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Original Article

(r, s)-(τ_{12}, τ^*_{12})-θ-Generalized double fuzzy closed sets in bitopological spaces

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Abstract In this paper, we introduce the notion of (r, s)-(i, j)-θ-generalized double fuzzy closed sets in double fuzzy bitopological spaces. A new θ-double fuzzy closure C^{θ}_{12} on double fuzzy bitopological spaces by using double supra fuzzy topological spaces are defined. Furthermore, generalized double fuzzy θ-continuous (resp. irresolute) and double fuzzy strongly θ-continuous mappings are introduced and some of their properties studied.

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1. Introduction and preliminaries

Mashhour et al. [1] introduced the so-called supra topology. El-Monsef and Ramadan [2] introduced the concept supra fuzzy topology, followed by Ghanim et al. [3] who introduced the supra fuzzy topology in Šostak sense. Abbas and Ramadan [4,5] generalized the supra fuzzy topology from fuzzy bitopological space in Šostak sense as an extension of supra fuzzy topology due to Kandil et al. [6].

Levine [7] introduced the first step of generalizing closed sets. Balasubramanian and Sundaram [8] introduced the concept of generalized closed sets within Chang's fuzzy topology [9] as an extension of generalized sets of Levine. Kim and Ko [10] defined r-generalized fuzzy closed sets in smooth topological spaces. Noiri [11] and Dontchev and Maki [12] introduced another new generalization of Levine generalized closed set by utilizing the θ-closure operator. Khedr and Al-Saadi [13] generalized the notion of θ-generalized sets to bitopological space. Recently, Tantawy et al. [14–17] introduced the notion of θ-generalized fuzzy closed sets in smooth bitopological spaces.

On the other hand, Atanassov [18] introduced the idea of intuitionistic fuzzy set. Recently, much work has been done with these concepts [18–20]. Çoker and coworker [21,22] introduced the idea of the topology of intuitionistic fuzzy sets. Samanta and Mondal [23,24] introduced the definition of the intuitionistic gradation of openness.

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In this paper, we introduce the notion of (r, s)-(i, j)-θ-generalized double fuzzy closed sets in double fuzzy bitopological spaces. A new θ-double fuzzy closure C_{12}^{θ} on double fuzzy bitopological spaces by using double supra fuzzy $(X, \tau_{12}, \tau_{12}^*)$ which is generated from double fuzzy bitopological spaces $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ are defined. By using (r, s)-(τ_{12}, τ_{12}^*)-generalized double fuzzy closed sets, we define a new double fuzzy closure operator which generates a new double fuzzy topology. Finally, generalized double fuzzy θ-continuous (resp. irresolute) and double fuzzy strongly θ-continuous mappings are introduced and some of their properties studied.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for each $x \in X$. The set of all fuzzy subsets of X are denoted by I^X. For $x \in X$ and $t \in I_0$ a fuzzy point denoted by

$$x_t(y) = \begin{cases} t, & \text{if } y = x \\ 0, & \text{if } y \neq x. \end{cases}$$

$x_t \in \lambda$ iff $t \le \lambda(x)$. We denote a fuzzy set λ which is quasi-coincident with a fuzzy μ by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise by $\lambda \bar{q} \mu$.

Definition 1.1 [23,25]**.** A double supra fuzzy topology on X is an ordered pair (τ, τ^*) of mappings from I^X to I such that

(1) $\tau(\lambda) + \tau^*(\lambda) \le 1, \ \forall \lambda \in I^X$,
(2) $\tau(\underline{0}) = \tau(\underline{1}) = 1, \ \tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,
(3) $\tau(\vee_{i \in \triangle} \lambda_i) \ge \wedge_{i \in \triangle} \tau(\lambda_i)$ and $\tau^*(\vee_{i \in \triangle} \lambda_i) \ge \vee_{i \in \triangle} \tau^*(\lambda_i), \forall \lambda_i \in I^X, i \in \triangle$.
The triplet (X, τ, τ^*) is called a double supra fuzzy topological space (dsfts, for short). A double supra fuzzy topology (τ, τ^*) is called double fuzzy topological space (dfts, for short) on X iff
(4) $\tau(\lambda_1 \wedge \lambda_2) \ge \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \ge \tau^*(\lambda_1) \vee \tau^*(\lambda_2), \forall \lambda_1, \lambda_2 \in I^X$.

τ and τ^* may interpreted as gradation of openness and gradation of nonopenness, respectively. The $(X, (\tau, \tau^*), (\nu, \nu^*))$ is called a double fuzzy bitopological space (dfbts, for short).

Definition 1.2 [25]**.** A map $C: I^X \times I_0 \times I_1 \to I^X$ is called a double supra fuzzy closure operator on X if for $\lambda, \mu \in I^X$ and $r \in I_0, s \in I_1$, it satisfies the following conditions:

(C1) $C(\underline{0}, r, s) = \underline{0}$.
(C2) $\lambda \le C(\lambda, r, s)$.
(C3) $C(\lambda, r, s) \vee C(\mu, r, s) \le C(\lambda \vee \mu, r, s)$.
(C4) $C(\lambda, r_1, s_1) \le C(\lambda, r_2, s_2)$, if $r_1 \le r_2$ and $s_1 \ge s_2$.
(C5) $C(C(\lambda, r, s), r, s) = C(\lambda, r, s)$.

The pair (X, C) is called a double supra fuzzy closure space. A double supra fuzzy closure space (X, C) is called double fuzzy closure space iff

(C) $C(\lambda, r, s) \vee C(\mu, r, s) = C(\lambda \vee \mu, r, s)$.

Theorem 1.1 [25]**.** *Let (X, τ, τ^*) be a dsfts. Then, for $\lambda \in I^X$, $r \in I_0, s \in I_1$, we define an operator $C_{\tau,\tau^*}: I^X \times I_0 \times I_1 \to I^X$ as follows:*

$$C_{\tau,\tau^*}(\lambda, r, s) = \wedge\{\mu \in I^X : \lambda \le \mu, \ \tau(\underline{1} - \mu) \ge r, \ \tau^*(\underline{1} - \mu) \le s\}. \quad (1)$$

Then (X, C_{τ,τ^*}) is a double supra fuzzy closure space. The mapping $I_{\tau,\tau^*}: I^X \times I_0 \times I_1 \to I^X$ defined by

$$I_{\tau,\tau^*}(\lambda, r, s) = \vee\{\mu \in I^X : \mu \le \lambda, \ \tau(\mu) \ge r, \ \tau^*(\mu) \le s\}, \quad (2)$$

is a double supra fuzzy interior space, and $I_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - C_{\tau,\tau^*}(\lambda, r, s)$.

If (X, τ, τ^*) is dfts, then the definition of the double fuzzy closure (resp. interior) for any fuzzy set is defined as (2.1) and (2.2), respectively.

Theorem 1.2 [25]**.** *Let (X, C) be a double (double supra) fuzzy closure space. Define the mappings $\tau_C, \tau_C^*: I^X \to I$ on X by*

$$\tau_C(\lambda) = \vee\{r \in I_0 : C(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\},$$
$$\tau_C^*(\lambda) = \wedge\{s \in I_1 : C(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}.$$

Then:

(1) *(τ_C, τ_C^*) is a double fuzzy (double supra fuzzy) topology on X.*
(2) *$C_{\tau_C,\tau_C^*} \le C$.*

Theorem 1.3 [25]**.** *Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be a dsfbts. We define the mappings $C_{12}, I_{12}: I^X \times I_0 \times I_1 \to I^X$ as follows:*

$$C_{12}(\lambda, r, s) = C_{\tau_1,\tau_1^*}(\lambda, r, s) \wedge C_{\tau_2,\tau_2^*}(\lambda, r, s),$$
$$I_{12}(\lambda, r, s) = I_{\tau_1,\tau_1^*}(\lambda, r, s) \vee I_{\tau_2,\tau_2^*}(\lambda, r, s),$$

for all $\lambda \in I^X, r \in I_0, s \in I_1$. Then ,

(1) *(X, C_{12}) is a double supra fuzzy closure space.*
(2) *$I_{12}(\underline{1} - \lambda, r, s) = \underline{1} - C_{12}(\lambda, r, s)$.*

Corollary 1.1 [25]**.** *Let (X, C_{12}) be a double supra fuzzy closure space. Then, the mappings $\tau_{C_{12}}, \tau_{C_{12}}^*: I^X \to I$ on X given by*

$$\tau_{C_{12}}(\lambda) = \vee\{r \in I_0 : C_{12}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\},$$
$$\tau_{C_{12}}^*(\lambda) = \wedge\{s \in I_1 : C_{12}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}.$$

is a dsfts on X.

Theorem 1.4 [25]**.** *Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be a dsfbts. Let (X, C_{12}) be a double supra fuzzy closure space. Define the mappings $\tau_s, \tau_s^*: I^X \to I$ on X by*

$$\tau_s(\lambda) = \vee\{\tau_1(\lambda_1) \wedge \tau_2(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2\},$$
$$\tau_s^*(\lambda) = \wedge\{\tau_1^*(\lambda_1) \wedge \tau_2^*(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2\},$$

where $\vee$ and $\wedge$ are taken over all families $\{\lambda_1, \lambda_2 : \lambda = \lambda_1 \vee \lambda_2\}$. Then,

(1) *$(\tau_s, \tau_s^*) = (\tau_{C_{12}}, \tau_{C_{12}}^*)$ is the coarsest double supra fuzzy topology on X which is finer than both of (τ_1, τ_1^*) and (τ_2, τ_2^*).*
(2) *$C_{12} = C_{\tau_s,\tau_s^*} = C_{\tau_{C_{12}},\tau_{C_{12}}^*}$.*

In this paper, we will denote to $\tau_{C_{12}}, \tau_{C_{12}}^*$ by τ_{12}, τ_{12}^*, respectively.

Definition 1.3 [26,27]**.** Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be a dfbts, $\mu \in I^X, r \in I_0, s \in I_1$ and $x_t \in Pt(X)$. μ is called an (r, s)-open Q_{τ_i,τ_i^*}-neighborhood of x_t if $x_t q \mu$ with $\tau_i(\mu) \ge r$ and $\tau_i^*(\mu) \le s$, we denote

$$Q_{\tau_i,\tau_i^*}(x_t, r, s) = \{\mu \in I^X : x_t q \mu, \ \tau_i(\mu) \ge r, \ \tau_i^*(\mu) \le s\}.$$

Definition 1.4 [24,28,29]**.** Let $f: (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (X, (\nu_1, \nu_1^*), (\nu_2, \nu_2^*))$ be a mapping. Then, f is called:

(1) DFP-continuous if and only if $\tau_i(f^{-1}(\mu)) \ge \nu_i(\mu)$ and $\tau_i^*(f^{-1}(\mu)) \le \nu_i^*(\mu), \ \forall \mu \in I^Y, \ i = 1, 2$.

(2) DFP*-continuous if and only if $f : (X, \tau_{12}, \tau^*_{12}) \to (X, \nu_{12}, \nu^*_{12})$ is DF-continuous, that is $\tau_{12}(f^{-1}(\mu)) \geq \nu_{12}(\mu)$ and $\tau^*_{12}(f^{-1}(\mu)) \leq \nu^*_{12}(\mu)$, $\forall\, \mu \in I^Y$.
(3) DFP*-open if and only if $f : (X, \tau_{12}, \tau^*_{12}) \to (X, \nu_{12}, \nu^*_{12})$ is DF-open, that is $\nu_{12}(f(\lambda)) \geq \tau_{12}(\lambda)$ and $\nu^*_{12}(f(\lambda)) \leq \tau^*_{12}(\lambda)$, $\forall\, \lambda \in I^X$.

2. (r, s)-(τ_{12}, τ^*_{12})-θ-Generalized double fuzzy closed sets

Definition 2.1. Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$. A fuzzy set λ is called:

(1) an (r, s)-(i, j)-generalized double fuzzy closed ((r, s)-(i, j)-gdfc, for short), if $C_{\tau_i,\tau^*_i}(\lambda, r, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq r$ and $\tau^*_i(\mu) \leq s$. The complement of (r, s)-(i, j))-gdfc is (r, s)-(i, j)-generalized double fuzzy open ((r, s)-(i, j)-gdfo, for short).
(2) an (r, s)-(τ_{12}, τ^*_{12})-generalized double fuzzy closed ((r, s)-(τ_{12}, τ^*_{12})-gdfc, for short), if $C_{12}(\lambda, r, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_{12}(\mu) \geq r$ and $\tau^*_{12}(\mu) \leq s$. The complement of (r, s)-(τ_{12}, τ^*_{12})-gdfc is (r, s)-(τ_{12}, τ^*_{12})-generalized double fuzzy open ((r, s)-(τ_{12}, τ^*_{12})-gdfo, for short).

The concepts of (r, s)-(τ_{12}, τ^*_{12})-gdfc and (r, s)-(i, j)-gdfc sets are independent.

Definition 2.2. Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$. Then:

(1) A fuzzy point $x_t \in Pt(X)$ is called (r, s)-(i, j)-θ -cluster point of λ if for every $\mu \in Q_{\tau_i,\tau^*_i}(x_t, r, s)$, $C_{\tau_j,\tau^*_j}(\mu, r, s)q\lambda$.
(2) An (i, j)-θ-closure is a mapping $T^{\tau_i,\tau^*_i}_{\tau_j,\tau^*_j} : I^X \times I_0 \times I_1 \to I^X$ defined as follows:

$$T^{\tau_i,\tau^*_i}_{\tau_j,\tau^*_j}(\lambda, r, s) = \vee\{x_t \in Pt(X) : x_t \text{ is } (r, s)\text{-}(i, j)\text{-}\theta\text{-cluster point of } \lambda\}.$$

(3) λ is called an (r, s)-(i, j)-double fuzzy θ -closed ((r, s)-(i, j)-dfθc, for short) iff $\lambda = T^{\tau_i,\tau^*_i}_{\tau_j,\tau^*_j}(\lambda, r, s)$. The complement of an (r, s)-(i, j)-dfθc is called (r, s)-(i, j) -double fuzzy θ-open ((r, s)-(i, j)-dfθo, for short).

Definition 2.3. Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda, \mu \in I^X$, $r \in I_0, s \in I_1$ and $x_t \in Pt(X)$. Then:

(1) $T^{\tau_i,\tau^*_i}_{\tau_j,\tau^*_j}(\lambda, r, s) = \wedge\{\mu \in I^X : I_{\tau_j,\tau^*_j}(\mu, r, s) \geq \lambda, \tau_i(\underline{1} - \mu) \geq r, \ \tau^*_i(\underline{1} - \mu) \leq s\}$, i.e. $T^{\tau_i,\tau^*_i}_{\tau_j,\tau*_j}(\lambda, r, s)$ is an (r, s)-(i, j)-double fuzzy closed set.
(2) x_t is an (r, s)-(i, j)-θ-cluster point of λ iff $x_t \in T^{\tau_i,\tau^*_i}_{\tau_j,\tau^*_j}(\lambda, r, s)$.

Definition 2.4. Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$. A fuzzy set λ is an (r, s)-(i, j)-θ-generalized double fuzzy closed ((r, s)-(i, j)-θ-gdfc, for short) if $T^{\tau_i,\tau^*_i}_{\tau_j,\tau^*_j}(\lambda, r, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq r$ and $\tau^*_i(\mu) \leq s$. The complement of (r, s)-(i, j)-θ -gdfc is an (r, s)-(i, j)-θ-generalized double fuzzy open ((r, s)- (i, j)-θ-gdfo, for short).

Definition 2.5. Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$ and $x_t \in Pt(X)$. Then:

(1) A fuzzy point x_t is said to be an (r, s)-(τ_{12}, τ^*_{12})-θ-cluster point if and only if $C_{12}(\mu, r, s)q\lambda$ for each $\mu \in Q_{\tau_{12},\tau^*_{12}}(x_t, r, s)$. $Q_{\tau_{12},\tau^*_{12}}(x_t, r, s) = \{\mu \in I^X : x_t q\mu, \ \tau_{12}(\mu) \geq r, \ \tau^*_{12}(\mu) \leq s\}$. The set of all (r, s)-(τ_{12}, τ^*_{12})-θ -cluster points of λ is called C^θ_{12} -fuzzy closure of λ, i.e. $C^\theta_{12} : I^X \times I_0 \times I_1 \to I^X$ defined as follows:

$$C^\theta_{12}(\lambda, r, s) = \vee\{x_t \in Pt(X) : x_t \text{ is } (r, s)\text{-}(\tau_{12}, \tau^*_{12})\text{-}\theta\text{-cluster point of } \lambda\}.$$

(2) λ is said to be an (r, s)-(τ_{12}, τ^*_{12}) -double fuzzy θ closed ((r, s)-(τ_{12}, τ^*_{12})-dfθc, for short) set iff $C^\theta_{12}(\lambda, r, s) = \lambda$. The complement of (r, s)-(τ_{12}, τ^*_{12})-df θc set is (r, s)-(τ_{12}, τ^*_{12})-double fuzzy θ open ((r, s)-(τ_{12}, τ^*_{12})-dfθo, for short) set.

Theorem 2.1. *Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$. Then:*

(1) $C_{12}(\lambda, r, s) \leq C^\theta_{12}(\lambda, r, s) \leq T^{\tau_i,\tau^*_i}_{\tau_j,\tau^*_j}(\lambda, r, s)$.
(2) *If λ is an (r, s)-(τ_{12}, τ^*_{12})-dfo set in X, then $C_{12}(\lambda, r, s) = C^\theta_{12}(\lambda, r, s)$.*

Proposition 2.1. *Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda, \lambda_1, \lambda_2 \in I^X, r \in I_0, s \in I_1$. Then:*

(1) $C^\theta_{12}(\underline{0}, r, s) = \underline{0}$.
(2) $\lambda \leq C^\theta_{12}(\lambda, r, s)$.
(3) *If $\lambda_1 \leq \lambda_2$, then $C^\theta_{12}(\lambda_1, r, s) \leq C^\theta_{12}(\lambda_2, r, s)$.*
(4) $C^\theta_{12}(\lambda_1, r, s) \vee C^\theta_{12}(\lambda_2, r, s) \leq C^\theta_{12}(\lambda_1 \vee \lambda_2, r, s)$.
(5) $C^\theta_{12}(\lambda, r_1, s_1) \leq C^\theta_{12}(\lambda, r_2, s_2)$, *if $r_1 \leq r_2$ and $s_1 \geq s_2$.*
(6) $C^\theta_{12}(\lambda_1 \wedge \lambda_2, r, s) \leq C^\theta_{12}(\lambda_1, r, s) \wedge C^\theta_{12}(\lambda_2, r, s)$.
(7) $C^\theta_{12}(\lambda, r, s) \leq C^\theta_{12}(C^\theta_{12}(\lambda, r, s), r, s)$.

Proof. The proof follows immediately from the definition of C^θ_{12}. □

Proposition 2.2. *Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$. Then:*

$$C^\theta_{12}(\lambda, r, s) = \wedge\{C_{12}(\rho, r, s) : \rho \geq \lambda, \tau_{12}(\rho) \geq r, \ \tau^*_{12}(\rho) \leq s\}.$$

Proof. Let $K = \wedge\{C_{12}(\rho, r, s) : \rho \geq \lambda, \tau_{12}(\rho) \geq r, \ \tau^*_{12}(\rho) \leq s\}$. Suppose $x_t \in C^\theta_{12}(\lambda, r, s)$ such that $x_t \notin K$. Then, there exists $\rho \in I^X$ such that $\rho \geq \lambda$ with $\tau_{12}(\rho) \geq r, \ \tau^*_{12}(\rho) \leq s$ and $x_t \notin C_{12}(\rho, r, s)$. Since $\lambda \leq \rho$ and $\tau_{12}(\rho) \geq r, \ \tau^*_{12}(\rho) \leq s$. Then by Proposition 2.1(2) and Theorem 2.1(2), $C^\theta_{12}(\lambda, r, s) \leq C^\theta_{12}(\rho, r, s) = C_{12}(\rho, r, s)$, this implies $x_t \notin C^\theta_{12}(\lambda, r, s)$ which is a contradiction. Thus, $C^\theta_{12}(\lambda, r, s) \leq K$.

Conversely, let $x_t \in K$ such that $x_t \notin C^\theta_{12}(\lambda, r, s)$. Then, there exists $\mu \in Q_{\tau_{12},\tau^*_{12}}(x_t, r, s)$ such that $C_{12}(\mu, r, s)\ \overline{q}\lambda$, implies $C_{12}(\mu, r, s) \leq \underline{1} - \lambda$ and hence $\lambda \leq \underline{1} - C_{12}(\mu, r, s)$ which is an (r, s)-dfo set in $(x, \tau_{12}, \tau^*_{12})$. Then, by assumption we have $x_t \in C_{12}(\underline{1} - C_{12}(\mu, r, s), r, s)$ and by Theorem 2.1, $x_t \in C^\theta_{12}(\underline{1} - C_{12}(\mu, r, s), r, s)$. Therefore, $C_{12}(\mu, r, s)q\underline{1} - C_{12}(\mu, r, s)$ which is a contradiction. Thus, $K \leq C^\theta_{12}(\lambda, r, s)$. Hence, we have the required result. □

Definition 2.6. Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$and $x_t \in Pt(X)$. A fuzzy point x_t is said to be an (r, s)-(τ_{12}, τ^*_{12})-θ-interior point of λ if there exists $\mu \in Q_{\tau_{12},\tau^*_{12}}(x_t, r, s)$ such that $C_{12}(\mu, r, s)\overline{q}\ \underline{1} - \lambda$. The set of all (r, s) -(τ_{12}, τ^*_{12})-θ-interior points of λ is called I^θ_{12}-double fuzzy interior of λ, i.e. $I^\theta_{12} : I^X \times I_0 \times I_1 \to I^X$ defined as follows:

$$I^\theta_{12}(\lambda, r, s) = \vee\{x_t \in Pt(X) : x_t \text{ is } (r, s)\text{-}(\tau_{12}, \tau^*_{12})\text{-}\theta\text{-interior point of } \lambda\}.$$

Equivalently,

$$I^{\theta}_{12}(\lambda,r,s)=\vee\{\mu\in I^X : C_{12}(\mu,r,s)\le\lambda,\ \tau_{12}(\mu)\ge r,\ \tau^*_{12}(\mu)\le s\}.$$

Definition 2.7. Let $f:(X,(\tau_1,\tau^*_1),(\tau_2,\tau^*_2))\to(X,(\nu_1,\nu^*_1),(\nu_2,\nu^*_2))$ be a mapping. Then, f is called:

(1) Generalized DFP*-continuous (GDFP*-continuous, for short) if and only if $f^{-1}(\mu)$ is (r,s)-(τ_{12},τ^*_{12})-gdfc with $\nu_{12}(\underline{1}-\mu)\ge r$ and $\nu^*_{12}(\underline{1}-\mu)\le s\ \forall\,\mu\in I^Y$.

(2) Generalized DFP*-irresolute (GDFP*-irresolute, for short) if and only if $f(\mu)$ is (r,s)-(ν_{12},ν^*_{12})-gdfc in Y for each (r,s)-(τ_{12},τ^*_{12})-gdfc μ in X.

Definition 2.8. Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts, $\lambda\in I^X, r\in I_0, s\in I_1$. Then:

(1) A fuzzy set λ is called an (r,s)-(τ_{12},τ^*_{12})-θ-generalized double fuzzy closed ((r,s)-(τ_{12},τ^*_{12})-θ-gdfc, for short) if $C^{\theta}_{12}(\lambda,r,s)\le\mu$ whenever $\lambda\le\mu$ such that $\tau_{12}(\mu)\ge r$ and $\tau^*_{12}(\mu)\le s$.

(2) A fuzzy set λ is called an (r,s)-(τ_{12},τ^*_{12})-θ-generalized double fuzzy open ((r,s)-(τ_{12},τ^*_{12})-θ-gdfo, for short) if $\underline{1}-\lambda$ is (r,s)-(τ_{12},τ^*_{12})-θ -gdfc.

Proposition 2.3. *Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts, $\lambda_1,\lambda_2\in I^X, r\in I_0, s\in I_1$. Then:*

(1) If λ_1,λ_2 are (r,s)-(τ_{12},τ^*_{12})-θ-gdfc sets, then $\lambda_1\vee\lambda_2$ is an (r,s)-(τ_{12},τ^*_{12})-θ-gdfc set.

(2) If λ_1,λ_2 are (r,s)-(τ_{12},τ^*_{12})-θ-gdfo sets, then $\lambda_1\wedge\lambda_2$ is an (r,s)-(τ_{12},τ^*_{12})-θ-gdfc set.

Proof. (1) Let $\lambda_1\vee\lambda_2\le\mu$ such that $\tau_{12}(\mu)\ge r$ and $\tau^*_{12}(\mu)\le s$. This implies that $\lambda_1\le\mu$ and $\lambda_2\le\mu$. Since λ_1 and λ_2 are (r,s)-(τ_{12},τ^*_{12})-θ -gdfc sets, then from (3) in Proposition 2.1, and in view of Definition 2.8(1), we have, $C^{\theta}_{12}(\lambda_1\vee\lambda_2,r,s)=C^{\theta}_{12}(\lambda_1,r,s)\vee C^{\theta}_{12}(\lambda_2,r,s)\ \le\mu\vee\mu$. Hence, $\lambda_1\vee\lambda_2$ is an (r,s)-(τ_{12},τ^*_{12})-θ-gdfc. Part (2), follows from the duality of (1). □

Remark 2.1. The finite intersection (resp., union) of (r,s)-(τ_{12},τ^*_{12})-θ-gdfc (resp., (r,s)-(τ_{12},τ^*_{12})-θ-gdfo) sets in a dfbts $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ need not to be (r,s)-(τ_{12},τ^*_{12})-θ-gdfc (resp., (r,s)-(τ_{12},τ^*_{12})-θ-gdfo).

Proposition 2.4. *Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts, $\lambda\in I^X, r\in I_0, s\in I_1$. If λ is an (r,s)-(τ_{12},τ^*_{12})-θ-dfc set, then λ is an (r,s)-(τ_{12},τ^*_{12})-θ-gdfc set.*

Proof. (1) Let $\lambda\le\mu$ such that $\tau_{12}(\mu)\ge r$ and $\tau^*_{12}(\mu)\le s$. Since λ is (r,s)-(τ_{12},τ^*_{12})-θ-dfc set, then $C^{\theta}_{12}(\lambda,r,s)=\lambda$ and from (4) in Proposition 2.1, for $r_1\le r_2$ and $s_1\ge s$, we have $C^{\theta}_{12}(\lambda,r_1,s_1)\le C^{\theta}_{12}(\lambda,r_2,s_2)=\lambda\le\mu$. Hence, λ is an (r,s)-(τ_{12},τ^*_{12})-θ-gdfc. □

The converse of Proposition 2.3 is not true from the following example.

Example 2.1. Let $X=\{a,b\}$. Define $\mu,\rho\in I^X$ as follows:

$$\mu(a)=0.2,\qquad \mu(b)=0.5$$

$$\rho(a)=0.5,\qquad \rho(b)=0.4$$

Define double fuzzy topologies $\tau_1,\ \tau^*_1,\ \tau_2,\ \tau^*_2 : I^X\to I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \lambda=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \lambda=\mu,\\ 0 & \text{otherwise},\end{cases}\quad,\quad \tau^*_1(\lambda)=\begin{cases}0 & \text{if } \lambda=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \lambda=\mu,\\ 1 & \text{otherwise},\end{cases}$$

$$\tau_2(\lambda)=\begin{cases}1 & \text{if } \lambda=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \lambda=\rho,\\ 0 & \text{otherwise.}\end{cases}\quad,\quad \tau^*_2(\lambda)=\begin{cases}0 & \text{if } \lambda=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \lambda=\rho,\\ 1 & \text{otherwise.}\end{cases}$$

The associated double supra fuzzy topology is defined as $\tau_{12},\tau^*_{12}: I^X\to I$ such that

$$\tau_{12}(\lambda)=\begin{cases}1 & \text{if } \lambda=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \lambda=\mu,\rho,\mu\vee\rho,\\ 0 & \text{otherwise.}\end{cases}$$

$$\tau^*_{12}(\lambda)=\begin{cases}0 & \text{if } \lambda=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \lambda=\mu,\rho,\mu\vee\rho,\\ 1 & \text{otherwise.}\end{cases}$$

Then, for $r=\frac{1}{2},\ s=\frac{1}{2}$, the fuzzy set $\lambda=\{0.3,0.4\}$ is a $(\frac{1}{2},\ \frac{1}{2})$-$(\tau_{12},\tau^*_{12})$-$\theta$-gdfc set but it is not a $(\frac{1}{2},\ \frac{1}{2})$-$(\tau_{12},\tau^*_{12})$-$\theta$-dfc set.

Proposition 2.5. *Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts, $\lambda\in I^X, r\in I_0, s\in I_1$. If λ is an (r,s)-(τ_{12},τ^*_{12})-θ-gdfc set, then λ is an (r,s)-(τ_{12},τ^*_{12})-gdfc set.*

Proof. The proof follows directly from (1) in Theorem 2.1. □

Proposition 2.6. *Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts, $\lambda\in I^X, r\in I_0, s\in I_1$. If λ is an (r,s)-(τ_{12},τ^*_{12})-dfc set, then λ is an (r,s)-(τ_{12},τ^*_{12})-gdfc set.*

Proposition 2.7. *Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts, $\lambda\in I^X, r\in I_0, s\in I_1$. If λ is an (r,s)-(τ_{12},τ^*_{12})-θ-dfc set, then λ is an (r,s)-(τ_{12},τ^*_{12})-dfc set.*

Proposition 2.8. *Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts, $\lambda\in I^X, r\in I_0, s\in I_1$. If λ is an (r,s)-(j,i)-dfθc set, then λ is an (r,s)-(τ_{12},τ^*_{12})-θ-dfc set.*

Proof. To prove λ is an (r,s)-(τ_{12},τ^*_{12})-θ-dfc set, we must prove $C^{\theta}_{12}(\lambda,r,s)=\lambda$. Clearly, $\lambda\le C^{\theta}_{12}(\lambda,r,s)$. On the other hand, by Theorem 2.1, $C^{\theta}_{12}(\lambda,r,s)\le T^{\tau_i,\tau^*_i}_{\tau_j,\tau*_j}(\lambda,r,s)$. Since λ is an (r,s)-(j,i)-dfθc set, then $T^{\tau_i,\tau^*_i}_{\tau_j,\tau*_j}(\lambda,r,s)=\lambda$. Consequently, $C^{\theta}_{12}(\lambda,r,s)\le\lambda$. Hence, λ is an (r,s)-(τ_{12},τ^*_{12})-θ-dfc. □

From the above discussion we have the following diagram:

$$\begin{array}{ccccccc}
(r,s)\text{-}(i,j)\text{-}\theta\text{-gdfc} & \Longleftarrow & (r,s)\text{-}(i,j)\text{-df}\theta\text{c} & \Longrightarrow & (r,s)\text{-}(\tau_{12},\tau^*_{12})\text{-}\theta\text{-dfc} & \Longrightarrow & (r,s)\text{-}(\tau_{12},\tau^*_{12})\text{-dfc}\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
(r,s)\text{-}(i,j)\text{-gdfc} & \Longleftarrow & (r,s)\text{-}(i,j)\text{-dfc} & & (r,s)\text{-}(\tau_{12},\tau^*_{12})\text{-}\theta\text{-gdfc} & \Longrightarrow & (r,s)\text{-}(\tau_{12},\tau^*_{12})\text{-gdfc}
\end{array}$$

3. Generalized C^{θ}_{12}-double fuzzy closure operator

Definition 3.1. Let $(X,\ (\tau_1,\tau^*_1),(\tau_2,\tau^*_2))$ be a dfbts. Then, for $\lambda\in I^X, r\in I_0, s\in I_1$, we define the GC^{θ}_{12}-fuzzy closure (interior) operators $GC^{\theta}_{12}, GI^{\theta}_{12}:\ I^X\times I_0\times I_1\to I^X$ defined as follows:

$$GC^{\theta}_{12}(\lambda,r,s)=\wedge\{\rho\in I^X :\ \rho\ge\lambda \text{ and } \rho \text{ is } (r,s)\text{-}(\tau_{12},\tau^*_{12})\text{-}\theta\text{-gdfc set}\},$$

$GI_{12}^{\theta}(\lambda, r, s) = \vee\{\rho \in I^X : \rho \leq \lambda$ and
ρ is (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfo set$\}$.

Proposition 3.1. *Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be a dfbts, $\lambda, \lambda_1, \lambda_2 \in I^X, r \in I_0, s \in I_1$. Then:*

(1) $GI_{12}^{\theta}(\underline{1} - \lambda, r, s) = \underline{1} - GC_{12}^{\theta}(\lambda, r, s)$.
(2) *If $\lambda_1 \leq \lambda_2$, then $GC_{12}^{\theta}(\lambda_1, r, s) \leq GC_{12}^{\theta}(\lambda_2, r, s)$.*
(3) *If $\lambda_1 \leq \lambda_2$, then $GI_{12}^{\theta}(\lambda_1, r, s) \leq GI_{12}^{\theta}(\lambda_2, r, s)$.*
(4) *If λ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc, then $GC_{12}^{\theta}(\lambda, r, s) = \lambda$.*
(5) *If λ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfo, then $GI_{12}^{\theta}(\lambda, r, s) = \lambda$.*

Proof.

(1) We prove (1) by using Definition 3.1:

$$\begin{aligned}\underline{1} - GC_{12}^{\theta}(\lambda, r, s) &= \underline{1} - \wedge\{\rho \in I^X : \rho \geq \lambda \text{ and } \\ &\quad \rho \text{ is } (r, s)\text{-}(\tau_{12}, \tau_{12}^*)\text{-}\theta \text{ -gdfc set}\} \\ &= \vee\{\underline{1} - \rho \in I^X : \underline{1} - \rho \leq \underline{1} - \lambda \text{ and } \\ &\quad \underline{1} - \rho \text{ is } (r, s)\text{ -}(\tau_{12}, \tau_{12}^*)\text{-}\theta\text{-gdfo set}\} \\ &= GI_{12}^{\theta}(\underline{1} - \lambda, r, s).\end{aligned}$$

(2) Suppose there exist $x \in X$ and $t \in (0, 1)$ such that

$$GC_{12}^{\theta}(\lambda_1, r, s)(x) > t > GC_{12}^{\theta}(\lambda_2, r, s)(x). \tag{3.1}$$

Since $GC_{12}^{\theta}(\lambda_2, r, s)(x) < t$, then there exists an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc set ρ with $\rho \geq \lambda_2$ such that $\rho(x) < t$. Since $\lambda_1 \leq \lambda_2$, then $GC_{12}^{\theta}(\lambda_1, r, s) \leq \rho$. It follows $GC_{12}^{\theta}(\lambda_1, r, s)(x) < t$. This contradicts (3.1). Hence, $GC_{12}^{\theta}(\lambda_1, r, s) \leq GC_{12}^{\theta}(\lambda_2, r, s)$.

(3) Taking the complement of (2) and using (1), we can prove (3).
(4) It can be proved from Definition 3.1.
(5) Similar to (3). □

Theorem 3.1. *Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$. Then:*

(1) *GC_{12}^{θ} (resp. GI_{12}^{θ}) is a double fuzzy closure (resp. interior) operator.*
(2) *Define $\tau_{12}^{G\theta} : I^X \to I$ as*

$$\tau_{12}^{G\theta}(\lambda) = \vee\{r \in I : GC_{12}^{\theta}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\},$$
$$\tau_{12}^{*G\theta}(\lambda) = \wedge\{s \in I : GC_{12}^{\theta}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}.$$

Then, $(\tau_{12}^{G\theta}, \tau_{12}^{*G\theta})$ is a double fuzzy on X such that $\tau_{12}^{\theta} \leq \tau_{12}^{G\theta}(\lambda)$ and $\tau_{12}^{*\theta}(\lambda) \geq \tau_{12}^{*G\theta}(\lambda)$

Proof. (1) To prove (1), we need to satisfy conditions (C1)–(C2) in Defininition 1.2.

(C1) Since $\underline{0}$ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc set in X, then from Proposition 3.1(4), $GC_{12}^{\theta}(\underline{0}, r, s) = \underline{0}$.
(C2) Follows from the definition of GC_{12}^{θ}.
(C3) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, from Proposition 3.1(2),

$GC_{12}^{\theta}(\lambda, r, s) \leq GC_{12}^{\theta}(\lambda \vee \mu, r, s)$ and
$GC_{12}^{\theta}(\mu, r, s) \leq GC_{12}^{\theta}(\lambda \vee \mu, r, s)$.

This implies, $GC_{12}^{\theta}(\lambda, r, s) \vee GC_{12}^{\theta}(\mu, r, s) \leq GC_{12}^{\theta}(\lambda \vee \mu, r, s)$.
Suppose $GC_{12}^{\theta}(\lambda \vee \mu, r, s) \nleq GC_{12}^{\theta}(\lambda, r, s) \vee GC_{12}^{\theta}(\mu, r, s)$. Consequently, $x \in X$ and $t \in (0, 1)$ exist such that

$$GC_{12}^{\theta}(\lambda, r, s)(x) \vee GC_{12}^{\theta}(\mu, r, s)(x) < t < GC_{12}^{\theta}(\lambda \vee \mu, r, s)(x). \tag{3.2}$$

Since $GC_{12}^{\theta}(\lambda, r, s)(x) < t$ and $GC_{12}^{\theta}(\mu, r, s)(x) < t$, there exist (r, s)-(τ_{12}, τ_{12}^*)-θ -gdfc sets ρ_1, ρ_2 with $\lambda \leq \rho_1$ and $\mu \leq \rho_2$ such that

$$\rho_1(x) < t, \ \rho_2(x) < t.$$

Since $\lambda \vee \mu \leq \rho_1 \vee \rho_2$ and $\rho_1 \vee \rho_2$ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc from Proposition 2.2(1), we have $GC_{12}^{\theta}(\lambda \vee \mu, r, s)(x) \leq (\rho_1 \vee \rho_2)(x) < t$. This, however, contradicts (3.2). Hence, $GC_{12}^{\theta}(\lambda, r, s) \vee GC_{12}^{\theta}(\mu, r, s) = GC_{12}^{\theta}(\lambda \vee \mu, r, s)$.

(C4) Let $r_1 \leq r_2$ and $s_1 \geq s_2$, $r_1, r_2 \in I_0$ and $s_1, s_2 \in I_1$, Suppose $GC_{12}^{\theta}(\lambda, r_1, s_1) \nleq GC_{12}^{\theta}(\lambda, r_2, s_2)$. Consequently, $x \in X$ and $t \in (0, 1)$ exist such that

$$GC_{12}^{\theta}(\lambda, r_2, s_2)(x) < t < GC_{12}^{\theta}(\lambda, r_1, s_1)(x). \tag{3.3}$$

Since $GC_{12}^{\theta}(\lambda, r_2, s_2)(x) < t$, there is an (r_2, s_2)-(τ_{12}, τ_{12}^*)-θ-gdfc set ρ with $\lambda \leq \mu$ such that $\rho(x) < t$. This yields $GC_{12}^{\theta}(\rho, r_3, s_3) \leq \mu$, whenever $\rho \leq \mu$ and $\tau_{12}(\mu) \geq r_3$ and $\tau_{12}^*(\mu) \leq s_3$. Since $r_1 \leq r_2$ and $s_1 \geq s_2$, then $GC_{12}^{\theta}(\rho, r_4, s_4) \leq \mu$, whenever $\rho \leq \mu$ and $\tau_{12}(\mu) \geq r_4$ and $\tau_{12}^*(\mu) \leq s_4$. This implies ρ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc. From Definition 3.1, we have $GC_{12}^{\theta}(\lambda, r_1, s_1)(x) \leq \rho(x) < t$. This contradicts (3.3). Hence, $GC_{12}^{\theta}(\lambda, r_1, s_1) \leq GC_{12}^{\theta}(\lambda, r_2, s_2)$.

(C5) Let ρ be any (r_2, s_2)-(τ_{12}, τ_{12}^*)-θ-gdfc containing λ. Then, from Definition 3.1, we have $GC_{12}^{\theta}(\lambda, r, s) \leq \rho$, from Proposition 3.1(2), we obtain $GC_{12}^{\theta}(\lambda, r, s) \leq GC_{12}^{\theta}(\rho, r, s) = \rho$. This mean that $GC_{12}^{\theta}(GC_{12}^{\theta}(\lambda, r, s), r, s)$ is contained in every (r_2, s_2)-(τ_{12}, τ_{12}^*)-θ-gdfc containing λ. Hence $GC_{12}^{\theta}(GC_{12}^{\theta}(\lambda, r, s), r, s) \leq GC_{12}^{\theta}(\lambda, r, s)$, however $GC_{12}^{\theta}(\lambda, r, s) \leq GC_{12}^{\theta}(GC_{12}^{\theta}(\lambda, r, s), r, s)$. Therefore $GC_{12}^{\theta}(GC_{12}^{\theta}(\lambda, r, s), r, s) = GC_{12}^{\theta}(\lambda, r, s)$. Thus GC_{12}^{θ} is a double fuzzy closure operator.

By similar way, we can prove that GI_{12}^{θ} is a double fuzzy interior operator.

By using (1) and Definition 1.2, we get $(\tau_{12}^{G\theta}, \tau_{12}^{*G\theta})$ is a double fuzzy topology on X. By Proposition 2.3, $C_{12}^{\theta}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda$ which yields $GC_{12}^{\theta}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda$. Thus $\tau_{12}^{\theta}(\lambda) \leq \tau_{12}^{G\theta}(\lambda)$ and $\tau_{12}^{*\theta}(\lambda) \geq \tau_{12}^{*G\theta}(\lambda)$ for all $\lambda \in I^X$. □

Proposition 3.2. *Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be a dfbts, $\lambda \in I^X, r \in I_0, s \in I_1$. If λ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc, then λ is an (r, s)-$(\tau_{12}^{G\theta}, \tau_{12}^{*G\theta})$-dfc set.*

Proof. Follows from Proposition 3.1(4) and Theorem 3.1(2). □

4. GDFP*-θ-continuous and GDFP*-θ-irresolute mappings

Definition 4.1. A mapping $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (X, (\sigma_1, \sigma_1^*), (\sigma_2, \sigma_2^*))$ is called:

(1) Generalized DFP*-θ-continuous (GDFP*-θ-continuous, for short) if $f^{-1}(\mu)$ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc in X for each (r, s)-$(\sigma_{12}, \sigma_{12}^*)$-dfc set μ in Y.

(2) Generalized DFP*-θ-irresolute (GDFP*-θ-irresolute, for short) if $f^{-1}(\mu)$ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc in X for each (r, s)-$(\sigma_{12}, \sigma_{12}^*)$-$\theta$-gdfc set μ in Y.

(3) DFP*-strongly θ-continuous (DFP*-Sθ -continuous, for short) if for each $x_t \in Pt(X)$ and $r \in I_0$, $s \in I_1$ if $\mu \in Q_{\sigma_{12},\sigma_{12}^*}(f(x_t), r, s)$, there exists $\nu \in Q_{\tau_{12},\tau_{12}^*}(x_t, r, s)$ such that $f(C_{12}(\nu, r, s)) \leq \mu$.

Proposition 4.1. *$f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (X, (\sigma_1, \sigma_1^*), (\sigma_2, \sigma_2^*))$ is GDFP*-θ-continuous, then GDFP *-continuous.*

Proof. Let $\mu \in I^Y$ such that μ is (r, s)-$(\sigma_{12}, \sigma_{12}^*)$-dfc set. Since f is GDFP*-θ -continuous, then we have $f^{-1}(\mu)$ is an (r, s)-(τ_{12}, τ_{12}^*)-θ-gdfc, and from Proposition 2.4, this yields $f^{-1}(\mu)$ is an (r, s)-(τ_{12}, τ_{12}^*)-gdfc. Hence, f is GDFP*-continuous. □

The converse of Proposition 4.1 is not true as the following example:

Example 4.1. Let $X = \{a, b\}$ and $Y = \{x, y, z\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1 = a_{\frac{2}{3}} \vee b_{\frac{1}{2}}, \qquad \lambda_2 = a_{\frac{3}{4}} \vee b_{\frac{1}{4}}$$
$$\mu_1 = x_{\frac{3}{4}} \vee y_{\frac{2}{3}} \vee z_{\frac{1}{2}}, \qquad \mu_2 = x_{\frac{2}{3}} \vee y_{\frac{3}{4}} \vee z_{\frac{1}{2}}$$

We define double fuzzy topologies $\tau_1, \tau_1^*, \tau_2, \tau_2^* : I^X \to I$ and $\sigma_1, \sigma_1^*, \sigma_2, \sigma_2^* : I^Y \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise,} \end{cases} \quad \tau_1^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise,} \end{cases} \quad \tau_2^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{3}{4} & \text{if } \lambda = \lambda_2, \\ 1 & \text{otherwise,} \end{cases}$$

$$\sigma_1(\lambda) = \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise,} \end{cases} \quad \sigma_1^*(\lambda) = \begin{cases} 0 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 1 & \text{otherwise,} \end{cases}$$

$$\sigma_2(\mu) = \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise.} \end{cases} \quad \sigma_2^*(\mu) = \begin{cases} 0 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{2}{3} & \text{if } \mu = \mu_2, \\ 1 & \text{otherwise.} \end{cases}$$

The associated double supra fuzzy topologies is defined as follows:

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{12}^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{3}{4} & \text{if } \lambda = \lambda_2, \\ \frac{3}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 1 & \text{otherwise,} \end{cases}$$

$$\sigma_{12}(\mu) = \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ \frac{1}{3} & \text{if } \mu = \mu_1 \vee \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\sigma_{12}^*(\lambda) = \begin{cases} 0 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ \frac{2}{3} & \text{if } \mu = \mu_2, \\ \frac{2}{3} & \text{if } \mu = \mu_1 \vee \mu_2, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the mapping $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (Y, (\sigma_1, \sigma_1^*), (\sigma_2, \sigma_2^*))$ defined by $f(a) = y$ and $f(b) = z$. Then f is GDFP*-continuous but is not GDFP*-θ -continuous because, there exists $\underline{1} - \mu$ is a $(\frac{1}{2}, \frac{1}{2}) - (\tau_{12}, \tau_{12}^*)$-dfc set but $f^{-1}(.\underline{1} - \mu)$ is not a $(\frac{1}{2}, \frac{1}{2}) - (\tau_{12}, \tau_{12}^*)$-$\theta$-gdfc set.

Proposition 4.2. *If $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (X, (\sigma_1, \sigma_1^*), (\sigma_2, \sigma_2^*))$ is DFP*-Sθ-continuous, then GDFP *-θ-continuous.*

Proof. Let $\lambda \in I^Y$ be an (r, s)-$(\sigma_{12}, \sigma_{12}^*)$-dfc set. Let $f^{-1}(\lambda) \leq \mu$, where $\tau_{12}(\mu) \geq r_1$ and $\tau_{12}^*(\mu) \leq s_1$ for $r_1 \leq r$ and $s_1 \geq s$. We must show that $C_{12}^{\theta}(f^{-1}(\lambda), r, s) \leq \mu$. Let $x_t \notin \mu$ this means, $x_t q \underline{1} - \mu$. In fact, $f^{-1}(\lambda) \leq \mu$, which implies that $\underline{1} - \mu \leq \underline{1} - f^{-1}(\lambda)$. Since $x_t q \underline{1} - \mu$ this yields, $x_t q \underline{1} - f^{-1}(\lambda)$. Thus, we have $f(x_t) q \underline{1} - \lambda$ such that $\underline{1} - \lambda$ is (r, s)-$(\sigma_{12}, \sigma_{12}^*)$-dfo set in Y. That is means $\underline{1} - \lambda \in Q_{\sigma_{12},\sigma_{12}^*}(f(x_t), r, s)$. Since f is DFP*-Sθ-continuous, then there exists $\eta \in Q_{\tau_{12},\tau_{12}^*}(x_t, r, s)$ such that $f(C_{12}(\eta, r, s)) \leq \underline{1} - \lambda$. This implies, $f(C_{12}(\eta, r, s)) \overline{q} \lambda$ and then $C_{12}(\eta, r, s) \overline{q} f^{-1}(\lambda)$. In view of Definition 2.6, we get $x_t \notin C_{12}^{\theta}(f^{-1}(\lambda), r, s)$. Since $r_1 \leq r$ and $s_1 \geq s$, then from Proposition 2.1(4), we have $x_t \notin C_{12}^{\theta}(f^{-1}(\lambda), r_1, s_1)$. Hence, we obtain $C_{12}^{\theta}(f^{-1}(\lambda), r_1, s_1) \leq \mu$. Thus, f is GDFP*-θ-continuous. □

The converse of Proposition 4.1 is not true as the following example:

Example 4.2. Let $X = \{a, b\}$ and $Y = \{x, y\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1 = a_{\frac{1}{2}} \vee b_{\frac{1}{3}}, \quad \lambda_2 = a_{\frac{1}{3}} \vee b_{\frac{1}{2}}$$
$$\mu_1 = x_{\frac{1}{2}} \vee y_{\frac{1}{4}}, \quad \mu_2 = x_{\frac{1}{4}} \vee y_{\frac{1}{2}}$$

We define double fuzzy topologies $\tau_1, \tau_1^*, \tau_2, \tau_2^* : I^X \to I$ and $\sigma_1, \sigma_1^*, \sigma_2, \sigma_2^* : I^Y \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise,} \end{cases} \quad \tau_1^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise,} \end{cases} \quad \tau_2^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{2}{3} & \text{if } \lambda = \lambda_2, \\ 1 & \text{otherwise,} \end{cases}$$

$$\sigma_1(\lambda) = \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise,} \end{cases} \quad \sigma_1^*(\lambda) = \begin{cases} 0 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 1 & \text{otherwise,} \end{cases}$$

$$\sigma_2(\mu) = \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise.} \end{cases} \quad \sigma_2^*(\mu) = \begin{cases} 0 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{2}{3} & \text{if } \mu = \mu_2, \\ 1 & \text{otherwise.} \end{cases}$$

The associated double supra fuzzy topologies is defined as follows:

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{12}^{*}(\lambda)=\begin{cases}0 & \text{if } \lambda=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \lambda=\lambda_1,\\ \frac{2}{3} & \text{if } \lambda=\lambda_2,\\ \frac{2}{3} & \text{if } \lambda=\lambda_1\vee\lambda_2,\\ 1 & \text{otherwise,}\end{cases}$$

$$\sigma_{12}(\mu)=\begin{cases}1 & \text{if } \mu=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \mu=\mu_1,\\ \frac{1}{3} & \text{if } \mu=\mu_2,\\ \frac{1}{3} & \text{if } \mu=\mu_1\vee\mu_2,\\ 0 & \text{otherwise.}\end{cases}$$

$$\sigma_{12}^{*}(\lambda)=\begin{cases}0 & \text{if } \mu=\underline{0},\ \underline{1},\\ \frac{1}{2} & \text{if } \mu=\mu_1,\\ \frac{2}{3} & \text{if } \mu=\mu_2,\\ \frac{2}{3} & \text{if } \mu=\mu_1\vee\mu_2,\\ 1 & \text{otherwise.}\end{cases}$$

Consider the mapping $f:(X,\ (\tau_1,\tau_1^*),(\tau_2,\tau_2^*))\rightarrow(Y,\ (\sigma_1,\sigma_1^*),(\sigma_2,\sigma_2^*))$ defined by $f(a)=y$ and $f(b)=x$. Then, f is GDFP*-θ-continuous but is not DFP* -Sθ-continuous because, there exists $a_{0.7}\in Pt(X), r=\frac{1}{3}$, $s=\frac{2}{3}$ and $\mu_1\in Q_{\sigma_{12},\sigma_{12}^*}(f(a_{0.7}),\frac{1}{3},\frac{2}{3})$ such that for any $\lambda\in Q_{\tau_{12},\tau_{12}^*}(a_{0.7},\frac{1}{3},\frac{2}{3}), f(C_{12}(\lambda,\frac{1}{3},\frac{2}{3})\not\leq\mu_1$.

Proposition 4.3. *If* $f:\ (X,\ (\tau_1,\tau_1^*),(\tau_2,\tau_2^*))\rightarrow(X,\ (\sigma_1,\sigma_1^*),(\sigma_2,\sigma_2^*))$ *is DFP*-continuous, then GDFP* -continuous.*

The converse of above proposition is not true in general.

Theorem 4.1. *A mapping* $f:\ (X,\ (\tau_1,\tau_1^*),(\tau_2,\tau_2^*))\rightarrow(X,\ (\sigma_1,\sigma_1^*),(\sigma_2,\sigma_2^*))$ *is DFP*-continuous iff for each* $x_t\in Pt(X)$ *and for each* $\mu\in Q_{\sigma_{12},\sigma_{12}^*}(f(x_t),\ r,\ s)$*, there exists* $\eta\in Q_{\tau_{12},\tau_{12}^*}(x_t,r,s)$ *such that* $f(\eta)\leq\mu$.

Proof. Suppose $f:(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))\rightarrow(X,\ (\sigma_1,\sigma_1^*),(\sigma_2,\sigma_2^*))$ is DFP*-continuous. Let $x_t\in Pt(X)$ and $\mu\in Q_{\sigma_{12},\sigma_{12}^*}(f(x_t),\ r,\ s)$. Since f is DFP*-continuous. Then $\tau_{12}(f^{-1}(\mu))\geq\sigma_{12}(\mu)$ and $\tau_{12}^*(f^{-1}(\mu))\leq\sigma_{12}^*(\mu)$. Since $f(x_t)\in\mu$ then $x_t\in f^{-1}(\mu)$ implies $f^{-1}(\mu)\in Q_{\tau_{12},\tau_{12}^*}(x_t,r,s)$ such that $f(f^{-1}(\mu))\leq\mu$.

Conversely, Let $\mu\in I^Y$ such that μ is $(r,\ s)$-$(\sigma_{12},\sigma_{12}^*)$-dfo set and let $x_t\in f^{-1}(\mu)$, this implies $f(x_t)\in\mu$, Therefore $\mu\in Q_{\sigma_{12},\sigma_{12}^*}(f(x_t),\ r,\ s)$, and from our assumption, there exists $\nu\in Q_{\tau_{12},\tau_{12}^*}(x_t,r,s)$ such that $f(\nu)\leq\mu$. Then, we get $\nu\leq f^{-1}(\mu)$. That is means $f^{-1}(\mu)$ contains $(r,\ s)$-$(\tau_{12},\ \tau_{12}^*)$-dfo set for each $f^{-1}(\mu)$ is an $(r,\ s)$-$(\tau_{12},\ \tau_{12}^*)$-dfo set. Hence, f is DFP*-continuous. □

Proposition 4.4. *If* $f:\ (X,\ (\tau_1,\tau_1^*),(\tau_2,\tau_2^*))\rightarrow(X,\ (\sigma_1,\sigma_1^*),(\sigma_2,\sigma_2^*))$ *is DFP*-Sθ-continuous, then DFP*-continuous.*

Proof. Let $x_t\in Pt(X)$ and $\mu\in Q_{\sigma_{12},\sigma_{12}^*}(f(x_t),\ r,\ s)$. Since f is DFP*-Sθ -continuous. Then there exists $\eta\in Q_{\tau_{12},\tau_{12}^*}(x_t,r,s)$ such that $f(C_{12}(\eta,\ r,\ s))\leq\mu$. Since $\eta\leq C_{12}(\eta,\ r,\ s)$, then $f(\eta)\leq f(C_{12}(\eta,\ r,\ s))\leq\mu$. Thus, in view of Theorem 4.1, f is DFP*-continuous. □

The converse of above Proposition is not true as seen in Example 4.1.

Also, Examples 4.1 and 4.2 show that the DFP*-continuous and DFP*-θ-continuous are independent. Therefore, we have the following implications.

$$\begin{array}{ccc}\text{DFP}^*\text{-}\theta\text{-continuous} & \Longrightarrow & \text{GDFP}^*\text{-continuous}\\ \Uparrow & & \Uparrow\\ \text{DFP}^*\text{-S}\theta\text{-continuous} & \Longrightarrow & \text{DFP}^*\text{-continuous}\end{array}$$

Theorem 4.2. *A mapping* $f:\ (X,\ (\tau_1,\tau_1^*),(\tau_2,\tau_2^*))\rightarrow(X,\ (\sigma_1,\sigma_1^*),(\sigma_2,\sigma_2^*))$ *is bijective, DFP*-θ -open and GDFP*-θ-continuous, then GDFP*-θ -irresolute.*

Proof. Let $\nu\in I^Y$ such that ν is $(r,\ s)$-$(\sigma_{12},\ \sigma_{12}{}^*)$-$\theta$-gdfc set and $f^{-1}(\nu)\leq\mu$ such that $\tau_{12}(\mu)\geq r_1$ and $\tau_{12}{}^*(\mu)\leq s_1$ for $r_1\leq r$ and $s_1\geq s$. Since $f^{-1}(\nu)\leq\mu$, then $\nu\leq f(\mu)$. From the fact that DFP*-open, we obtain $f(\mu)$ is an $(r_1,\ s_1)$-$(\sigma_{12},\ \sigma_{12}{}^*)$-dfo. Now, we have ν is an $(r,\ s)$-$(\sigma_{12},\ \sigma_{12}{}^*)$-$\theta$-gdfc set and $\nu\leq f(\mu)$. From Definition 2.1(1) we get, $(C_{12}{}^*)_\theta(\nu,\ r_1,\ s_1)\leq f(\mu)$ and thus, $f^{-1}((C_{12}{}^*)_\theta(\nu,\ r_1,\ s_1))\leq\mu$. Since $(C_{12}{}^*)_\theta(\nu,\ r_1,\ s_1)$ is an $(r,\ s)$-$(\sigma_{12},\ \sigma_{12}{}^*)$-$\theta$-gdfc in Y and f is GDFP*-θ-continuous. Then $C_{12}{}^\theta(\ f^{-1}((C_{12}{}^*)_\theta(\nu,\ r_1,\ s_1)),\ r_1,\ s_1)\leq\mu$ this yields $C_{12}{}^\theta(f^{-1}(\nu),\ r_1,\ s_1))\leq\mu$. Therefore, $f^{-1}(\nu)$ is an $(r,\ s)$-$(\tau_{12},\ \tau_{12}^*)$-$\theta$-gdfc. Hence, f is GDFP *-θ-irresolute. □

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