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Original Article

On semi separation axioms in L-fuzzifying bitopological spaces

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$ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2)$

Abstract The main purpose of this paper is to introduce a concept of semi separation axioms in L-fuzzifying bitopological spaces (here L is a complete residuated lattice some times used double negation law) and discuss some of their basic properties and the structures. We shown that the category of semi-separation axioms in L- fuzzifying bitopological. Finlay we prove that its some a relation between a semi-separation axioms in L-fuzzifying bitopological spaces.

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1. Introduction

The concept of L-fuzzifying topology appeared in Höhle [1] under the name "L-fuzzy topology" (cf. Definition 4.6, Proposition 4.11 in Höhle [1] where L is a completely distributive complete lattice. In the case of $L = [0, 1]$ this terminology traces back to (Ying [2–4]) who studied the fuzzifying topology and elementarily developed fuzzy topology from a new direction with semantic method of continuous valued logic. Fuzzifying topology (resp. L-Fuzzifying topology) in the sense of Ying (resp. U. Höhle) was introduced as a fuzzy subset (resp. an L-Fuzzy subset) of the power set of an ordinary set.

Höhle [5] introduced and studied a characterization of stratified and transitive L- topology by stratified and transitive L-interior operator K, where L is a complete MV-algebra.

A characterization of L-fuzzifying topology by L-fuzzifying neighborhood system, where L is a completely distributive, was given also in Höhle [5]. Finally, Höhle [5] introduced a characterization of stratified and transitive L- topology by L-contiguity and L -fuzzifying topology, where L is a completely distributive complete MV-algebra. Many separation axioms in fuzzy to pological spaces in the sense of Chang [6] or in the sense

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of Lowen [7] are introduced and studied by many authors. Ying [2], introduced and characterized the concept of T_2(Hausdorff)-separation axiom as a fuzzy subset on the family of all [0, 1]−fuzzifying topological spaces. Shen [8], introduced and studied T_0, T_1, T_3(regular), T_4(normal)-separation axioms. Kheder, et al. [9], introduced R_0, R_1-separation axioms in fuzzifying topology and studied their relations with T_1 and T_2-separation axioms. In 2003 Zhang et al. [10], studied the concept of fuzzy (*i*; *j*) -closed, (*i*; *j*)-open sets in fuzzifying bitopological spaces. In this paper, we introduce and study the concepts of $semi-R_0$- and $semi-R_1$-separation axioms in fuzzifying bitopology and study their relations with $semi-T_1$-and $semi-T_2$-separation axioms . Furthermore, we discuss $semi-T_0$, $semi-T_3$(semi-regularity) and $semi-T_4$ (semi-normality)-separation axioms in fuzzifying bitopological spaces and give some of their characterizations as well as the relations of these axioms and other semi separation axioms in fuzzifying bitopology introduced. In Section 1, we recall some notions and results in separation axioms in *L*- fuzzifying bitopology, Also, in Section 2, semi separation axioms in *L*- fuzzifying bitopology are introduced and studied. Finally In Section 3, relations among semi separation axioms are discussed.

Definition 1.1 (Höhle [5]). The double negation law in a complete residuated lattice *L* is given as follows:

$$\forall a, b \in L,\ (a \to \bot) \to \bot = a.$$

Definition 1.2 (Höhle [5], Rosenthal [11]). A structure $(L, \vee, \wedge, *, \to, \bot, \top)$ is called a strictly two-sided commutative quantale iff

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,
(2) $(L, *, \top)$ is a commutative monoid,
(3) (a) $*$ is distributive over arbitrary joins, i.e., $a * \bigvee_{j\in J} b_j = \bigvee_{j\in J} (a * b_j)\ \forall a \in L, \forall \{b_j | j \in J\} \subseteq L$, .
(b) $\to$ is a binary operation on L defined by : $a \to b = \bigvee_{\lambda * a \le b} \lambda \quad \forall a, b \in L$.

Definition 1.3 (Ying [4]). A structure $(L, \vee, \wedge, *, \to, \bot, \top)$ is called a complete residuated lattice iff

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,
(2) $(L, *, \top)$ is a commutative monoid, i.e.,
(a) $*$ is a commutative and associative binary operation on L, and
(b) $\forall\, a \in L, a * \top = \top *\ a = a$,
(3) (a) $*$ is isotone,
(b) $\to$ is a binary operation on L which is antitone in the first and isotone in the second variable,
(c) $\to$ is couple with $*$ as: $a*b \le c$ iff $a \le b \to c\ \ \forall\, a, b, c \in L$.

Definition 1.4 [4]. Let (X, τ) be an *L*-fuzzifying topological space, let $x \in X$.

(1) The fuzzifying neighborhood system of x, denoted by $N_x \in I^{P(X)}$, is defined as follows:

$$N_x(A) = \bigvee_{x\in B\subseteq A} \tau(B).$$

2- The family of all fuzzifying closed sets is denoted by $\mathcal{F} \in \Im(P(X))$ and defined as follows:

$$A \in \mathcal{F} := X \sim A \in \tau$$

3- The closure $cl(A)$ of $A \subseteq X$ is defined as follows:

$$cl(A)(x) = 1 - \ N_x(X \sim A).$$

Definition 1.9 [12]. Let (X, τ_1) and (X, τ_2) be two an *L*-fuzzifying topological space. Then a system (X, τ_1, τ_2) consisting of a universe of discourse X with two *L*-fuzzifying topologies τ_1 and τ_2 on X is called an *L*-fuzzifying bitopological space.

Definition 1.10 [12]. Let (X, τ_1, τ_2) be an *L*-fuzzifying bitopological space.

(1) A set A is said to be a pairwise open if and only if $A \in \tau_1 \cap \tau_2$. i.e., $\bigcirc^P(A) = \min(\tau_1(A), \tau_2(A))$.
(1) A set B is said to be a pairwise closed if and only if $X - B \in \bigcirc^P$. i.e., $B \in F^P = X - B \in \bigcirc^P$.

Lema 1.1 [12]. Let (X, τ_1, τ_2) be an *L*-fuzzifying bitopological space. Then

$$F^P(B) = \min(F_1(B), F_2(B)).$$

Lema 1.1 [12]. Let (X, τ_1, τ_2) be an *L*-fuzzifying bitopological space. Then

1- $\bigcirc^P \subseteq \tau_i, \quad i = 1, 2$
2- $F^P \subseteq F_i \quad i = 1, 2$.

2. Semi separation axioms in *L*-fuzzifying bitopological space

Definition 2.1. Let (X, τ_1, τ_2) be an *L* -fuzzifying bitopological space.

(1) The family of all *L*-fuzzifying (i, j)-semiopen sets, denoted by $s\tau_{(i,j)} \in \Im(P(X))$, is defined as follows:

$$s\tau_{(i,j)}(A) \ = \bigwedge_{x\in A} cl_j(int_i(A)(x).$$

(2) The family of all fuzzifying (i, j)-semiclosed sets, denoted by $sF_{(i,j)} \in \Im(P(X))$, is defined as follows:

$$sF_{(i,j)}(A) \ = s\tau_{(i,j)}(A) \to \bot.$$

Example 2.1. If $L = [0, 1]$ and Let $X = \{a, b\}$, τ_1, τ_2 be two fuzzifying topologies defined as follows:

$$\tau_1(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{2} & \text{if} \quad A = \{a\} \\ 0 & \text{if} \quad A = \{b\} \end{cases}, \quad \tau_2(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{4} & \text{if} \quad A = \{b\} \\ 0 & \text{if} \quad A = \{a\} \end{cases}$$

and

$$F_1(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{2} & \text{if} \quad A = \{b\} \\ 0 & \text{if} \quad A = \{a\} \end{cases}, \quad F_2(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{4} & \text{if} \quad A = \{a\} \\ 0 & \text{if} \quad A = \{b\} \end{cases}$$

Note that

$$\bigcirc^{P}(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ 0 & \text{if} \quad o.w.\end{cases},$$

$$S\tau_{(i,j)}(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{4} & \text{if} \quad A=\{a\}\\ 0 & \text{if} \quad A=\{b\}\end{cases}.$$

Definition 2.2. Let (X,τ_1,τ_2) be an L-fuzzifying bitopological space and $x\in A$

(1) The (i,j)-semi neighborhood system of x is denoted by $SN_x^{(i,j)}\in\Im(P(X))$ and defined as follows:

$$SN_x^{(i,j)}(A)=\bigvee_{x\in B\subseteq A} s\tau_{(i,j)}(B).$$

(2) The (i,j)-semi derived set $sd_{(i,j)}(A)$ of A is defined as follows:

$$sd_{(i,j)}(A)(x)=\bigwedge_{B\cap(A-\{x\})\,=\,\phi}(SN_x^{(i,j)}(B)\rightarrow\perp).$$

(3) The (i,j)-semi closure of $A\subseteq X$, is denoted by $scl_{(i,j)}(A)(x)$ and defined as follows:

$$scl_{(i,j)}(A)(x)=\bigwedge_{x\notin B\supseteq A}(sF_{(i,j)}(B)\rightarrow\perp).$$

(4) The (i,j)-semi interior of $A\subseteq X$, is denoted by $scl_{(i,j)}(A)(x)$ and defined as follows:

$$sint_{(i,j)}(A)(x)=\bigwedge_{x\notin B\supseteq A}(SN_x^{(i,j)}(A)).$$

For simplicity we put the following notations:

$$SK_{x,y}^{(i,j)}=\left(\bigvee_{y\notin A}SN_x^{(i,j)}(A)\right)\vee\left(\bigvee_{x\notin A}SN_y^{(i,j)}(A)\right),$$

$$SH_{x,y}^{(i,j)}=\left(\bigvee_{y\notin B}SN_x^{(i,j)}(B)\right)\wedge\left(\bigvee_{x\notin C}SN_y^{(i,j)}(C)\right),$$

$$SM_{x,y}^{(i,j)}=\bigvee_{C\cap B=\phi}(SN_x^{(i,j)}(B)\wedge SN_y^{(i,j)}(C)),$$

$$SV_{x,D}^{(i,j)}=\bigvee_{A\cap B=\phi,D\subseteq B}(SN_x^{(i,j)}(A)\wedge S\tau_{(i,j)}(B)),$$

$$SW_{A,B}^{(i,j)}=\bigvee_{A\subseteq C,\ B\subseteq D,\ C\cap D=\phi}\min(S\tau_{(i,j)}(C),S\tau_{(i,j)}(D)),$$

where $x,y\in X,A,B,C,D\in P(X)$ and $(X,\tau_1,\ \tau_2)$ is an L-fuzzifying bitopological space.

Definition 2.3. Let Ω be the class of all L-fuzzifying bitopological spaces. The unary L-predicates $ST_{R_i}^{(i,j)}$, $ST_{R_j}^{(i,j)}$ and $ST_R^{(i,j)}\in L^{\Omega}$ are defined as follows:

$$ST_{R_i}^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\notin u}((F_{\tau_i}(u)\rightarrow\bigvee_{A\cap B=\phi,\ u\subseteq B}\min\left(SN_x^{(i,j)}(A),\left(\bigwedge_{y\in u}S\tau_{(i,j)}(B)\right)\right)$$

$$ST_{R_j}^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\notin u}((F_{\tau_j}(u)\rightarrow\bigvee_{A\cap B=\phi,\ u\subseteq B}\min\left(SN_x^{(i,j)}(A),\left(\bigwedge_{y\in u}S\tau_{(i,j)}(B)\right)\right),$$

$$ST_R^{(i,j)}(X,\tau_1,\tau_2)=ST_{R_i}^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_{R_j}^{(i,j)}(X,\tau_1,\tau_2).$$

Example 2.2. If $L=[0,1]$ and Let $X=\{a,b\}$, τ_1,τ_2 be two fuzzifying topologies defined as follows:

$$\tau_1(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{5} & \text{if} \quad A=\{a\}\\ \frac{1}{2} & \text{if} \quad A=\{b\}\end{cases},\quad \tau_2(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{4} & \text{if} \quad A=\{a\}\\ \frac{1}{8} & \text{if} \quad A=\{b\}\end{cases}$$

Note that

$$\bigcirc^{P}(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{5} & \text{if} \quad A=\{a\}\\ \frac{1}{8} & \text{if} \quad A=\{b\}\end{cases},$$

$$S\tau_{(1,2)}(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{5} & \text{if} \quad A=\{a\}\\ \frac{1}{2} & \text{if} \quad A=\{b\}\end{cases}$$

$$ST_R^{(i,j)}(X,\tau_1,\tau_2)=ST_{R_i}^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_{R_j}^{(i,j)}(X,\tau_1,\tau_2)=\frac{7}{10}$$

Definition 2.4. Let Ω be the class of all L-fuzzifying bitopological spaces. The unary L-predicates $ST_n^{(i,j)}\in L^{\Omega}, n=0,1,2,3,4$ and $SR_n^{(i,j)}\in L^{\Omega}$, $n=0,1$ are defined as follows:

$$ST_0^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\neq y}SK_{x,y}^{(i,j)},$$

$$ST_1^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\neq y}SH_{x,y}^{(i,j)},$$

$$ST_2^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\neq y}SM_{x,y}^{(i,j)},$$

$$ST_3^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\notin D}\left(F_{\tau_{(i,j)}}(D)\rightarrow SV_{x,\ D}^{(i,j)}\right),$$

$$ST_4^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{A\cap B=\phi}\left(\min(F_{\tau_{(i,j)}}(A),F_{\tau_{(i,j)}}(B))\rightarrow SW_{A,B}^{(i,j)}\right)),$$

$$SR_0^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\neq y}(SK_{x,y}^{(i,j)}\rightarrow SH_{x,y}^{(i,j)}),$$

$$SR_1^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\neq y}(SK_{x,y}^{(i,j)}\rightarrow SM_{x,y}^{(i,j)}).$$

Example 2.3. If $L=[0,1]$ and Let $X=\{a,b\}$, τ_1,τ_2 be two fuzzifying topologies defined as follows:

$$\tau_1(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{5} & \text{if} \quad A=\{a\}\\ \frac{1}{2} & \text{if} \quad A=\{b\}\end{cases},\quad \tau_2(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{4} & \text{if} \quad A=\{a\}\\ \frac{1}{8} & \text{if} \quad A=\{b\}\end{cases}$$

Note that

$$\bigcirc^{P}(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{5} & \text{if} \quad A=\{a\}\\ \frac{1}{8} & \text{if} \quad A=\{b\}\end{cases},$$

$$S\tau_{(1,2)}(A)=\begin{cases}1 & \text{if} \quad A\in\{\phi,X\}\\ \frac{1}{5} & \text{if} \quad A=\{a\}\\ \frac{1}{2} & \text{if} \quad A=\{b\}\end{cases}$$

$$ST_3^{(1,2)}(X,\tau_1,\tau_2)=\bigwedge_{x\notin D}\left(F_{\tau_{(1,2)}}(D)\to SV_{x,D}^{(1,2)}\right)=\frac{7}{10}$$

$$ST_{R_1}^{(1,2)}(X,\tau_1,\tau_2)=\bigwedge_{x\notin u}((F_{\tau_1}(u)$$
$$\to\bigvee_{A\cap B=\phi,\ u\subseteq B}\min\left(SN_x^{(1,2)}(A),\left(\bigwedge_{y\in u}S\tau_{(1,2)}(B)\right)\right)=\frac{7}{10}$$

$$ST_{R_2}^{(1,2)}(X,\tau_1,\tau_2)=\bigwedge_{x\notin u}((F_{\tau_2}(u)$$
$$\to\bigvee_{A\cap B=\phi,\ u\subseteq B}\min\left(SN_x^{(1,2)}(A),\left(\bigwedge_{y\in u}S\tau_{(1,2)}(B)\right)\right)=\frac{7}{10}$$

Remark 2.1. Let (X,τ_1,τ_2) be an L-fuzzifying bitopological space. From remark 3.1 in [15], we have $ST_i^{(1,2)}(X,\tau_1,\tau_2)\neq ST_i^{(2,1)}(X,\tau_1,\tau_2),\ i=0,1,2,3,4$

Lemma 2.1. *Let $(X,\tau)\in\Omega$. Then for any $x,y\in X$,*

(1) $SM_{x,y}^{(i,j)}\leq SH_{x,y}^{(i,j)}$,
(2) $SH_{x,y}^{(i,j)}\leq SK_{x,y}^{(i,j)}$,
(3) $SM_{x,y}^{(i,j)}\leq SK_{x,y}^{(i,j)}$.

Proof.

1- $SM_{x,y}^{(i,j)}=\bigvee_{C\cap B=\phi}\min(SN_x^{(i,j)}(B),SN_y^{(i,j)}(C))\leq\bigvee_{y\notin B,\ x\notin C}\min(SN_x^{(i,j)}(B),SN_y^{(i,j)}(C))=SH_{x,y}^{(i,j)}$,

2- $SK_{x,y}^{(i,j)}=\max(\bigvee_{y\notin A}SN_x^{(i,j)}(A)),(\bigvee_{x\notin A}SN_y^{(i,j)}(A))\geq\bigvee_{y\notin A}SN_x^{(i,j)}(A)$

$$\geq\bigvee_{y\notin A,\ x\notin B}\min(SN_x^{(i,j)}(A),SN_y^{(i,j)}(B))=SH_{x,y}^{(i,j)}$$

3- It is obtained from (1) and (2). □

Theorem 2.1. *For any $(X,\tau_1,\tau_2)\in\Omega$, $SR_1^{(i,j)}(X,\tau_1,\tau_2)\leq SR_0^{(i,j)}(X,\tau_1,\tau_2)$.*

Proof. $SR_1^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\neq y}(SK_{x,y}^{(i,j)}\to SM_{x,y}^{(i,j)})$. Since $\to$ is isotone in the second, then

$$SR_1^{(i,j)}(X,\tau_1,\tau_2)\leq\bigwedge_{x\neq y}(SK_{x,y}^{(i,j)}\to SH_{x,y}^{(i,j)})=SR_0^{(i,j)}(X,\tau_1,\tau_2).$$

□

Theorem 2.2. *For any $(X,\tau_1,\tau_2)\in\Omega$, the following statements are satisfied:*

(1) $ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq SR_0^{(i,j)}(X,\tau_1,\tau_2)$,
(2) $ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq ST_0^{(i,j)}(X,\tau_1,\tau_2)$,
(3) $ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq SR_0^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_0^{(i,j)}(X,\tau_1,\tau_2)$,
(4) If $ST_0^{(i,j)}(X,\tau_1,\tau_2)=\top$, then $ST_1^{(i,j)}(X,\tau_1,\tau_2)=SR_0^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_0^{(i,j)}(X,\tau_1,\tau_2)$.

Proof.

(1) Since $SH_{x,y}^{(i,j)}\leq SK_{x,y}^{(i,j)}\to SH_{x,y}^{(i,j)}$ so that $ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq SR_0^{(i,j)}(X,\tau_1,\tau_2)$.
(2) From Lemma 2.1 (2) one can deduce that $ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq ST_0^{(i,j)}(X,\tau_1,\tau_2)$.
(3) The proof follows from (1) and (2) .
(4) Since $\top\to\alpha=\alpha\ \forall\alpha\in L$ (Indeed $\top\to\alpha=\bigvee_{\lambda*\top\leq\alpha}\lambda=\bigvee_{\lambda\leq\alpha}\lambda=\alpha$.) then $ST_1^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\neq y}SH_{x,y}^{(i,j)}=(\bigwedge_{x\neq y}(SK_{x,y}^{(i,j)}\to SH_{x,y}^{(i,j)}))\wedge\top=SR_0^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_0^{(i,j)}(X,\tau_1,\tau_2)$
because $\forall x,y\in X\ s.t.\ x\neq y$, if $ST_0^{(i,j)}(X,\tau_1,\tau_2)=\top$, then $SK_{x,y}^{(i,j)}=\top$. □

Theorem 2.3. *For any $(X,\tau_1,\tau_2)\in\Omega$, the following statements are satisfied:*

(1) $ST_2^{(i,j)}(X,\tau_1,\tau_2)\leq SR_1^{(i,j)}(X,\tau_1,\tau_2)$,
(2) $ST_2^{(i,j)}(X,\tau_1,\tau_2)\leq ST_1^{(i,j)}(X,\tau_1,\tau_2)$,
(3) $ST_2^{(i,j)}(X,\tau_1,\tau_2)\leq SR_1^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_0^{(i,j)}(X,\tau_1,\tau_2)$
(4) *If* $ST_0^{(i,j)}(X,\tau_1,\tau_2)$, *then* $ST_2^{(i,j)}(X,\tau_1,\tau_2)=SR_1^{(i,j)}(X,\tau_1,\tau_2)$.

Proof. The proof is similar to the proof of Theorem 2.2. □

Theorem 2.4. *For any $(X,\tau_1,\tau_2)\in\Omega$, $ST_2^{(i,j)}(X,\tau_1,\tau_2)\leq ST_0^{(i,j)}(X,\tau_1,\tau_2)$.*

Proof. The conclusion is obtained from Theorem 2.2(2) and Theorem 2.3(2). □

Theorem 2.5. *If L satisfies the completely distributive law, then for any $(X,\tau_1,\tau_2)\in\Omega$,*

$$ST_1^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x\in X}SF_{(i,j)}(\{x\}).$$

Proof. Let $x_1,x_2\in X\ s.t.\ x_1\neq x_2$.
Then

$$\bigwedge_{x\in X}SF_{(i,j)}(\{x\})=\bigwedge_{x\in X}S\tau_{(i,j)}(X-\{x\})$$
$$=\bigwedge_{x\in X}\left(\bigwedge_{y\in X-\{x\}}SN_y^{(i,j)}(X-\{x\})\right)$$
$$\leq\bigwedge_{y\in X-\{x_2\}}SN_y^{(i,j)}(X-\{x_2\})\leq SN_{x_1}^{(i,j)}(X-\{x_2\})$$
$$=\bigvee_{x_2\notin A}SN_{x_1}^{(i,j)}(A)$$

Similary, we have

$$\bigwedge_{x\in X}SF_{(i,j)}(\{x\})\leq\bigvee_{x_1\notin B}SN_{x_2}^{(i,j)}(B)$$

Then

$$\bigwedge_{x\in X}SF_{(i,j)}(\{x\})\leq\bigwedge_{x_1\neq x_2}\bigvee_{x_1\notin B,x_2\notin A}(SN_{x_1}^{(i,j)}(A)\wedge SN_{x_2}^{(i,j)}(B))$$
$$=ST_1(X,\tau).$$

For the other hand,

$$ST_1^{(i,j)}(X,\tau_1,\tau_2)=\bigwedge_{x_1,x_2\in X,x_1\neq x_2}\left(\left(\bigvee_{x_2\notin A}SN_{x_1}^{(i,j)}(A)\right)\wedge\left(\bigvee_{x_1\notin B}SN_{x_2}^{(i,j)}(B)\right)\right)$$

$$= \bigwedge_{x_1 \neq x_2} ((SN_{x_1}^{(i,j)}(X - \{x_2\}) \wedge (SN_{x_2}^{(i,j)}(X - \{x_1\}))$$
$$\leq \bigwedge_{x_1 \neq x_2} SN_{x_1}^{(i,j)}(X - \{x_2\})$$
$$= \bigwedge_{x_2 \in X} \bigwedge_{x_1 \in X - \{x_2\}} SN_{x_1}^{(i,j)}(X - \{x_2\})$$
$$= \bigwedge_{x_2 \in X} \bigwedge_{x \in X} \tau_{(i,j)}(X - \{x_2\})$$
$$= \bigwedge_{x \in X} \tau_{(i,j)}(X - \{x\}) = \bigwedge_{x \in X} \bigwedge_{x \in X} SF_{(i,j)}(\{x\})$$
$$= \bigwedge_{x_2 \in X} \bigwedge_{x \in X} \tau_{(i,j)}(X - \{x_2\})$$
$$= \bigwedge_{x \in X} S\tau_{(i,j)}(X - \{x\}) = \bigwedge_{x \in X} \bigwedge_{x \in X} SF_{(i,j)}(\{x\}).$$

□

Definition 2.3. Let Ω be the class of all L -fuzzifying bitopological spaces. We define

$${}^1ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \notin u}((F_{\tau_i}(u) \to \bigvee_{A \in P(X)} \min\left(SN_x^{(i,j)}(A), \bigwedge_{y \in u}(Scl_{(i,j)}(A)(y) \to \bot)\right),$$

$${}^1ST_{R_j}^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \notin u}((F_{\tau_j}(u) \to \bigvee_{A \in P(X)} \min\left(SN_x^{(i,j)}(A), \bigwedge_{y \in u}(S\tau_{(i,j)}(A)(y) \to \bot)\right),$$

$${}^1ST_R^{(i,j)}(X, \tau_1, \tau_2) = {}^1ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) \wedge {}^1ST_{R_j}^{(i,j)}(X, \tau_1, \tau_2)$$

$${}^2ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \in u}((\tau_i(u) \to \bigvee_{B \in P(X)} \min(SN_x^{(i,j)}(B), \wedge[[Scl_{(i,j)}(B), u[[]),$$

$${}^2ST_{R_j}^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \in u}((\tau_j(u) \to \bigvee_{B \in P(X)} \min(SN_x^{(i,j)}(B), \wedge[[Scl_{(i,j)}(B), u[[\ ),$$

$${}^2ST_R^{(i,j)}(X, \tau_1, \tau_2) = {}^2ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) \wedge {}^2ST_{R_j}^{(i,j)}(X, \tau_1, \tau_2).$$

Theorem 2.6. *Let* $(X, \tau_1, \tau_2) \in \Omega$ *, then,*

1- $ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = {}^nST_{R_i}^{(i,j)}(X, \tau_1, \tau_2), \quad n = 1, 2$
2- $ST_{R_j}^{(i,j)}(X, \tau_1, \tau_2) = {}^nST_{R_j}^{(i,j)}(X, \tau_1, \tau_2), \quad n = 1, 2$
3- $ST_R^{(i,j)}(X, \tau_1, \tau_2) = {}^nST_R^{(i,j)}(X, \tau_1, \tau_2), \quad n = 1, 2$

Proof.

(1) ${}^1ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \notin u}((F_{\tau_i}(u) \to \bigvee_{A \in P(X)} \min (SN_x^{(i,j)}(A), \bigwedge_{y \in u}(Scl_{(i,j)}(A)(y) \to \bot)) = \bigwedge_{x \notin u}((SF_{\tau_i}(u) \to \bigvee_{A \in P(X)} \min(SN_x^{(i,j)}(A), (\bigwedge_{y \in u} SN_y^{(i,j)}(X - A))))$ and $ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \notin u}((F_{\tau_i}(u) \to \bigvee_{A \cap B = \phi, u \subseteq B} \min(SN_x^{(i,j)}(A), (\bigwedge_{y \in u} S\tau_{(i,j)}(B)))$. So, the result holds if we prove that

$$\bigvee_{A \in P(X)} \min(SN_x^{(i,j)}(A), \left(\bigwedge_{y \in u} SN_y^{(i,j)}(X - A)\right)$$
$$= \bigvee_{A \cap B = \phi,\ u \subseteq B} \min(SN_x^{(i,j)}(A), \left(\bigwedge_{y \in u} S\tau_{(i,j)}(B)\right).(*)$$

In the left side of $(*)$ if $A \cap u \neq \phi, \ \exists\, y \in u\ s.t.\ y \notin X - A$ so that $\bigwedge_{y \in u} SN_y^{(i,j)}(X - A) = \bot$. Second,

$$\bigvee_{A \in P(X)} \min(SN_x^{(i,j)}(A), \left(\bigwedge_{y \in u} SN_y^{(i,j)}(X - A)\right)$$
$$= \bigvee_{A \cap u = \phi,\ A \in P(X)} \min(SN_x^{(i,j)}(A), \left(\bigwedge_{y \in u}\left(\bigvee_{y \in B \subseteq X - A} S\tau_{(i,j)}(B)\right)\right).$$

Now we prove that $\bigwedge_{y \in u} \bigvee_{y \in B \subseteq X - A} S\tau_{(i,j)}(B) = \bigvee_{A \cap B = \phi, u \subseteq B} S\tau_{(i,j)}(B)$. Let $y \in u$. Assume $S = \{H \in P(X) | H \cap A = \phi$ and $u \subseteq H\}$ and $\wp_y = \{M \in P(X) | y \in M \subseteq X - A\}$.
Then $S \subseteq \wp_y$ so that $\bigvee_{B \in \wp_y} S\tau_{(i,j)}(B) \geq \bigvee_{B \in S} S\tau_{(i,j)}(B)$ so that $\bigwedge_{y \in D} \bigvee_{y \in B \subseteq X - A} S\tau_{(i,j)}(B) \geq \bigvee_{A \cap B = \phi,\ D \subseteq B} S\tau_{(i,j)}(B)$. Let $\wp_y^{\tau_{(i,j)}} = \{\tau(M) | M \in \wp_y\}$. Then $\bigwedge_{y \in D} \bigvee \wp_y^{\tau_{(i,j)}} = \bigwedge_{y \in D} \bigvee_{B \in \wp_y} S\tau_{(i,j)}(B) = \bigvee_{f \in \prod_{y \in D} \wp_y^{\tau_{(i,j)}}} \bigwedge_{y \in D} f(y)$. Then for each $f \in \prod_{y \in D} \wp_y^{\tau_{(i,j)}}$, $\exists\, K = \cup\{f(y) | f(y) \in \wp_y^{\tau_{(i,j)}}, y \in D\}\ s.t.\ D \subseteq K \subseteq X - A$
and $\bigwedge_{y \in D} f(y) \leq \tau(\cup\{f(y) | f(y) \in \wp_y^{\tau_{(i,j)}},\ y \in D\}) = S\tau_{(i,j)}(K)$ so that
$\bigwedge_{y \in D} f(y) \leq S\tau_{(i,j)}(K) \leq \bigvee_{A \cap B = \phi,\ D \subseteq B} S\tau_{(i,j)}(B)$ so that $\bigwedge_{y \in D} \bigvee_{y \in B \subseteq X - A} S\tau_{(i,j)}(B) = \bigvee_{f \in \prod_{y \in D} \wp_y^{S\tau_{(i,j)}}} \bigwedge_{y \in D} f(y) \leq \bigvee_{A \cap B = \phi,\ D \subseteq B} S\tau_{(i,j)}(B)$. So shoud prove that $\bigvee_{A \in P(X)} \min(SN_x^{(i,j)}(A), \bigwedge_{y \in u}(Scl_{(i,j)}(A)(y) \to \bot) = \bigvee_{A \cap B = \phi,\ u \subseteq B} \min(SN_x^{(i,j)}(A), (\bigwedge_{y \in u} S\tau_{(i,j)}(B))$

(2) It is similar to (1)
(3) It is clear.

□

Theorem 2.7. *If L satisfies the completely distributive law, for any* $(X, \tau_1, \tau_2) \in \Omega$.
Then ${}^2ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2)$

Proof.

$$ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = {}^1ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \notin u}((F_{\tau_i}(u) \to \bigvee_{A \in P(X)} \min(SN_x^{(i,j)}(A), \bigwedge_{y \in u}(Scl_{(i,j)}(A)(y) \to \bot))$$
$$= \bigwedge_{x \notin X - B}((F_{\tau_i}(X - B) \to \bigvee_{A \in P(X)} \min\left(SN_x^{(i,j)}(A), \left(\bigwedge_{y \in X - B}(Scl_{(i,j)}(A)(y) \to \bot)\right)\right)$$
$$= \bigwedge_{x \in B}(S\tau_{(i,j)}(B) \to \bigvee_{A \in P(X)} \min(SN_x^{(i,j)}(A), \left(\bigwedge_{y \in X - B}(Scl_{(i,j)}(A)(y) \to \bot)\right)$$

$$= \bigwedge_{x\in u}((\tau_i(u) \to \bigvee_{B\in P(X)} \min(SN_x^{(i,j)}(B),$$

$$\wedge[[Scl_{(i,j)}(A), B[[]) =^2 ST_{R_i}^{(i,j)}(X, \tau_1, \tau_2)$$

Note that $\bigwedge_{y\in X}(Scl_{(i,j)}(A)(y) \to B(y)) = (\bigwedge_{y\in B}(Scl_{(i,j)}(A)(y) \to \top) \wedge (\bigwedge_{y\in X-B}(Scl_{(i,j)}(A)(y) \to \bot)) = \top \wedge \bigwedge_{y\in X-B}(Scl_{(i,j)}(A)(y) \to \bot) = \bigwedge_{y\in X-B}(Scl_{(i,j)}(A)(y) \to \bot)$ □

Theorem 2.8. *Let $(X, \tau_1, \tau_2) \in \Omega$. If L satisfies the double negation law, then,*

$$ST_0^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x\neq y}((cl_{(i,j)}(\{y\})(x) \to \bot) \vee (cl_{(i,j)}(\{x\})(y) \to \bot)).$$

Proof.

$$ST_0^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x\neq y}\left(\left(\bigvee_{y\notin A} SN_x^{(i,j)}(A)\right) \vee \left(\bigvee_{x\notin A} SN_y^{(i,j)}(A)\right)\right) = \bigwedge_{x\neq y}((SN_x^{(i,j)}(X - \{y\}) \vee (SN_y^{(i,j)}(X - \{x\}))$$

$$= \bigwedge_{x\neq y}((cl_{(i,j)}(\{y\})(x) \to \bot) \vee (cl_{(i,j)}(\{x\})(y) \to \bot)).$$

□

Theorem 2.9. *Let $(X, \tau_1, \tau_2) \in \Omega$, and let A be a finite subset of X. If L satisfies the completely distributive, then,*

$$ST_1^{(i,j)}(X, \tau_1, \tau_2) \le \bigwedge_{y\in X} SN_y^{(i,j)}((X - A) \cup \{y\}).$$

Proof. Now,

$$\bigwedge_{y\in X-A} SN_y^{(i,j)}((X - A) \cup \{y\}) = \bigwedge_{y\in X-A} SN_y^{(i,j)}(X - A)$$

$$= \bigwedge_{y\in X-A} SN_y^{(i,j)}\left(\left(X - \bigcup_{x\in A}\{x\}\right) = \bigwedge_{y\in X-A} SN_y^{(i,j)}\left(\bigcap_{x\in A}(X - \{x\}\right)\right)$$

$$= \bigwedge_{y\in X-A}\left(\bigwedge_{x\in A} SN_y^{(i,j)}(X - \{x\})\right) \ge \bigwedge_{x\neq y} SN_y^{(i,j)}(X - \{x\}),$$

and

$$\bigwedge_{y\in A} SN_y^{(i,j)}((X - A) \cup \{y\}) = \bigwedge_{y\in A} SN_y^{(i,j)}(X - (A - \{y\}))$$

$$= \bigwedge_{y\in A} SN_y^{(i,j)}\left(X - \left(\bigcup_{x\in A-\{y\}}\{x\}\right)\right)$$

$$= \bigwedge_{y\in A} SN_y^{(i,j)}\left(\bigcap_{x\in A-\{y\}}(X - \{x\})\right)$$

$$= \bigwedge_{y\in A}\left(\bigwedge_{x\in A-\{y\}} SN_y^{(i,j)}(X - \{x\})\right) \ge \bigwedge_{x\neq y} SN_y^{(i,j)}(X - \{x\}).$$

Then $\bigwedge_{y\in X} SN_y^{(i,j)}((X - A) \cup \{y\}) \ge \bigwedge_{x\neq y} SN_y^{(i,j)}(X - \{x\}) = \bigwedge_{x\in X}(\bigwedge_{y\in X-\{x\}} SN_y^{(i,j)}(X - \{x\})) = \bigwedge_{x\in X} S\tau_{(i,j)}(X - \{x\}) = \bigwedge_{x\in X} SF\tau_{(i,j)}(\{x\}) = ST_1^{(i,j)}(X, \tau_1, \tau_2)$. □

Definition 2.4. Let $(X, \tau_1, \tau_2) \in \Omega$ and let $A \subseteq X$. The L-fuzzifying derived set of A is denoted by $d_\tau(A) \in L^X$ and defined as follows:

$$d_{(i,j)}(A)(x) = SN_x^{(i,j)}((X - A) \cup \{x\}) \to \bot \ \ \forall x \in X.$$

Theorem 2.10. *Let $(X, \tau) \in \Omega$ and let A be a finite subset of X. If L satisfies the completely distributive and the double negation law, then*

$$ST_1^{(i,j)}(X, \tau_1, \tau_2) \le [[d_{(i,j)}(A), 1_\phi]].$$

Proof. It follows from Theorem 2.7 and since

$$[[d_{(i,j)}(A), 1_\phi]] = \bigwedge_{y\in X}(d_{(i,j)}(A)(y) \to 1_\phi(y)) \wedge (1_\phi(y) \to d_{(i,j)}(y)))$$

$$= \bigwedge_{y\in X}(d_{(i,j)}(A)(y) \to 1_\phi(y)) = \bigwedge_{y\in X} SN_y^{(i,j)}((X - A) \cup \{y\}.$$

□

Theorem 2.11. *Let $(X, \tau) \in \Omega$, and let*

(1) ${}^1ST_N^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{A\subseteq B}((F_i(A) \wedge \tau_j(B)) \to \bigvee_{A\subseteq u\subseteq v\subseteq B}(\min(S\tau_{(i,j)}(u), SF_{(i,j)}(v)))$

(2) ${}^2ST_N^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{A\cap B=\phi}((F_i(A) \wedge \tau_j(B)) \to \bigvee_{A\subseteq u}(\min(S\tau_{(i,j)}(u), Scl_{(i,j)}(u)))$

(3) ${}^3ST_N^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{A\subseteq B}((F_i(A) \wedge \tau_j(B)) \to \bigvee_{A\subseteq B}(\min(S\tau_{(i,j)}(u), Scl_{(i,j)}(v)))$

(4) $ST_N^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{A\cap B=\phi}(\min(F_{\tau_i}(A), F_{\tau_j}(B)) \to \bigvee_{U\cap V=\phi,\ A\subseteq V,\ B\subseteq U}(\min(S\tau_{(i,j)}(u), S\tau_{(i,j)}(v)))).$

Then $ST_N^{(i,j)}(X, \tau_1, \tau_2) = {}^nST_N^{(i,j)}(X, \tau_1, \tau_2), \quad n = 1, 2, 3$

Proof. (1) From Lema 1.1

$$ST_N^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{A\cap B=\phi}(\min(F_{\tau_i}(A), F_{\tau_j}(B)) \to \bigvee_{u\cap v=\phi,\ A\subseteq v,\ B\subseteq u}(\min(S\tau_{(i,j)}(u), S\tau_{(i,j)}(v)))).$$

$$= \bigwedge_{(X-A\cap B)=\phi}\left(\min(F_{\tau_{(i,j)}}(X - A), F_{\tau_{(i,j)}}(B)) \to \bigvee_{(u\cap X-v)=\phi u\subseteq X-v,\ A\subseteq X-v,\ B\subseteq u}(\min(S\tau_{(i,j)}(u), S\tau_{(i,j)}(X - v)))\right)$$

$$= \bigwedge_{B\subseteq A}(\min(F_{\tau_{(i,j)}}(A), F_{\tau_{(i,j)}}(B)) \to \bigvee_{u\subseteq v,\ v\subseteq A,\ B\subseteq u}(\min(S\tau_{(i,j)}(u), SF_{(i,j)}(v)))$$

$$= \bigwedge_{B\subseteq A}(\min(F_{\tau_{(i,j)}}(A), F_{\tau_{(i,j)}}(B)) \to \bigvee_{B\subseteq u\subseteq v\subseteq A}(\min(S\tau_{(i,j)}(u), SF_{(i,j)}(v))) =^1 ST_N^{(i,j)}(X, \tau_1, \tau_2)$$

(2) and (3) are similar □

Theorem 2.12. *Let $(X, \tau_1, \tau_2) \in \Omega$, Then* $ST_N^{(i,j)}(X, \tau_1, \tau_2) = ST_4^{(i,j)}(X, \tau_1, \tau_2)$.

Proof. From Lema 1.1 (2) . It is obtained. □

The following example shows that generally the reverse of the Theorem 2.12 need not be true.

Example 2.4. [12] If $L = [0, 1]$ and Let $X = \{a, b\}$, τ_1, τ_2 be two fuzzifying topologies defined as follows:

$$\tau_1(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{2} & \text{if} \quad A = \{a\} \\ 0 & \text{if} \quad A = \{b\} \end{cases},$$

$$\tau_2(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{4} & \text{if} \quad A = \{b\} \\ 0 & \text{if} \quad A = \{a\} \end{cases}$$

and

$$F_1(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{2} & \text{if} \quad A = \{b\} \\ 0 & \text{if} \quad A = \{a\} \end{cases},$$

$$F_2(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{4} & \text{if} \quad A = \{a\} \\ 0 & \text{if} \quad A = \{b\} \end{cases}$$

Note that

$$\bigcirc^P(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ 0 & \text{if} \quad o.w. \end{cases},$$

$$S\tau_{(i,j)}(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X\} \\ \frac{1}{4} & \text{if} \quad A = \{a\} \\ 0 & \text{if} \quad A = \{b\} \end{cases}.$$

Hence $ST_4^{(i,j)}(X, \tau_1, \tau_2) = 1 \nleq \frac{3}{4} = ST_N^{(i,j)}(X, \tau_1, \tau_2)$

3. Relations among semi-separation axioms in fuzzifying bitopological spaces

Theorem 3.1. *If L satisfies the completely distributive law, then for any* $(X, \tau) \in \Omega$, $ST_3^{(i,j)}(X, \tau_1, \tau_2) * ST_1^{(i,j)}(X, \tau_1, \tau_2) \leq ST_2^{(i,j)}(X, \tau_1, \tau_2)$.

Proof. Now,

$$\bigwedge_{x \neq y}(S\tau_{(i,j)}(X - \{y\}))$$

$$\to \bigvee_{A \cap B = \phi,\ y \in B}\left(\bigwedge_{y \in B}(SN_x^{(i,j)}(A) \wedge SN_y^{(i,j)}(B))\right)\Bigg)$$

$$\leq \bigwedge_{x \neq y}\left(\bigwedge_{y \in X} S\tau_{(i,j)}(X - \{y\})\right.$$

$$\to \bigvee_{A \cap B = \phi,\ y \in B}\left(\bigwedge_{y \in B}(SN_x^{(i,j)}(A) \wedge SN_y^{(i,j)}(B))\right)\Bigg)$$

$$= \bigwedge_{x \neq y} ST_1^{(i,j)}(X, \tau_1, \tau_2))$$

$$\to \bigvee_{A \cap B = \phi,\ y \in B}\left(\bigwedge_{y \in B}(SN_x^{(i,j)}(A) \wedge SN_y^{(i,j)}(B))\right)\Bigg)$$

$$= ST_1^{(i,j)}(X, \tau_1, \tau_2)$$

$$\to \bigwedge_{x \neq y}\left(\bigvee_{A \cap B = \phi,}\left(\bigwedge_{y \in B}(SN_x^{(i,j)}(A) \wedge SN_y^{(i,j)}(B))\right)\right)$$

$$\leq ST_1^{(i,j)}(X, \tau_1, \tau_2) \to \bigwedge_{x \neq y}\left(\bigvee_{A \cap B = \phi}(SN_x^{(i,j)}(A) \wedge SN_y^{(i,j)}(B))\right)$$

$$= ST_1^{(i,j)}(X, \tau_1, \tau_2) \to ST_2^{(i,j)}(X, \tau_1, \tau_2).$$

Since $ST_3^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{x \notin D}(S\tau_{(i,j)}(X - D) \to \bigvee_{A \cap B = \phi, D \subseteq B}(SN_x^{(i,j)}(A) \wedge \tau_{(i,j)}(B)))$
$= \bigwedge_{x \notin D}(S\tau_{(i,j)}(X - D) \to \bigvee_{A \cap B = \phi, D \subseteq B}(SN_x^{(i,j)}(A) \wedge (\bigwedge_{y \in B}(SN_y^{(i,j)}(B))))$
$\leq \bigwedge_{x \notin \{y\}}(S\tau_{(i,j)}(X - \{y\}) \to \bigvee_{A \cap B = \phi,\ y \in B}(\bigwedge_{y \in B}(SN_x^{(i,j)}(A) \wedge SN_y^{(i,j)}(B))))$
$= \bigwedge_{x \neq y}(S\tau_{(i,j)}(X - \{y\}) \to \bigvee_{A \cap B = \phi,\ y \in B}(\bigwedge_{y \in B}(SN_x^{(i,j)}(A) \wedge SN_y^{(i,j)}(B))))$, then from above, $ST_3^{(i,j)}(X, \tau_1, \tau_2) \leq ST_1^{(i,j)}(X, \tau_1, \tau_2) \to ST_2^{(i,j)}(X, \tau_1, \tau_2)$, so that $ST_3^{(i,j)}(X, \tau_1, \tau_2) * ST_1^{(i,j)}(X, \tau_1, \tau_2) \leq ST_2^{(i,j)}(X, \tau_1, \tau_2)$. □

Theorem 3.2. *For any* $(X, \tau) \in \Omega$,

1- $ST_1^{(i,j)}(X, \tau_1, \tau_2) \leq SR_0^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2)$
2- If $ST_1^{(i,j)}(X, \tau_1, \tau_2) = \top$, then $ST_1^{(i,j)}(X, \tau_1, \tau_2) \leq SR_0^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2)$
3- $SR_1^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2) \leq ST_2^{(i,j)}(X, \tau_1, \tau_2)$
4- If $ST_0^{(i,j)}(X, \tau_1, \tau_2) = \top$, then $ST_2^{(i,j)}(X, \tau_1, \tau_2) = SR_0^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2)$

Proof. It obvious □

Theorem 3.3. *For any* $(X, \tau) \in \Omega$,

1- $ST_2^{(i,j)}(X, \tau_1, \tau_2) \leq SR_1^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2)$
2- If $ST_2^{(i,j)}(X, \tau_1, \tau_2) = \top$, then $ST_2^{(i,j)}(X, \tau_1, \tau_2) = SR_1^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2)$
3- $SR_1^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2) \leq ST_2^{(i,j)}(X, \tau_1, \tau_2)$
4- If $ST_0^{(i,j)}(X, \tau_1, \tau_2) = \top$, then $ST_2^{(i,j)}(X, \tau_1, \tau_2) = SR_1^{(i,j)}(X, \tau_1, \tau_2) \wedge ST_0^{(i,j)}(X, \tau_1, \tau_2)$

Proof. It obvious □

Theorem 3.4. *If L satisfies the completely distributive law, then for any* $(X, \tau) \in \Omega$, $ST_4^{(i,j)}(X, \tau_1, \tau_2) * ST_1^{(i,j)}(X, \tau_1, \tau_2) \leq ST_3^{(i,j)}(X, \tau_1, \tau_2)$.

Proof.

$$ST_4^{(i,j)}(X, \tau_1, \tau_2) = \bigwedge_{E \cap D = \phi}\left(\min(SF_{\tau_{(i,j)}}(E), SF_{\tau_{(i,j)}}(D))\right.$$

$$\left.\to \bigvee_{E \subseteq A,\ D \subseteq B,\ A \cap B = \phi} \min(S\tau_{(i,j)}(A) \wedge S\tau_{(i,j)}(B))\right)$$

$$\leq \bigwedge_{x \notin D}\left(\min(SF_{\tau_{(i,j)}}(\{x\}), SF_{\tau_{(i,j)}}(D))\right.$$

$$\left.\to \bigvee_{x \in A,\ D \subseteq B,\ A \cap B = \phi} \min(S\tau_{(i,j)}(A) \wedge S\tau_{(i,j)}(B))\right)$$

$$= \bigwedge_{x \notin D}\Big(\min(SF_{\tau_{(i,j)}}(\{x\}), SF_{\tau_{(i,j)}}(D))$$

$$\to \bigvee_{D \subseteq B,\ A \cap B = \phi}\left(\bigvee_{x \in A} S\tau_{(i,j)}(A) \wedge S\tau_{(i,j)}(B)\right)\Bigg)$$

$$\leq \bigwedge_{x\notin D}\Bigg(\min(SF_{\tau_{(i,j)}}(\{x\}), SF_{\tau_{(i,j)}}(D)) \rightarrow \bigvee_{D\subseteq B,\ A\cap B=\phi}\Big(\bigvee_{x\in K\subseteq A} S\tau_{(i,j)}(K)\wedge S\tau_{(i,j)}(B)\Big)\Bigg)$$

$$= \bigwedge_{x\notin D}\Bigg(\min(SF_{\tau_{(i,j)}}(\{x\}), SF_{\tau_{(i,j)}}(D)) \rightarrow \bigvee_{D\subseteq B,\ A\cap B=\phi}(SN_x^{i,j}(A)\wedge S\tau_{(i,j)}(B))\Bigg)$$

$$\leq \bigwedge_{x\notin D}\Bigg(\Big(\Big(\Big(\bigwedge_{x\in X} SF_{\tau_{(i,j)}}(\{x\})\wedge SF_{\tau_{(i,j)}}(D)\Big) \rightarrow \bigvee_{D\subseteq B,\ A\cap B=\phi}(SN_x^{(i,j)}(A)\wedge S\tau_{(i,j)}(B))\Bigg)$$

$$= \bigwedge_{x\notin D}(((ST_1^{i,j}(X,\tau_1,\tau_2)\wedge SF_{\tau_{(i,j)}}(D)) \rightarrow \bigvee_{D\subseteq B,\ A\cap B=\phi}(SN_x^{(i,j)}(A)\wedge S\tau_{(i,j)}(B)))$$

$$\leq \bigwedge_{x\notin D}(((ST_1^{(i,j)}(X,\tau_1,\tau_2)* SF_{\tau_{(i,j)}}(D)) \rightarrow \bigvee_{D\subseteq B,\ A\cap B=\phi}(SN_x^{(i,j)}(A)\wedge S\tau_{(i,j)}(B)))$$

$$\leq ST_1^{(i,j)}(X,\tau_1,\tau_2)\rightarrow \bigwedge_{x\notin D}((SF_{\tau_{(i,j)}}(D) \rightarrow \bigvee_{D\subseteq B,\ A\cap B=\phi}(SN_x^{(i,j)}(A)\wedge S\tau_{(i,j)}(B)))$$

$$= ST_1^{(i,j)}(X,\tau_1,\tau_2)\rightarrow ST_3^{(i,j)}(X,\tau_1,\tau_2) \quad \text{so that}$$

$$ST_4^{(i,j)}(X,\tau_1,\tau_2)* ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq ST_3^{(i,j)}(X,\tau_1,\tau_2).$$

(Indeed, put $ST_1^{(i,j)}(X,\tau_1,\tau_2)=\alpha$, $j=(x,D)$, $J=\{(x,D)|x\in X,\ D\in P(X),\ x\notin D\}$, $B_{(x,D)}=(F_{\tau_i}(D)$, $M_{(x,D)}=\bigvee_{D\subseteq B,\ A\cap B=\phi}(SN_x^{(i,j)}(A)\wedge S\tau_{(i,j)}(B))$ and $A_j=\{\lambda|\lambda*\alpha\leq B_j\rightarrow M_j\}$. Then $\bigwedge_{j\in J}((\alpha*B_j)\rightarrow M_j)=\bigwedge_{j\in J}\bigvee_{\lambda*(\alpha*B_j)\leq M_j}\lambda$
$=\bigwedge_{j\in J(}\bigvee_{\lambda*\alpha)*B_j\leq M_j}\lambda=\bigwedge_{j\in J}\bigvee_{\lambda*\alpha\leq(B_j\rightarrow M_j)}\lambda=\bigwedge_{j\in J}\bigvee A_j$ $=\bigvee_{f\in\prod_{j\in J}A_j}\bigwedge_{j\in J}f(j)$. Now, $\forall f\in\prod_{j\in J}A_j$, there exists $K_f=\bigwedge_{j\in J}f(j)$ *s.t.*

$K_f*\alpha=(\bigwedge_{j\in J}f(j))*\alpha \leq \bigwedge_{j\in J}(f(j)*\alpha)\leq\bigwedge_{j\in J}(B_j\rightarrow M_j)$. Then $\bigwedge_{j\in J}((\alpha*B_j)\rightarrow M_j)\leq\bigvee_{f\in\prod_{j\in J}A_j}K_f \leq \bigvee_{\lambda*\alpha\leq\bigwedge_{j\in J}(B_j\rightarrow M_j)}\lambda=\alpha\rightarrow\bigwedge_{j\in J}(B_j\rightarrow M_j).)$. □

We have the following results which their proof are obvious.

Theorem 3.5. *If L satisfies the completely distributive law, then for any* $(X,\tau_1,\tau_2)\in\Omega$, $ST_4^{(i,j)}(X,\tau_1,\tau_2)=SR_i^{(i,j)}(X,\tau_1,\tau_2)$.

Theorem 3.6. *If L satisfies the completely distributive law, then for any* $(X,\tau_1,\tau_2)\in\Omega$, $ST_4^{(i,j)}(X,\tau_1,\tau_2)=SR_j^{(i,j)}(X,\tau_1,\tau_2)$.

Theorem 3.7. *For any* $(X,\tau_1,\tau_2)\in\Omega$,

1- $SR_i^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq ST_2^{(i,j)}(X,\tau_1,\tau_2)$
2- $SR_j^{(i,j)}(X,\tau_1,\tau_2)\wedge ST_1^{(i,j)}(X,\tau_1,\tau_2)\leq ST_2^{(i,j)}(X,\tau_1,\tau_2)$

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