)\prod_{i=s}^{n}\Gamma(t_i - t_{i-1})\right]^{-1}$. For $Exp(\mu, \beta)$, we have $\ell(\mu,\beta) = \ln L(\mu,\beta) \propto -t_n \ln\beta + t_{s-1}\ln(x_s - \mu) - \frac{k}{\beta}(x_n - \mu)$. Therefore, the maximum likelihood estimates (MLE's) of μ and β, respectively, are

$$\hat{\mu} = \frac{t_n X_{t_s}^{(k)} - t_{s-1}X_{t_n}^{(k)}}{t_n - t_{s-1}}, \quad \text{and} \quad \hat{\beta} = \frac{k\left(X_{t_n}^{(k)} - X_{t_s}^{(k)}\right)}{t_n - t_{s-1}}. \tag{2.5}$$

However, the MLE's are biased estimators and the corrected unbiased estimators are

$$\hat{\beta}_c = \frac{k\left(X_{t_n}^{(k)} - X_{t_s}^{(k)}\right)}{t_n - t_s} \quad \text{and} \quad \hat{\mu}_c = \hat{\mu} - \frac{\hat{\beta}_c t_n}{k}\left(\frac{t_s - t_{s-1}}{t_n - t_{s-1}}\right). \tag{2.6}$$

The estimated lower limit, $\hat{L}_P$, as well as the point reconstructor, $\tilde{\hat{X}}_{t_r}^{(k)}$, based on the pivotal quantity P can be obtained from Corollary 2.1 by replacing μ with $\hat{\mu}_c$.

3. Reconstructing past fractional kth lower record values

In the following two theorems, two pivotal quantities are developed and their distributions are obtained to reconstruct past fractional kth lower record values.

Theorem 3.1. *Suppose that* $Y_{t_1}^{(k)}, Y_{t_2}^{(k)}, \ldots, Y_{t_r}^{(k)}$ *are unobserved fractional kth lower record values with fractional indices,* $0 = t_0 < t_1 < t_2 < \cdots < t_n$*, from an absolutely continuous distribution with cdf F and pdf f. Furthermore, let* $Y_{t_s}^{(k)}, \ldots, Y_{t_n}^{(k)}$ *be observed from the same population. Then, the pivotal quantity* $P^\star = \left(Y_s^\star - Y_r^\star\right)/Y_r^\star$*, follows beta distribution with parameters* $t_s - t_r$ *and* t_r*, respectively. Furthermore, a* $100(1-\delta)\%$ *RCI for* $Y_{t_r}^{(k)}$*, is* $(L, U_{P^\star})$ *where,*

$$L = Y_{t_s}^{(k)}, \qquad U_{P^\star} = F^{-1}\left(\left(F\left(Y_{t_s}^{(k)}\right)\right)^{1-p_\delta^\star}\right),$$

$p_\delta^\star$ *satisfies* $Pr(P^\star \leqslant p_\delta^\star) = 1-\delta$ *and* $Y_i^\star = \log[F(Y_{t_i}^{(k)})]$.

Theorem 3.2. *With the same conditions of Theorem 3.1, the cdf of the pivotal quantity,* $Q^\star = \dfrac{Y_r^\star - Y_s^\star}{Y_s^\star - Y_n^\star}$*, is given by (2.1). Consequently, a* $100(1-\delta)\%$ *RCI for* $Y_{t_r}^{(k)}$*, is* $(L, U_{Q^\star})$ *where,*

$$L = Y_{t_s}^{(k)}, \qquad U_{Q^\star} = F^{-1}\left(F\left(Y_{t_s}^{(k)}\right)\left(\frac{F\left(Y_{t_s}^{(k)}\right)}{F\left(Y_{t_n}^{(k)}\right)}\right)^{q_\delta^\star}\right),$$

and $q_\delta^\star$ *satisfies the non linear equation* $F_{Q^\star}(q_\delta^\star) = 1-\delta$.

Corollary 3.1. *Under the conditions of Theorem 3.1, if the population distribution is Fréchet, with cdf* $F(y) = \exp\left[-\left(\frac{y}{\beta}\right)^{-\alpha}\right]$, $y > 0,\ \alpha, \beta > 0$, *then an observed and expected limits of a* $100(1-\delta)\%$ *RCI for the lower record values with fractional indices from an absolutely continuous distribution,* $Y_{t_r}^{(k)}$*, based on the pivotal quantity* $P^\star$*, are respectively, given by* $(L, U_{P^\star})$ *and* $(E[L], E[U_{P^\star}])$ *where,*

$$L = Y_{t_s}^{(k)},\ U_{P^\star} = (1-p_\delta^\star)^{-\frac{1}{\alpha}}\ Y_{t_s}^{(k)},$$

$$E[L] = \frac{\beta}{\Gamma(t_s)]}\Gamma\left(t_s - \alpha^{-1}\right),\ E[U_{P^\star}] = \frac{\beta(1-p_\delta^\star)^{-\frac{1}{\alpha}}}{\Gamma(t_s)]}\Gamma\left(t_s - \alpha^{-1}\right).$$

Moreover, an unbiased point reconstructor for $Y_{t_r}^{(k)}$*, is*

$$\tilde{Y}_{t_r}^{(k)} = \frac{\Gamma(t_s)\Gamma\left(t_r - \alpha^{-1}\right)}{\Gamma(t_r)\Gamma\left(t_s - \alpha^{-1}\right)}\ Y_{t_s}^{(k)}, \tag{3.1}$$

and its MSRE is given by

$$\mathrm{MSRE}(\tilde{Y}_{t_r}^{(k)}) = \frac{\beta^2}{\Gamma^2(t_r)\Gamma^2\left(t_s - \alpha^{-1}\right)}\left[\Gamma(t_r)\Gamma\left(t_r - 2\alpha^{-1}\right)\right.$$
$$\left.\Gamma^2\left(t_s - \alpha^{-1}\right) - \Gamma(t_s)\Gamma\left(t_s - 2\alpha^{-1}\right)\Gamma^2\left(t_r - \alpha^{-1}\right)\right], \tag{3.2}$$

for $t_r > \frac{2}{\alpha}$.

As in the case of upper record values, we derive the likelihood function of the fractional kth lower record values, $Y_{t_s}^{(k)}, \ldots, Y_{t_n}^{(k)}$, namely,

$$L^\star(\mathbf{y};\Theta) = f(y_s,...,y_n;\Theta) = C(-\log(F(y_s;\Theta)))^{t_{s-1}}(F(y_n;\Theta))^k \prod_{i=s}^{n}\left(\left[\ln\left(\frac{F(y_{i-1};\Theta)}{F(y_i;\Theta)}\right)\right]^{t_i-t_{i-1}-1}\frac{f(y_i;\Theta)}{F(y_i;\Theta)}\right), \quad (3.3)$$

where $y_s > y_{s+1} > \cdots > y_n$ for $s < n$, are observed fractional kth lower record values. For Fréchet distribution, the MLE's can be obtained by (3.3), namely, $\hat{\beta} = \left(\frac{t_n}{k}\right)^{\frac{1}{\hat{\alpha}}} Y_{t_n}^{(k)}$ where $\hat{\alpha}$ is the solution of the nonlinear equation,

$$t_{s-1}\ln(y_s) + \sum_{i=s}^{n}\ln(y_i) + \sum_{i=s}^{n}\left[(t_i - t_{i-1}+1)\frac{y_i^{-\alpha}\ln(y_i) - y_{i-1}^{-\alpha}\ln(y_{i-1})}{y_i^{-\alpha} - y_{i-1}^{-\alpha}}\right] - \frac{n-s+1}{\alpha} - t_n\ln(y_n) = 0.$$

Remark 3.1

1. The point reconstructor does not depend on the scale parameter or the significance level δ.
2. The point reconstructor $\hat{X}_{t_r}^{(k)}$ based on Q for $Exp(\mu,\beta)$, is exactly the best linear unbiased reconstructor (see Khatib and Ahmadi [18] for ordinary record values) and it is free of the distribution parameters.
3. The ordinary upper (lower) record values are obtained as special cases from the preceding results by setting $t_i = i$, for all $i = 1, 2, ..., n$.
4. It is not difficult to verify, $\mathtt{MSRE}\left(\tilde{X}_{t_r}^{(k)}\right) < \mathtt{MSRE}\left(\hat{X}_{t_r}^{(k)}\right)$.
5. For $Exp(\mu,\beta)$, if we replace μ and β with $\hat{\mu}_c$ and $\hat{\beta}_c$, we have $\tilde{\tilde{X}}_{t_r}^{(k)} = \hat{X}_{t_r}^{(k)}$.

The proof of the results given here, can be accomplished by following the same argument of Barakat et al. [15,21], or by a standard method of transformations of random variables after routine mathematical calculations and some effort. We only explain how one can obtain a point reconstructor from the limits of the RCI. A simple method for this goal is to find the constant c such that $\tilde{X}_{t_r}^{(k)} = L + c(U - L)$ with $0 < c < 1$, is an unbiased for $X_{t_r}^{(k)}$ or has minimum variance, where L and U denote the lower and upper limits for RCI. For $Exp(\mu,\beta)$, based on the pivotal quantity P(the case of upper record values), the value of c is $c = c_p = 1 - (t_s - t_r)/p_\delta t_s$, and $c_Q = 1 - (t_s - t_r)/[q_\delta(t_n - t_s)]$, based on the pivotal quantity P. The proof of (3.1) is similar.

4. Simulation

In this section, simulation experiments are carried out to demonstrate the efficiency of the proposed method. The following two algorithms are essential to accomplish our simulation. In view of the results of Rider [22], Rahman [23], Cramer [24] and Burkschat et al. [25], we can generate ordinary and fractional record values from any continuous cdf F by using Algorithm 1.

Algorithm 1. (Generation of fractional kth upper record values)

Step 1. Determine the cdf F and choose the values of n, k,
Step 2. generate a random sample, $B_1, B_2, ..., B_n$, of size n from beta distribution, $B(k, 1)$,
Step 3. find the rth ordinary upper kth record value $X_r^{(k)}$ from the relation $X_r^{(k)} = F^{-1}\left(1 - \prod_{j=1}^{r} B_j\right)$,
Step 4. generate $W_t^{(k)}$ from $Exp(1)$ by Theorems 2.1 or 2.2 and 3.1 of Bieniek and Szynal [5],
Step 5. compute $X_{t_i}^{(k)}$ based on F from the relation $X_{t_i}^{(k)} = F^{-1}\left(1 - e^{-W^{(k)}(t_i)}\right)$ $i = 1, 2, ..., n$.

The fractional kth lower record values can be generated similarly.

Algorithm 2.

Step 1. Determine k, r, s, n, the fractional indices, $0 = t_0 < t_1 < \cdots < t_n$, and the number of replicates M,
Step 2. select a continuous distribution and its parameter(s),
Step 3. generate M arrays, each array include n of $X_{t_i}^{(k)}$ and then store them,
Step 4. find the numerical values of $p_\delta(q_\delta)$ by solving the nonlinear equations $F_P(p_\delta) = Pr(P \leqslant p_\delta) = 1 - \delta$ $(F_Q(q_\delta) = 1 - \delta)$ for fractional kth upper record values, or $F_{P^\star}(p_\delta^\star) = 1 - \delta$ $(F_{Q^\star}(q_\delta^\star) = 1 - \delta)$, for fractional kth lower record values,
Step 5. compute the point reconstructors, lower and upper limits for the RCI's based on the pivotal quantities P $(P^\star)$ and Q $(Q^\star)$ by Theorems 2.1, 2.2, 3.1, and 3.2,
Step 6. check whether, the observed value of $X_{t_r}^{(k)}$ $(Y_{t_r}^{(k)})$ did belong to the RCI,
Step 7. repeat Steps 5 and 6 after replacing the parameters with their MLE's, which can be obtained by (2.4) or (3.3),
Step 8. repeat Steps 5, 6 and 7, M times,
Step 9. compute the percentage of the coverage probability, and the average of lower (upper) limits,
Step 10. compute the expected value of upper (lower) limits based on $P(Q)$ and $P^\star(Q^\star)$, and the root mean square reconstructive errors (RMSRE's) by using Corollaries 2.1, 2.2 and 3.1.

The results are shown in Table 1, where they are based on $M = 10^5$ replicates of $n = 17$ kth upper records (including 9 ordinary records and 8 fractional records) corresponding to $t_i = 1, 1.5, 2, 2.5, ..., 9$, $k = 5$, from Exp(8, 2.5). In this study, we assume that the upper records, $X_{t_s}^{(5)}, \ldots, X_{t_n}^{(5)}$, have been observed and the first upper records, $X_{t_2}^{(5)}, \ldots, X_{t_r}^{(5)}$, are to be reconstructed. In Tables 1 and 2, we consider the following two situations: **(i)** The distribution parameters are known and **(ii)** the distribution parameters are unknown and in this case, the MLE's (or their corrected unbiased estimators) are obtained by (2.5) (2.6) or by (3.3) for the kth lower records. For each value of t_r, the first line includes average values based on $M = 10^5$ replicates, while the RMSRE's are given in the second line between parentheses. Table 2, contains similar results but for lower records with $n = 20$, $k = 2$ (including 10 ordinary records and 10 fractional records) from Fréchet distribution with $\alpha = 2$ and $\beta = 5$ based on the pivotal quantity $P^\star$. It is worth to mention here that, $\bar{L}_P(\bar{\bar{L}}_P)$ and $\bar{\tilde{X}}_{t_r}^{(5)}(\bar{\tilde{\tilde{X}}}_{t_r}^{(5)})$ denote the average lower limit

Table 1 90% coverage probability, limits of RCI, point reconstructor, and RMSRE's of past ordinary and fractional kth upper record values from $Exp(8, 2.5)$ based on $M = 10^5$ replicates.

t_s	MLE's	t_r	$CP_Q\%$	$CP_P\%$	$\hat{CP}_P\%$	$\bar{L}_Q = \bar{\hat{L}}_Q$	$\bar{L}_P$	$\bar{\hat{L}}_P$	$\bar{X}_{t_r}^{(5)}$	$\bar{\hat{X}}_{t_r}^{(5)}$	$\bar{\hat{\hat{X}}}_{t_r}^{(5)} = \bar{\hat{X}}_{t_r}^{(5)}$	$\bar{U}$
4.0	$\bar{\hat{\mu}}_c = 7.995$	3.5	90.150	90.068	86.947	9.175	9.320	9.319	9.748	9.748	9.747	9.997
	$\bar{\hat{\beta}}_c = 2.503$					(0.700)	(0.477)	(0.613)	(0.187)	(0.331)	(0.371)	(1.000)
		3.0	90.165	90.045	83.326	8.533	8.927	8.924	9.498	9.498	9.497	9.997
						(1.148)	(0.605)	(0.844)	(0.173)	(0.433)	(0.548)	(1.000)
		2.5	90.157	90.034	79.532	7.949	8.630	8.626	9.250	9.248	9.246	9.997
						(1.562)	(0.650)	(1.008)	(0.158)	(0.484)	(0.698)	(1.000)
		2.0	90.174	89.984	75.522	7.389	8.391	8.387	8.999	8.999	8.996	9.997
						(1.958)	(0.629)	(1.130)	(0.141)	(0.500)	(0.837)	(1.000)
		1.5	90.144	89.944	70.960	6.841	8.203	8.198	8.749	8.749	8.746	9.997
						(2.342)	(0.546)	(1.220)	(0.122)	(0.484)	(0.968)	(1.000)
3.5	$\bar{\hat{\mu}}_c = 7.996$	3.0	90.116	90.056	87.324	8.940	9.073	9.071	9.498	9.498	9.497	9.748
	$\bar{\hat{\beta}}_c = 2.503$					(0.681)	(0.469)	(0.604)	(0.173)	(0.327)	(0.369)	(0.935)
		2.5	90.036	90.046	83.428	8.316	8.696	8.693	9.250	9.248	9.247	9.748
						(1.101)	(0.578)	(0.819)	(0.158)	(0.423)	(0.544)	(0.935)
		2.0	90.107	89.963	79.062	7.750	8.422	8.418	8.999	8.999	8.997	9.748
						(1.491)	(0.594)	(0.965)	(0.141)	(0.463)	(0.691)	(0.935)
		1.5	90.046	90.020	74.317	7.208	8.215	8.211	8.749	8.749	8.746	9.748
						(1.860)	(0.532)	(1.066)	(0.122)	(0.463)	(0.826)	(0.935)
3.0	$\bar{\hat{\mu}}_c = 7.997$	2.5	90.095	90.087	87.447	8.704	8.827	8.826	9.250	9.249	9.248	9.498
	$\bar{\hat{\beta}}_c = 2.502$					(0.664)	(0.455)	(0.595)	(0.158)	(0.323)	(0.368)	(0.866)
		2.0	90.022	90.059	82.979	8.094	8.474	8.472	8.999	8.999	8.998	9.498
						(1.064)	(0.538)	(0.792)	(0.141)	(0.408)	(0.540)	(0.866)
		1.5	89.992	90.041	77.831	7.543	8.234	8.232	8.749	8.749	8.748	9.498
						(1.431)	(0.511)	(0.915)	(0.122)	(0.433)	(0.685)	(0.866)
2.5	$\bar{\hat{\mu}}_c = 7.999$	2.0	90.015	90.160	87.304	8.466	8.585	8.585	8.999	9.000	9.000	9.250
	$\bar{\hat{\beta}}_c = 2.501$					(0.653)	(0.434)	(0.589)	(0.141)	(0.316)	(0.367)	(0.791)
		1.5	89.934	90.036	81.940	7.868	8.269	8.269	8.749	8.750	8.750	9.250
						(1.033)	(0.474)	(0.760)	(0.122)	(0.387)	(0.537)	(0.791)
2.0	$\bar{\hat{\mu}}_c = 7.999$	1.5	90.071	89.926	87.059	8.224	8.351	8.351	8.749	8.750	8.749	8.999
	$\bar{\hat{\beta}}_c = 2.501$					(0.642)	(0.398)	(0.577)	(0.122)	(0.306)	(0.366)	(0.707)

of the RCI and the average of the point reconstructor based on the pivotal quantity P whenever the parameters are assumed to be known(unknown), respectively.

5. Illustrative examples

In this section, two real data sets are analyzed for illustrative purposes.

Example 5.1. The first real data set represents the times (in minutes) between 48 consecutive telephone calls to a company's switchboard which was obtained from Castillo et al. [26] and have been analyzed by Khatib and Ahmadi [18]. It was shown that the two-parameter exponential distribution is a suitable model for this data. The vector of observed upper records (for $k = 1$) is $\mathbf{x}^{(1)} = (1.34, 1.68, 1.86, 2.2, 3.2, 3.25)$. The MLE's and their corrections are computed from (2.5) and (2.6), respectively. Table 3 summarizes the reconstruction results of Example 5.1.

Example 5.2. The second set of data consists of, 720 monthly maximum wind speed of Boston in the United States during the period from 1950 to 2009. The data are obtained from the Mathematica Documentation Center. Relying on Kolmogorov–Smirnov (K–S) test, we can judge whether the data follows two parameter-Fréchet distribution or not. The observed value of K–S test statistic is 0.0790207, which indicates that two parameter-Fréchet distribution is adequate model for this data. The vector of observed lower records is $\mathbf{y}^{(1)} = (66.6, 55.8, 51.84, 50.04, 42.48, 38.88, 37.08, 33.48, 29.52)$. The MLE's, of $\hat{\alpha}$ and $\hat{\beta}$, as well as the reconstruction results are shown in Table 4.

6. Concluding remarks

In this paper, we have proposed reconstruction method for fractional kth upper (lower) records based on pivotal quantities. Point reconstructors and RCI's with comparisons based on the MSRE have obtained for two-parameter exponential distribution in upper case and two-parameter Fréchet distribution for lower case. Moreover, a simple method has given to get an unbiased point reconstructor from the reconstruction interval for the exponential and Fréchet distributions. In addition, the likelihood function based on the available fractional kth upper (lower) records, $X_{t_s}^{(k)}, \ldots, X_{t_n}^{(k)}$ ($Y_{t_s}^{(k)}, \ldots, Y_{t_n}^{(k)}$), has derived and the MLE's of parameters have obtained which reveal that the results are satisfactory when the parameters are replaced by their estimators. To demonstrate the efficiency of the proposed method simulation experiments have carried out and two real

Table 2 90% coverage probability, limits of RCI, point reconstructor, and RMSRE's of past ordinary and fractional kth lower record values from Fréchet distribution with $\alpha = 2,\ \beta = 5$ based on $M = 10^5$ replicates.

t_s	MLE' s	t_r	CP_{P_*}	$\hat{C}P_{P_*}$	$\bar{L}_{P_*}$	$\bar{Y}_{t_r}^{(2)}$	$\tilde{Y}_{t_r}^{(2)}$	$\tilde{\bar{Y}}_{t_r}^{(2)}$	$\bar{U}_{P_*}$	$\tilde{\bar{U}}_{P_*}$
4.0	$\bar{\hat{\alpha}} = 2.565$	3.5	89.958	86.145	3.9159	4.2569	4.2546	4.2654	4.8164	4.8384
	$\bar{\hat{\beta}} = 5.117$						(0.5779)	(0.6144)	(1.0156)	(1.1491)
		3.0	90.078	84.165	3.9159	4.6998	4.6991	4.7366	5.7478	5.8297
							(1.0085)	(1.1385)	(1.6908)	(2.1404)
		2.5	90.090	82.632	3.9159	5.3195	5.3183	5.4214	6.9740	7.1934
							(1.6319)	(1.9801)	(2.8675)	(3.8650)
		2.0	89.922	81.163	3.9159	6.2786	6.2655	6.5546	8.8497	9.4110
							(2.7765)	(3.9044)	(5.5472)	(7.5654)
		1.5	89.892	80.132	3.9159	7.9967	7.9775	9.0726	12.2891	13.8956
							(5.4767)	(29.9364)	(23.4942)	(26.3142)
3.5	$\bar{\hat{\alpha}} = 2.521$	3.0	90.125	86.367	4.2569	4.6998	4.7016	4.7240	5.4337	5.4811
	$\bar{\hat{\beta}} = 5.120$						(0.7787)	(0.8289)	(1.4619)	(1.6261)
		2.5	90.080	84.548	4.2569	5.3195	5.3211	5.4000	6.7467	6.9131
							(1.4574)	(1.6538)	(2.6746)	(3.2697)
		2.0	89.939	82.917	4.2569	6.2786	6.2687	6.5071	8.6647	9.1318
							(2.6311)	(3.2832)	(5.4108)	(6.8060)
		1.5	89.844	81.770	4.2569	7.9967	7.9816	8.9188	12.1328	13.5259
							(5.3661)	(11.1138)	(23.4343)	(25.3600)
3.0	$\bar{\hat{\alpha}} = 2.485$	2.5	90.097	86.637	4.6998	5.3195	5.3191	5.3655	6.3266	6.4222
	$\bar{\hat{\beta}} = 5.113$						(1.1672)	(1.2488)	(2.4183)	(2.6263)
		2.0	89.959	84.571	4.6998	6.2786	6.2664	6.4479	8.3576	8.7116
							(2.4180)	(2.8093)	(5.2241)	(6.0715)
		1.5	89.851	83.270	4.6998	7.9967	7.9787	8.7435	11.8811	13.0482
							(5.2099)	(10.9339)	(23.3556)	(24.6433)
2.5	$\bar{\hat{\alpha}} = 2.455$	2.0	89.779	86.357	5.3195	6.2786	6.2669	6.3764	7.7748	7.9827
	$\bar{\hat{\beta}} = 5.116$						(1.9927)	(2.1808)	(4.9076)	(5.1954)
		1.5	89.833	84.765	5.3195	7.9967	7.9793	8.5881	11.4605	12.3689
							(4.9018)	(8.1952)	(23.2172)	(24.0134)
2.0	$\bar{\hat{\alpha}} = 2.429$	1.5	90.016	86.698	6.2786	7.9967	7.9942	8.3788	10.5923	11.1454
	$\bar{\hat{\beta}} = 5.114$						(4.1803)	(6.3613)	(22.9636)	(23.1844)

Table 3 Point and interval reconstruction with relative errors of past ordinary upper records for $X_{t_r}^{(1)}$ with $r = s-1, ..., 1$, for $s = 4, 3, 2$.

t_s	$\hat{\mu}_c$, $\hat{\beta}_c$	t_r	$X_{t_r}^{(1)}$	$\tilde{X}_{t_r}^{(1)}$	$\hat{X}_{t_r}^{(1)} = \tilde{\tilde{X}}_{t_r}^{(1)}$	$RE\left(\tilde{X}_{t_r}^{(1)}\right)$	$RE\left(\hat{X}_{t_r}^{(1)}\right)$	95% $\tilde{\mathtt{R}}$CI$_P$	95% $\tilde{\tilde{\mathtt{R}}}CI_P$
4.0	$\hat{\mu}_c = 0.1$	3.0	1.86	1.938	1.675	0.0417	0.0995	(1.537 , 2.20)	(0.874 , 2.20)
	$\hat{\beta}_c = 0.525$	2.0	1.68	1.675	1.150	0.0030	0.3155	(1.292 , 2.20)	(0.384 , 2.20)
		1.0	1.34	1.413	0.625	0.0541	0.5336	(1.168 , 2.20)	(0.136 , 2.20)
3.0	$\hat{\mu}_c = 0.47$	2.0	1.68	1.628	1.397	0.0308	0.1687	(1.320 , 1.86)	(0.781 , 1.86)
	$\hat{\beta}_c = 0.463$	1.0	1.34	1.397	0.933	0.0423	0.3035	(1.183 , 1.86)	(0.505 , 1.86)
2.0	$\hat{\mu}_c = 0.895$	1.0	1.34	1.523	1.288	0.1366	0.0392	(1.382 , 1.68)	(0.934 , 1.68)
	$\hat{\beta} = 0.3925$								

data sets have analyzed. The results of Example 4.1 are satisfactory compared with Khatib and Ahmadi [18]. In view of the results given in the preceding sections, we have noticed the following:

1. The lower (upper) limits of RCI's as well as the point reconstructor and their estimates are closed to each other when we use the MLE's of parameters. Theoretically this is expected, since MLE's are consistent and satisfy the asymptotic normality.
2. In most cases the RCI contains the exact value of $X_{t_r}^{(1)}$ or $Y_{t_r}^{(1)}$ (see Tables 3, 4).
3. In all cases the mean square error decreases as $s - r$ decreases.
4. An application of the pivotal quantity Q or $Q^{\star}$, requires the restrictive condition,

$$\bar{F}\left(X_{t_s}^{(k)}\right)\left(\frac{\bar{F}\left(X_{t_s}^{(k)}\right)}{\bar{F}\left(X_{t_n}^{(k)}\right)}\right)^{q_\delta} < 1 \quad \text{or} \quad F\left(Y_{t_s}^{(k)}\right)\left(\frac{F\left(Y_{t_s}^{(k)}\right)}{F\left(Y_{t_n}^{(k)}\right)}\right)^{q_\delta^\star} < 1, \tag{6.1}$$

which is not always satisfied.

Table 4 Point and interval reconstruction with relative error of past ordinary lower records for $Y_{t_r}^{(1)}$ with $n = 9,\ r = s - 1, \ldots, 2$, for $s = 8, 5, 3$.

t_s	MLE's by (3.3)	t_r	$Y_{t_r}^{(1)}$	$\tilde{Y}_{t_r}^{(1)}$	$RE\left(\tilde{Y}_{t_r}^{(k)}\right)$	95%RCI based on P
8.0	$\hat{\alpha} = 1.9860$	7.0	37.08	36.075	0.0271	(33.48 , 41.531)
	$\hat{\beta} = 89.248$	6.0	38.88	39.380	0.0129	(33.48 , 48.485)
		5.0	42.48	43.789	0.0308	(33.48 , 57.529)
		4.0	50.04	50.095	0.0011	(33.48 , 70.903)
		3.0	51.84	60.199	0.1613	(33.48 , 93.980)
		2.0	55.80	80.455	0.4418	(33.48 , 146.418)
5.0	$\hat{\alpha} = 2.0415$	4.0	50.04	48.408	0.0326	(42.48 , 61.306)
	$\hat{\beta} = 86.602$	3.0	51.84	57.854	0.1160	(42.48 , 84.000)
		2.0	55.80	76.619	0.3731	(42.48 , 132.786)
3.0	$\hat{\alpha} = 2.1801$	2.0	55.80	67.269	0.2055	(51.84 , 103.054)
	$\hat{\beta} = 80.869$					

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