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Original Article

Strong semilattices of topological groups

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Abstract The notion of partial groups and their basic properties have been given in [1,2]. In this paper, we introduce the concept of topological partial groups and discuss some of their basic properties. So, the category of topological partial groups Tpg, as objects, and the homomorphisms of topological partial groups, as arrows, have some deficiencies. To get over these deficiencies, we introduced the category of locally compact partial groups denoted by Lcpg. Finally, we introduced the category of strong semilattices of topological groups denoted by Sstg.

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1. Preliminaries

We collect for sake of reference the needed definitions and results appeared in the given references.

Definition 1.1 ([3,4]). A topological group G is a pair (G, τ), where G is a group and τ is a topology on G which satisfies the continuity of the following maps:

(i) $\mu: G \times G \to G; (x, y) \mapsto xy$;
(ii) $\gamma: G \to G; x \mapsto x^{-1}$.

Theorem 1.1 [3]. *If G is a topological group, then γ is a homoeomorphism.*

Theorem 1.2 [4]. *A group G with a topology τ is a topological group if and only if the map f: $G \times G \to G$, $(x, y) \mapsto x^{-1}y$ is continuous.*

Definition 1.2 [3]. Let G and H be topological groups, then ϕ: $G \to H$ is called a morphism if ϕ is continuous and a group homomorphism.

Definition 1.3 [4]. Let G be a topological group and B be a subgroup of G. Then B with the relative topology is called a topological subgroup.

Theorem 1.3 [4]. *B is a topological subgroup of a topological group G if and only if the inclusion map i: $B \to G$ is a morphism.*

Definition 1.4 [5]. Let S be a semigroup. Then $x \in S$ is called an idempotent element if $x \cdot x = x$. The set of all idempotent elements in S is denoted by $E(S)$.

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Definition 1.5 [2]. Let S be a semigroup and $x \in S$. Then $e \in S$ is called a partial identity of x if

(i) $ex = xe = x$;
(ii) If $e'x = xe' = x$, $e' \in S$, then $ee' = e'e = e$.

Theorem 1.4 [2]. *If S is a semigroup, then*

(i) *If $x \in S$ has a partial identity, then it is unique.*
(ii) *$E(S)$ is the set of all partial identities of the elements of S.*

We will denote by e_x the partial identity of the element $x \in S$.

Definition 1.6 [2]. Let S be a semigroup and $x \in S$ has a partial identity e_x. The element $y \in S$ is called a partial inverse of x if

(i) $xy = yx = e_x$.
(ii) $e_x y = ye_x = y$.

Theorem 1.5 [2]. *Let S be a semigroup and $x \in S$ has a partial identity e_x. If x has a partial inverse y, then it is unique.*

We will denote by x^{-1} the partial inverse of $x \in S$.

Theorem 1.6 [2]. *Let S be a semigroup and $x \in S$. Then:*

(i) $(e_x)^{-1} = e_x, \forall\, e_x \in E(S)$.
(ii) $e_{x^{-1}} = e_x$.
(iii) $(x^{-1})^{-1} = x$.

Definition 1.7 [2]. A semigroup S is called a partial group if:

(i) Every $x \in S$ has a partial identity e_x.
(ii) Every $x \in S$ has a partial inverse x^{-1}.
(iii) The map e_S: $S \to S$; $x \mapsto e_x$ is a semigroup homomorphism.
(iv) The map γ: $S \to S$, $x \mapsto x^{-1}$ is a semigroup anti-homomorphism $[(xy)^{-1} = y^{-1}x^{-1}]$.

From this definition we have every group is a partial group. So, the notion of partial group is a good generalization of that of group. So, it is important to study a reasonable topology on a partial group to satisfy the nice properties of topological groups.

Definition 1.8 [2]. If S is a partial group and $x \in S$, then we define

$$S_x = \{y \in S : e_x = e_y\}.$$

Theorem 1.7 [2]. *Let S be a partial group and $x \in S$, then*

(i) *S_x is a maximal subgroup of S which has identity e_x.*
(ii) $S = \cup\{S_x : x \in S\} = \cup\{S_{e_x} : e_x \in E(S)\}$.

Corollary 1.1 [2]. *Every partial group is a disjoint union of a family of groups.*

Theorem 1.8 [2]. *Let S be a partial group, then $E(S)$ is commutative and central.*

Definition 1.9 [1]. A subsemigroup B of a partial group S is called a subpartial group, denoted by $B \leq S$, if $\forall\, x \in B$ we have $x^{-1} \in B$ and $e_x \in B$.

Theorem 1.9 [5]. *Let S be a partial group and $B \subseteq S$, then $B \leq S$ if and only if $x^{-1}y \in B$, $\forall\, x, y \in B$.*

Definition 1.10 [1]. Let S and T be partial groups, then ϕ: $S \to T$ is called a partial group homomorphism if $\phi(xy) = \phi(x)\phi(y)$, $\forall\, x, y \in S$.

Definition 1.11 [1]. Let ϕ: $S \to T$ be a partial group homomorphism, then $ker\ \phi = \{x \in S : \phi(x) = e_{\phi(x)}\}$ and $Im\ \phi = \{\phi(x) : x \in S\}$.

Definition 1.12 [1]. A partial group homomorphism ϕ: $S \to T$ is called an isomorphism if it is bijective.

Definition 1.13 [1]. If S is a partial group and $B \leq S$, then B is called normal, denoted by $B \trianglelefteq S$, if B is wide ($E(S) \subseteq B$) and $xyx^{-1} \in B$, $\forall\, x \in S, y \in B$.

Definition 1.14 [1]. Let S be a partial group and $B \trianglelefteq S$. The set $\{xB: x \in S\}$ is called the quotient set, denoted by $S|B$, where $xB = \{y \in S : x^{-1}y \in B, e_x = e_y\}$ is called the left coset of B by x .

Theorem 1.10 [1]. *Let S be a partial group and $N \trianglelefteq S$. Then $S|N$ with the map μ: $S|N \times S|N \to S|N$, $(xN, yN) \mapsto (xy)N$ is a partial group.*

Definition 1.15 [5]. Let $(S_i)_{i \in Y}$ be a family of groups indexed by a semilattice Y of the identities of the groups such that if $i \geq j$, $i, j \in Y$, there exists a group homomorphism $\phi_{i,j}$: $S_i \to S_j$, satisfies:

(i) $\phi_{i,i}$ is the identical automorphism;
(ii) $\phi_{j,k}\phi_{i,j} = \phi_{i,k}$, where $i \geq j \geq k$, $i, j, k \in Y$.

Then the disjoint union $S = \bigcup_{i \in Y} S_i$, with the binary operation $S \times S \to S$, $(x_i, y_j) \mapsto x_i y_j = (\phi_{i,ij}x_i)(\phi_{j,ij}y_j)$, $\forall\, x_i \in S_i$, $y_j \in S_j$, is called a strong semilattice of groups, denoted by $S = \mathcal{L}(S_i, Y, \phi_{i,j})$.

Theorem 1.11 [2]. *S is a partial group if and only if S is a strong semilattice of groups.*

Definition 1.16 [5]. Let ϕ: $S \to T$ be a partial group homomorphism. Then ϕ is called idempotent separating if $\phi(e_x) = \phi(e_y)$ implies that $e_x = e_y$, $\forall\, e_x, e_y \in E(S)$.

The following results are the fundamental theorems of isomorphisms.

Theorem 1.12 [1]. *Let ϕ: $S \to T$ be an idempotent separating surjective partial group homomorphism and $K = ker\ \phi$. Then there exists a unique isomorphism α: $S|K \to T$ such that $\phi = \alpha\rho_K$, where ρ_K: $S \to S|K$; $x \mapsto xK$ is the quotient map.*

Theorem 1.13 [1]. *Let $M, N \trianglelefteq S$ be such that $M \subseteq N$. Then*

(i) $N|M \trianglelefteq S|M$;
(ii) *There exists a unique isomorphism α: $(S|M)|(N|M) \to S|N$ such that $\rho_N = \alpha\rho_{N|M}\rho_M$, where ρ_N: $S \to S|N$ and $\rho_{N|M}$: $S|M \to (S|M)|(N|M)$ are the quotient maps.*

Definition 1.17 [2]. Let S be a partial group, and $A, B \subseteq S$. Then, we define $AB = \{ab : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$.

Definition 1.18 [6]. Let $X = \sqcup_{\lambda \in L} X_\lambda$ be the sum of the underlying sets of the family $(X_\lambda)_{\lambda \in L}$ of topological spaces, and let i_λ: $X_\lambda \to X$ be the inclusions. The final topology on X with respect to $(i_\lambda)_{\lambda \in L}$ is called the sum topology. Clearly, a map $f : X = \sqcup_{\lambda \in L} X_\lambda \to Y$ is continuous if and only if fi_λ is continuous, for all $\lambda \in L$.

Definition 1.19 [6]. Let $X = \sqcup_{n \in N} X_n$ and $Y = \sqcup_{m \in M} Y_m$ and $X \times Y$ be the cartesian product of X and Y. Then, $X \times Y$ with the final topology with respect to the inclusions $(i_n \times i_m)_{n \in N, m \in M}$ is called the weak product of X and Y, denoted by $X \times_W Y$.

2. Topological partial groups

In this article, we introduce the notion of topological partial groups. The category of topological partial groups Tpg and its continuous homomorphisms has some deficiencies.

Definition 2.1. Let S be a partial group and τ be a topology on S. Then S is called a topological partial group if the following maps are continuous:

(i) $\mu: S \times S \to S, (x, y) \mapsto xy$;
(ii) $\gamma: S \to S, x \mapsto x^{-1}$;
(iii) $e_S: S \to S, x \mapsto e_x$.

Every topological group is a topological partial group.

Theorem 2.1. *A partial group S with a topology τ is a topological partial group if and only if the map $f: S \times S \to S, (x, y) \mapsto x^{-1}y$ is continuous.*

Proof. Let S be a topological partial group. Then f is continuous, since $f = \mu(\gamma \times I)$, where I is the identity map on S.

Conversely, let $f: S \times S \to S, (x, y) \mapsto x^{-1}y$ be continuous. Then the maps e_S, γ and μ are continuous, since, $e_S = f\Delta$, $\gamma = f(I \times e_S)\Delta$ and $\mu = f(\gamma \times I)$, respectively, where Δ is the diagonal map which is continuous. □

Definition 2.2. Let S and T be topological partial groups. The map $\phi: S \to T$ is called a morphism if ϕ is continuous and partial group homomorphism.

Theorem 2.2. *If S is a topological partial group, then γ is a homeomorphism.*

Proof. It is clear. □

Definition 2.3. Let S be a topological partial group and B be a subpartial group of S. Then B with the relative topology is a topological partial group, called a topological subpartial group, denoted by $B \leq S$

Theorem 2.3. *B is a topological subpartial group of a topological partial group S if and only if the inclusion map $i: B \to S$ is a morphism.*

Proof. It is clear. □

Theorem 2.4. *Let S be a topological partial group, then the closure $\bar{N}$ of a topological subpartial group N of S is a topological subpartial group of S.*

Proof. Let $a, b \in \bar{N}$. Since S is a topological partial group, then $f: S \times S \to S, (x, y) \mapsto x^{-1}y$ is continuous. Now, $f(\bar{N} \times \bar{N}) = f(\overline{N \times N}) \subseteq \overline{f[N \times N]} = \bar{N}$. So, $a^{-1}b \in \bar{N}$. □

Definition 2.4. Let S be a topological partial group and $a \in S$. Then, the map $r_a: S \to S$, $x \mapsto xa$ is called a right transformation, and the map $\ell_a: S \to S, x \mapsto ax$ is called a left transformation.

Theorem 2.5. *The maps r_a and ℓ_a are continuous.*

Proof. The map r_a is continuous since $r_a = \mu(I_S, I_a)$, where I_S is the identity map and I_a is the constant map on S with value a. Similarly, ℓ_a is continuous since $\ell_a = \mu(I_a, I_S)$. □

Theorem 2.6. *Let S be a topological partial group. Then*

(i) $(\bar{A})^{-1} = \overline{(A^{-1})}$,
(ii) $(A^\circ)^{-1} = (A^{-1})^\circ$, *where $\bar{A}$ and A° are the closure and interior of the set A, respectively.*

Proof. Since $\gamma: S \to S$ is a homeomorphism, then

(i) $\gamma(\bar{A}) = \overline{\gamma(A)}$. So, $(\bar{A})^{-1} = \overline{(A^{-1})}$ and
(ii) $\gamma(A^\circ) = (\gamma(A))^\circ$. So, $(A^\circ)^{-1} = (A^{-1})^\circ$. □

3. External direct product of topological partial groups

Let $\{S_i : i = 1, 2, \ldots, n\}$ be a family of topological partial groups and $S = \bigotimes_{i=1}^n S_i$ be the cartesian product of the underlying sets S_i. That is, $S = \{x = \langle x_i \rangle : x_i \in S_i, \ \forall \ i = 1, 2, \ldots, n\}$.

Theorem 3.1. *The set $S = \bigotimes_{i=1}^n S_i$ with the map $\mu: S \times S \to S$; $(\langle x_i \rangle, \langle y_i \rangle) \mapsto \langle x_i y_i \rangle$ is a partial group.*

Proof. Clearly, S is a semigroup, that is, μ is a well defined associative binary operation. The element $e_x = \langle e_{x_i} \rangle$ is the partial identity of the element $x = \langle x_i \rangle$ because $xe_x = \langle x_i \rangle \langle e_{x_i} \rangle = \langle x_i e_{x_i} \rangle = \langle x_i \rangle = x$. Similarly, $e_x x = x$. If $xe = ex = x$, $e = \langle e_i \rangle \in E(S)$, then $\langle x_i e_i \rangle = \langle e_i x_i \rangle = \langle x_i \rangle$. That is, $x_i e_i = e_i x_i = x_i$, $i = 1, 2, \cdots, n$. Now, $ee_x = \langle e_i \rangle \langle e_{x_i} \rangle = \langle e_i e_{x_i} \rangle = \langle e_i x_i x_i^{-1} \rangle = \langle x_i x_i^{-1} \rangle = \langle e_{x_i} \rangle = e_x$. Similarly, $e_x e = e_x$.

The partial inverse of the element $x = \langle x_i \rangle$ is $x^{-1} = \langle x_i^{-1} \rangle$, since $xx^{-1} = \langle x_i \rangle \langle x_i^{-1} \rangle = \langle x_i x_i^{-1} \rangle = \langle e_{x_i} \rangle = e_x$. Similarly, $x^{-1}x = e_x$. Also, $e_x x^{-1} = \langle e_{x_i} \rangle \langle x_i^{-1} \rangle = \langle e_{x_i} x_i^{-1} \rangle = \langle x_i^{-1} \rangle = x^{-1}$, and $x^{-1}e_x = x^{-1}$.

$e_{xy} = e_{\langle x_i \rangle \langle y_i \rangle} = e_{\langle x_i y_i \rangle} = \langle e_{x_i y_i} \rangle = \langle e_{x_i} e_{y_i} \rangle = \langle e_{x_i} \rangle \langle e_{y_i} \rangle = e_x e_y$.

Finally, $(xy)^{-1} = (\langle x_i \rangle \langle y_i \rangle)^{-1} = \langle x_i y_i \rangle^{-1} = \langle (x_i y_i)^{-1} \rangle = \langle y_i^{-1} x_i^{-1} \rangle = \langle y_i^{-1} \rangle \langle x_i^{-1} \rangle = y^{-1}x^{-1}$. Hence, S is a partial group. □

Theorem 3.2. *The partial group $S = \bigotimes_{i=1}^n S_i$ with the cartesian product topology is a topological partial group.*

Proof. The maps μ, γ and e_S are continuous since $\mu = < \mu_i(P_i \times P_i) >$, $\gamma = < \gamma_i P_i >$ and $e_S = < e_{S_i} P_i >$, respectively, where $P_i : \otimes_{i=1}^n S_i \to S_i$ are the projection maps. □

Definition 3.1. The topological partial group $\bigotimes_{i=1}^n S_i$ is called the external direct product of $\{S_i : i = 1, 2, \ldots, n\}$.

Theorem 3.3. *Let $S = \bigotimes_{i=1}^n S_i$ be an external direct product of topological partial groups and let $A_{i_o} = \{< x_i : x_{i_o} = e_{x_{i_o}} >\}$, where $e_{x_{i_o}}$ is the partial identity of x_{i_o} in S_{i_o}. Then $A_{i_o} \trianglelefteq S$.*

Proof. Let $x, y \in A_{i_o}$, be such that $x = < x_i : x_{i_o} = e_{x_{i_o}} >$, $y = < y_i : y_{i_o} = e_{y_{i_o}} >$. Then $x^{-1}y = < x_i^{-1} y_i : x_{i_o}^{-1} y_{i_o} = e_{x_{i_o}} e_{y_{i_o}} > \in A_{i_o}$. Also, let $x = < x_i : x_{i_o} = e_{x_{i_o}} > \in A_{i_o}$ and $y = < y_i > \in S$. Then, $yxy^{-1} = < y_i x_i y_i^{-1} : y_{i_o} x_{i_o} y_{i_o}^{-1} = e_{x_{i_o}} e_{y_{i_o}} >$. So, $yxy^{-1} \in A_{i_o}$, since $y_{i_o} x_{i_o} y_{i_o}^{-1} = e_{x_{i_o}} e_{y_{i_o}} = e_{x_{i_o} y_{i_o}}$ Hence, $A_{i_o} \trianglelefteq S$. □

4. Neighborhoods of the partial identity elements

Let S be a topological partial group and $x \in S$, then the system of open neighborhoods of x is denoted by N_x and a subfamily β_x of N_x is called a base of open neighborhoods of x if $\forall \ N \in N_x \ \exists \ B \in \beta_x$ such that $x \in B \subseteq N$.

Theorem 4.1. *Let S be a topological partial group and $x \in S$, then β_{e_x} has the following properties:*

(i) *If $U, V \in \beta_{e_x}$, then $\exists \ W \in \beta_{e_x}$ such that $W \subseteq U \cap V$.*
(ii) *If $U \in \beta_{e_x}$, then $\exists \ V \in \beta_{e_x}$ such that $V^{-1}V \subseteq U$.*

(iii) *If* $U \in \beta_{e_x}$ *, then* $\exists\ V \in \beta_{e_x}$ *such that* $x^{-1}Vx \subseteq U$.

Proof.

(i) Let $U, V \in \beta_{e_x}$, then $U \cap V \in \beta_{e_x}$. So $\exists\ W \in \beta_{e_x}$ such that $W \subseteq U \cap V$.

(ii) Since f: $S \times S \to S$, $(x, y) \mapsto x^{-1}y$ is continuous and U is open in S, then $f^{-1}[U]$ is open in $S \times S$. Since $(e_x, e_x) \in f^{-1}[U]$, then there exists $N_1, N_2 \in N_{e_x}$ such that $(e_x, e_x) \in N_1 \times N_2 \subseteq f^{-1}[U]$. But $N_1, N_2 \in N_{e_x}$, then $N_1 \cap N_2 \in N_{e_x}$, and so $\exists\ V \in \beta_{e_x}$ such that $V \subseteq N_1 \cap N_2$. Now, $V \times V \subseteq (N_1 \cap N_2) \times (N_1 \cap N_2) \subseteq N_1 \times N_2 \subseteq f^{-1}[U]$. Hence, $f[V \times V] \subseteq U$. Then $V^{-1}V \subseteq U$.

(iii) The map f_x: $S \to S$, $y \mapsto x^{-1}yx$ is continuous because $f_x = \ell_{x^{-1}}r_x$. So, $f_x^{-1}[U]$ is open in S for each $U \in \beta_{e_x}$. But $e_x \in f_x^{-1}[U]$, since $f_x(e_x) = e_x$. Then, $f_x^{-1}[U]$ is an open neighborhood of e_x, and so, $\exists\ V \in \beta_{e_x}$ such that $V \subseteq f_x^{-1}[U]$. So, $f_x[V] \subseteq U$. Hence, $x^{-1}Vx \subseteq U$. □

Theorem 4.2. *Let S be a topological partial group,* $x \in S$ *and* $U \in \beta_{e_x}$. *Then*

(i) $\exists\ V \in \beta_{e_x}$ *such that* $V^{-1}e_x \subseteq U$;

(ii) $\exists\ V \in \beta_{e_x}$ *such that* $VV \subseteq U$;

(iii) $U^{-1} \in N_{e_x}$.

Proof.

(i) From (ii) above.

(ii) Since μ: $S \times S \to S$, $(x, y) \mapsto xy$ is continuous and U is open in S, then $\mu^{-1}[U]$ is open in $S \times S$. But $(e_x, e_x) \in \mu^{-1}[U]$, since $\mu(e_x, e_x) = e_x$. Then, $\exists\ N_1, N_2 \in N_{e_x}$ such that $(e_x, e_x) \in N_1 \times N_2 \subseteq \mu^{-1}[U]$. Since $N_1, N_2 \in N_{e_x}$, then $N_1 \cap N_2 \in N_{e_x}$, and so, $\exists\ V \in \beta_{e_x}$ such that $V \subseteq N_1 \cap N_2$. Now, $V \times V \subseteq (N_1 \cap N_2) \times (N_1 \cap N_2) \subseteq N_1 \times N_2 \subseteq \mu^{-1}[U]$. Thus, $\mu(V \times V) \subseteq U$. Hence, $VV \subseteq U$.

(iii) Since γ: $S \to S$, is a homeomorphism and U is open in S, then $\gamma^{-1}[U] = U^{-1}$ is open in S and $e_x \in U^{-1}$. Therefore, $U^{-1} \in N_{e_x}$. □

We note that ℓ_a and r_a may be neither injective nor surjective and so may not be a homeomorphism, as clear from the following example.

Example 4.1. Let $S_e = \{e, a\}$ and $S_f = \{f, g, h\}$ be isomorphic to Z_2 and Z_3, respectively. Consider the semilattice $e \geq f$, that means $ef = f$, one can define $\phi_{e,f}$: $S_e \to S_f$; $e \mapsto f$, $a \mapsto f$. Then, the corresponding partial group $S = \{e, a, f, g, h\}$ is given by the table:

$\cdot$	e	a	f	g	h
e	e	a	f	g	h
a	a	e	f	g	h
f	f	f	f	g	h
g	g	g	g	h	f
h	h	h	h	f	g

It is clear that r_f is not surjective and is not injective.

Let $\tau_S = \{S, \phi, S_e, S_f, \{e\}, \{a\}, \{e, f, g, h\}, \{a, f, g, h\}\}$. Then S is a topological partial group. We note that $S = S_e \sqcup S_f$ and $\phi_{e,f}$ is continuous.

We have the following deficiency: If $x \in S$, then the maps r_x and ℓ_x may not be open. In the above example, we have that $\{a\}$ is open in S but $r_f\{a\} = \{a\}f = \{f\}$ is not open in S.

Definition 4.1. If S is a topological partial group and $N \leq S$, then $S|N$ with the identification topology, with respect to the quotient map ρ_N: $S \to S|N$, is called the coset space.

Also, we have the following deficiencies

(i) The quotient map ρ_N: $S \to S|N$, $N \leq S$ may not be open, in general.

(ii) If S is a topological partial group and $N \trianglelefteq S$, then $S|N$ may not be a topological partial group, because $\rho_N \times \rho_N$ may not be an identification map.

We note that if S is a locally compact space, then S/N is also locally compact. Thus, $\rho_N \times \rho_N$ is an identification map. So, $S|N$ is a topological partial group. Therefore, one can say that the category of locally compact partial groups *Lcpg* has a quotient (limit) and a product (co-limit). Hence, we have the following result.

Theorem 4.3. *Let S, T be locally compact partial groups,* ϕ: $S \to T$ *be an idempotent separating surjective morphism, and* $K = ker\ \phi$. *Then, there exists a unique bijective morphism* α: $S|K \to T$ *such that* $\phi = \alpha\rho_K$.

Proof. Since ρ_K is identification and ϕ is continuous, then α is continuous. That is all we need. □

5. Strong semilattices of topological groups

In this section, we suggest the notion of strong semilattices of topological groups to get over the above deficiencies. Also, we replace the cartesian product $S \times S$ by the weak product $S \times_W S$.

Definition 5.1. Let $(S_i)_{i \in Y}$ be a family of topological groups indexed by a semilattice Y of the identities of the groups such that if $i \geq j$, $i, j \in Y$, , there exist a continuous and open group homomorphism $\phi_{i,j}$: $S_i \to S_j$, satisfies:

(i) $\phi_{i,i}$ is the identical automorphism;

(ii) $\phi_{j,k}, \phi_{i,j} = \phi_{i,k}$, $\forall\ i, j, k \in Y$, $i \geq j \geq k$. Then $S = \bigsqcup_{i \in Y} S_i$ with the binary operation $S \times S \to S$; $(x_i, y_j) \mapsto x_iy_j = (\phi_{i,ij}x_i)(\phi_{j,ij}y_j)$, $\forall x_i \in S_i$, $y_j \in S_j$, is called a strong semilattice of topological groups, and is denoted by $S = \mathcal{L}(S_i, Y, \phi_{i,j})$.

Theorem 5.1. *If S is a strong semilattice of topological groups, then S is a topological partial group.*

Proof. Since S is a strong semilattice of groups, then S is a partial group (Theorem 1.11). We prove the continuity of structure maps. The maps μ, γ and e_S are continuous since $\mu(i_\alpha \times i_\beta) = i_{\alpha\beta}\mu_{\alpha\beta}(\phi_{\alpha,\alpha\beta} \times \phi_{\beta,\alpha\beta})$, $\gamma i_\alpha = i_\alpha\gamma_\alpha$, and $e_S i_\alpha = i_\alpha e_{S_\alpha}$, respectively. □

Theorem 5.2. *Let S be a strong semilattice of topological groups, then the maps* r_a *and* ℓ_a *are open.*

Proof. Let $U \subseteq S$ be open, so $U_\alpha = U \cap S_\alpha$ is open in S_α for each α. If $a \in S$, then $a \in S_\beta$, for some β. Now, $r_a(U) = Ua = (\cup_\alpha U_\alpha)a = \cup_\alpha(U_\alpha a)$. Since $U_\alpha a = \phi_{\alpha,\alpha\beta}(U_\alpha)\phi_{\beta,\alpha\beta}(a)$, then:

(i) If $\alpha > \beta$, then $\alpha\beta = \beta$, and so $U_\alpha a = \phi_{\alpha,\beta}(U_\alpha)\phi_{\beta,\beta}(a) = \phi_{\alpha,\beta}(U_\alpha)a = r_a(\phi_{\alpha,\beta}(U_\alpha))$ is open in S_β. So $U_\alpha a$ is open in S.
(ii) If $\alpha = \beta$, then $U_\alpha a = U_\beta a = r_a(U_\beta)$ is open in S_β. So, $U_\alpha a$ is open in S.
(iii) If $\beta > \alpha$, then $\beta\alpha = \alpha$, then $U_\alpha a == \phi_{\alpha,\alpha}(U_\alpha)\phi_{\beta,\alpha}(a) = U_\alpha\phi_{\beta,\alpha}(a) = r_{\phi_{\beta,\alpha}(a)}(U_\alpha)$ is open in S_α. So, $U_\alpha a$ is open in S.
Hence $r_a(U)$ is open in S. Similarly, ℓ_a is open. □

Example 5.1. Let $S_e = \{e, a\}$ and $S_f = \{f, g, h\}$ be isomorphic to Z_2 and Z_3, respectively. Consider the semilattice $f \geq e$ that means $fe = e$, one can define $\phi_{f,e}$: $S_f \to S_e$, $f \mapsto e$, $g \mapsto e$, $h \mapsto e$. Then the corresponding partial group $S = \{e, a, f, g, h\}$ is given by the next table:

$\cdot$	e	a	f	g	h
e	e	a	e	e	e
a	a	e	a	a	a
f	e	a	f	g	h
g	e	a	g	h	f
h	e	a	h	f	g

Let $\tau_{S_e} = \{S_e, \phi, \{e\}, \{a\}\}$ and $\tau_{S_f} = \{S_f, \phi\}$. Hence, S_e and S_f are topological groups and $S = S_e \sqcup S_f$ is a strong semilattice of topological groups, since $\phi_{f,e}$ is continuous and open.

We note that in Example (4.1) S is not a strong semilattice of topological groups even $S = S_e \sqcup S_f$ and $\phi_{e,f}$ is continuous, since $\phi_{e,f}$ is not open.

Theorem 5.3. *If S is a strong semilattice of topological groups, $x \in S$ and A, $B \subseteq S$. Then*

(i) *If A is open in S, then xA and Ax are also open in S.*
(ii) *If A is open in S, then AB and BA are also open in S.*

Proof.

(i) The result follows since r_a and ℓ_a are open maps.
(ii) Since $AB = \cup_{b\in B} r_b(A)$, then AB is open. Similarly, BA is open. □

We note that every topological group is a strong semilattice of topological groups and every strong semilattice of topological groups which is not a topological group, is disconnected.

Theorem 5.4. *If S_1 and S_2 are strong semilattices of topological groups, then $S_1 \times_W S_2$ is also a strong semilattice of topological groups.*

Proof. Let $S_1 = \mathcal{L}(S_i, Y, \phi_{i,j})$ and $S_2 = \mathcal{L}(S'_{i'}, Y', \phi'_{i',j'})$ be strong semilattices of topological groups, then $S_1 \times_W S_2 = \mathcal{L}(S_i \times S'_{i'}, Y \times Y', \phi_{i,j} \times \phi'_{i',j'})$ is a strong semilattice of topological groups, where $(S_i \times S'_{i'})_{(i,i')\in Y\times Y'}$ is a family of topological groups indexed by the semilattice $Y \times Y'$ and $\phi_{i,j} \times \phi'_{i',j'} = \phi''_{(i,i'),(j,j')} : S_i \times S'_{i'} \to S_j \times S'_{j'}$, are morphisms and open which satisfies:

(i) $\phi''_{(i,i'),(i,i')}$ is the identical automorphism.
(ii) $\phi''_{(j,j'),(k,k')}\phi''_{(i,i')(j,j')} = \phi''_{(i,i'),(k,k')}$. □

Let A be a wide topological subpartial group of the strong semilattice of topological groups $S = \mathcal{L}(S_i, Y, \phi_{i,j})$. Then $A_i = A \cap S_i$ is a topological subgroup of S_i. If $\phi_{i,j}(A_i) \subseteq A_j$, then $\phi'_{i,j} = \phi_{i,j}|_{A_i A_j}$ is a morphism of topological groups. If $\phi'_{i,j}$ are open, then $A = (A_i, Y, \phi'_{i,j})$ is a substrong semilattice of topological groups, called a substrong semilatic of topological group of S.

Theorem 5.5. *Let S be a strong semilattice of topological groups. Then, every open substrong semilattice of topological groups is closed.*

Proof. Let $N \leq S$ be open, then xN is open in S for each $x \in S$. Since $S - N = \cup_{x\notin N} xN$, then $S - N$ is open. So, N is closed. □

Theorem 5.6. *Let N be a substrong semilattice of topological groups of the strong semilattice of topological groups S, then the quotient map ρ_N: $S \to S|N$ is open.*

Proof. Let U be open in S. Then, $\rho_N^{-1}[\rho_N[U]] = UN$ is open in S. Hence, $\rho_N[U]$ is open in $S|N$. □

Theorem 5.7. *If S is a strong semilattice of topological groups and $N \trianglelefteq S$, then $S|N$ is a strong semilattice of topological groups.*

Proof. Let $S = \mathcal{L}(S_i, Y, \phi_{i,j})$ be a strong semilattice of topological groups and $N \trianglelefteq S$. It is easily to show that $N \cap S_i = N_i \trianglelefteq S_i$. So, $S_i|N_i$ is a topological group. So, $(S_i|N_i)_{e'_i\in Y'}$ is a family of topological groups indexed by the semilattice Y' of the identities of the groups. Let $\phi_{e'_i,e'_j} : S_i|N_i \to S_j|N_j$, $xN_i \mapsto \phi_{i,j}(x)N_j$. We have $\phi_{e'_i,e'_j}$ is continuous, since $\phi_{e'_i,e'_j}\rho_{N_i} = \rho_{N_j}\phi_{i,j}$ and ρ_{N_i} is an identification map. From the properties of the quotient of topological groups [3] there exists a continuous homomorphism and open map ψ: $S_i/N_i \to S_j$ such that $\psi\rho_{N_i} = \phi_{i,j}$. Since $\phi_{e'_i,e'_j} = \rho_{N_j}\psi$, then $\phi_{e'_i,e'_j}$ is open. Finally $\phi_{e'_i,e'_i}$ is the identical automorphism and $\phi_{e'_j,e'_k}\phi_{e'_i,e'_j} = \phi_{e'_i,e'_k}$. Therefore $S/N = \mathcal{L}(S_i/N_i, Y, \phi_{e'_i,e'_j})$, is a strong semilattice of topological groups. □

The sum of two strong semilattices of topological groups can be constructed but it may not be unique, in general.

Theorem 5.8. *Let ϕ: $S \to T$ be an idempotent separating morphism of strong semilattice of topological groups. Let $N \trianglelefteq S$ such that $N \subseteq ker\phi$, then, there exists a unique injective morphism α: $S|N \to T$ such that $\phi = \alpha\rho_N$.*

Proof. We only prove the continuity of α as follows. We have that α is continuous since ϕ is continuous and ρ_N is an identification map. □

In particular, if $N = ker\phi$ in the above theorem, then α: $S|ker\phi \to T$ is the unique injective morphism.

Theorem 5.9. *Let S be a strong semilattice of topological groups and M, $N \leq S$ with $M \subseteq N$. Then the identification topology on $N|M$ with respect to the quotient map $\rho'_M : N \to N|M$ is the relative topology on $N|M$ as a subspace of $S|M$.*

Proof. Let ρ_M: $S \to S|M$ be the quotient map and ρ'_M : $N \to N|M$ be the restriction ρ_M on N. So, ρ'_M is continuous and surjective . Let $N|M$ be a subspace of $S|M$, then the inclusion i': $N|M \to S|M$ is continuous. We need only to prove that ρ'_M is open to show that it is an identification map. Let $U \subseteq N$ be open, there exists an open set V in S such that $U = V \cap N$. Now, $\rho'_M[U] = \rho_M[U] = \rho_M[V \cap N] = \rho_M[V] \cap \rho_M[N] = \rho_M[V] \cap N|M$.

Since $\rho_M[V]$ is open in S, then $\rho'_M[U]$ is open in $N|M$.

Conversely, let ρ'_M be an identification map. Since $i'\rho'_M = \rho_M i$ and i is continuous, then i' is continuous, where i: $N \to S$ is the inclusion map. So, $N|M$ is a subspace of $N|M$. □

Theorem 5.10. *Let S be a strong semilattice of topological groups and M, $N \trianglelefteq S$ such that $M \subseteq N$, then*

(i) $N|M \trianglelefteq S|M$;
(ii) *There exists a unique bijective morphism* $\alpha: (S|M)|(N|M) \to S|N$, *such that* $\rho_N = \alpha\rho_{N|M}\rho_M$.

Proof.

(i) See [1].
(ii) Let ρ_N: $S \to S|N$ and ρ_M: $S \to S|M$ be the quotient maps. Since ρ_N is an idempotent separating surjective morphism and $ker\rho_N = \{x \in S : \rho_N(x) = e_xN\} = \{x \in S : xN = e_xN\} = \{x \in S : x \in N\} = N$, that is, $M \subseteq ker\rho_N$. So, from Theorem 5.8, there exists a unique bijective morphism ϕ: $S|M \to S|N$, $xM \mapsto xN$ such that $\phi\rho_M = \rho_N$. Now, $ker\phi = \{xM \in S|M : \phi(xM) = e_xN\} = \{xM \in S|M : xN = e_xN\} = N|M$ is a strong semilattice of topological groups, from Theorem 5.7. Then, from Theorem 5.8, there exists a unique bijective morphism α: $(S|M)|(N|M) \to S|N$ such that $\alpha\rho_{N|M} = \phi$. □

Theorem 5.11. *Let $(S_i)_{i \in I}$ be strong semilattices of topological groups and $N_i \trianglelefteq S_i$, $\forall\ i \in I$. Then,*

(i) $\otimes_{i=1}^n N_i \trianglelefteq \otimes_{i=1}^n S_i$
(ii) *There exists a unique bijective morphism* $\beta : \otimes_{i=1}^n S_i| \otimes_{i=1}^n N_i \to \otimes_{i=1}^n (S_i|N_i)$.

Proof.

(i) Since the inclusions j_i: $N_i \to S_i$, $\forall\ i \in I$ are morphisms, then the inclusion $j = \otimes_{i=1}^n j_i : \otimes_{i=1}^n N_i \to \otimes_{i=1}^n S_i$ is a morphism, and so $\otimes_{i=1}^n N_i \leq \otimes_{i=1}^n S_i$. Let $x =< x_i >\in \otimes_{i=1}^n S_i$ and $y =< y_i >\in \otimes_{i=1}^n N_i$. Now, $xyx^{-1} =< x_i >< y_i >< x_i^{-1} >=< x_iy_ix_i^{-1} >\in \otimes_{i=1}^n N_i$, where $x_iy_ix_i^{-1} \in N_i$, $\forall\ i \in I$. So, $\otimes_{i=1}^n N_i \trianglelefteq \otimes_{i=1}^n S_i$.
(ii) Let ϕ_i: $S_i \to S_i\,|\,N_i$, $x_i \mapsto x_iN_i$. Then ϕ_i are continuous surjective maps. Now, we define $\alpha : \otimes_{i=1}^n S_i \to \otimes_{i=1}^n (S_i|N_i)$, $< x_i > \mapsto < \phi_i(x_i) >$, $\forall\ i \in I$. Then, α is an idempotent separating surjective morphism. Hence by Theorem 5.8, we have that there exists a unique bijective morphism $\beta : \otimes_{i=1}^n S_i|ker\alpha \to \otimes_{i=1}^n (S_i|N_i)$. Now, we prove that $ker\alpha = \otimes_{i=1}^n N_i$. $ker\alpha = \{< x_i >\in \otimes_{i=1}^n S_i : \alpha(< x_i >) =< e_{x_i}N_i >\} = \{< x_i >\in \otimes_{i=1}^n S_i :< \phi_i(x_i) >=< e_{x_i}N_i >\} = \{< x_i >\in \otimes_{i=1}^n S_i : x_iN_i = e_{x_i}N_i\} = \{< x_i >\in \otimes_{i=1}^n S_i : x_i^{-1}e_{x_i} \in N_i, e_{x_i} = e_{e_{x_i}}\} = \{< x_i >\in \otimes_{i=1}^n S_i : x_i^{-1} \in N_i\} = \{< x_i >\in \otimes_{i=1}^n S_i : x_i \in N_i\} = \otimes_{i=1}^n N_i$. □

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References

[1] A.M. Abd-Allah, M.E.G.M. Abdallah, Quotients in partial groups, Delta J. Sci. 8 (2) (1984) 470–480.
[2] A.M. Abd-Allah, M.E.G.M. Abdallah, On clifford semigroups, Pure Math. Manuscr. 7 (1988) 1–17.
[3] P.J. Higgins, An introduction to topological groups, Lecture Notes Series, 15, Cambridge University Press, 1974.
[4] T. Husain, Introduction to topological groups, W. B. Saunders company, 1966.
[5] J.M. Howie, An introduction to semigroup theory, Academic Press, 1976.
[6] R. Brown, Topology and groupoids, Oxon, McGraw-Hill, 2006.