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Original Article

Topological representation and quantic separation axioms of semi-quantales

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Abstract An adjunction between the category of semi-quantales and the category of lattice-valued quasi-topological spaces is established. Some characterizations of quantic separation axioms, for semi-quantales and lattice-valued quasi-topological spaces, are obtained and some relations among these axioms are established.

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1. Introduction

Quantales were first introduced in the eighties by Mulvey [1] in the ambitious aim of providing a possible common lattice-theoretic setting for constructive foundations for quantum mechanics, as well as a non-commutative analogue of the maximal spectrum of a C^*-algebra, and for non-commutative logics. The study of such ordered algebraic structures goes back to a series of papers by Ward and Dilworth [2–4] in the 1930s. They were motivated by the ideal theory of commutative rings. Following Mulvey, various types and aspects of quantales have been considered by many authors [5–8].

Since quantale theory provides a powerful tool in studying non-commutative structures, it has a wide applications, especially in studying non-commutative C^*-algebra theory [6,9], the ideal theory of commutative ring [10], linear logic [11] which supports part of the foundation of theoretic computer science [12,13] and so on.

In 1989 Borceux and van den Bossche [14] proposed a duality between spatial right-sided idempotent quantales and sober quantum spaces. In 2015, Höhle [15] established two adjunctions based on right-sided idempotent quantales. The first adjunction based on quantum spaces as an extension of the duality between spatial right-sided idempotent quantales and sober quantum spaces. The second adjunction between the category of right-sided idempotent quantales and the category of three-valued topological spaces. Both adjunctions restricts to the well known Papert–Papert–Isbell adjunction [16,17] between topological spaces and locales. In 2014 Demirci [18] established an

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abstract categorical analogue of famous Papert–Papert–Isbell adjunction to a general adjunction $X \dashv C^{op}$ in which C is an abstract category and X is a suitable category of such counterparts. Also he formulated two main categorical theorems: Fundamental Categorical Adjunction Theorem (FCAT) and Fundamental Categorical Duality Theorem (FCDT).

In this paper we aim to introduce and study a more general adjunction between the category of semi-quantales [19] and the category of lattice-valued quasi-topological spaces [20]. Also, we aim to study some separation axioms for semi-quantales with applications to lattice-valued quasi-topological spaces.

The present paper has been prepared in four sections. After this introductory section, the next section overviews the some useful concepts about semi-quantales, quantic nucleus and L-quasi-topologies. In Section 3, as one of the main contribution of this paper, we construct a dual adjunction between the category **SQuant** of semi-quantales and the category ***L*-QTop** of lattice-valued quasi-topological spaces. Also, by defining L-Qspatiality in the given category **SQuant** and L-Qsobriety in ***L*-QTop**, we show that the full subcategory of **SQuant** of all L-Qspatial objects and the full subcategory of ***L*-QTop** of all L-Qsober objects are dually equivalent. The results of this section can be obtained as applications of Fundamental Categorical Adjunction Theorem (FCAT) and Fundamental Categorical Duality Theorem (FCDT) [18]. Finally in Section 4, we will discuss the counterparts of the quantic regularity and normality axioms of objects in the category **SQuant** with applications to objects in the category L-**QTop**.

2. Preliminaries

By a $\bigvee$-semilattice we mean a partially ordered set $(L, \leq)$ having arbitrary $\bigvee$. A $\bigvee$-semilattice homomorphism is a map preserving arbitrary $\bigvee$.

Definition 2.1 ([19]). (lattice structures and associated categories).

(1) A semi-quantale $(L, \leq, \otimes)$, abbreviated as s-quantale, is a $\bigvee$-semilattice $(L, \leq)$ equipped with a binary operation $\otimes : L \times L \longrightarrow L$, with no additional assumptions, called a tensor product. The category **SQuant** comprises all semi-quantales together with s-quantale morphisms (i.e., mappings preserving $\otimes$ and arbitrary $\bigvee$). By **SSQuant** [20], we mean a non-full subcategory of **SQuant** comprising all semi-quantales and all ss-quantale morphisms (i.e., mappings preserving $\otimes$, arbitrary $\bigvee$ and $\top$). **SSQuant** and **SQuant** clearly share the same objects.

(2) A quantale $(L, \leq, \otimes)$ is an s-quantale whose multiplication is associative and distributes across $\bigvee$ from both sides [7]. **Quant** is the full subcategory of **SQuant** of all quantales.

(3) An ordered semi-quantale $(L, \leq, \otimes)$, abbreviated as os-quantale, is an s-quantale in which $\otimes$ is isotone in both variables. **OSQuant** is the full subcategory of **SQuant** of all os-quantales.

(4) A unital semi-quantale $(L, \leq, \otimes)$, abbreviated as us-quantale, is an s-quantale in which $\otimes$ has an identity element $e \in L$ called the unit. **USQuant** comprises all us-quantales together with all mappings preserving arbitrary $\bigvee$, $\otimes$, and e.

(5) A commutative semi-quantale $(L, \leq, \otimes)$, abbreviated as cs-quantale, is an s-quantale in which, $\otimes$ that is, $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in L$. **CSQuant** is the full subcategory of **SQuant** of all commutative semi-quantales.

(6) A complete quasi-monoidal lattice $(L, \leq, \otimes)$, abbreviated as cqml, is an os-quantale having $\top$ idempotent i.e., $\top \otimes \top = \top$. **CQML** comprises all cqml together with mappings preserving arbitrary $\bigvee$, $\otimes$, and $\top$ [21,22]. Note that **CQML** is a subcategory of **OSQuant**.

(7) A semi-frame [22] is a us-quantale whose multiplication and unit are $\wedge$ and $\top$ respectively. **SFrm** is the category of all semi-frames together with mappings preserving finite $\wedge$ and arbitrary $\bigvee$. **SFrm** is a full subcategory of **CQML**.

(8) A frame [23] is a unital quantale whose multiplication and unit are $\wedge$ and $\top$ respectively. **Frm** is the subcategory of **Quant** of all frames and morphisms preserving finite $\wedge$ and arbitrary $\bigvee$.

Definition 2.2 ([24]). An s-quantale is called distributive (ds-quantale) provided that its multiplication distributes across finite $\vee$ from both sides. **DSQuant** is the category of ds-quantales.

Definition 2.3 ([20]). Let $L = (L, \leq, \otimes)$ be an s-quantale. A subset $K \subseteq L$ is a subsemi-quantale of L iff it is closed under the tensor product $\otimes$ and arbitrary $\bigvee$. A subsemi-quantale K of L is said to be strong iff $\top$ belongs to K. If L is a us-quantale with the identity e, then a subsemi-quantale K of L is called a unital subsemi-quantale of L iff e belongs to K.

Definition 2.4 ([25]). Let Q be a semi-quantale. An element $\top \neq p \in Q$ is said to be prime if $a \otimes b \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in Q$. The set of all prime elements of Q, denoted by $Pr(Q)$.

Definition 2.5 (see [7]). Let $Q \in |\mathbf{SQuant}|$. A quantic nucleus on Q is a closure operator j: $Q \rightarrow Q$ such that $j(a) \otimes j(b) \leq j(a \otimes b)$ for all $a, b \in Q$.

A subset $S \subseteq Q$ is called a quantic quotient if $S = Q_j$ for some quantic nucleus j, where $Q_j = \{a \in Q : j(a) = a\}$.

Let X be a non-empty set and let L be a complete lattice or $L \in |\mathbf{SQuant}|$. An L-fuzzy subset (or L-set) of X is a mapping A: $X \rightarrow L$. The family of all L-fuzzy subsets on X will be denoted by L^X. The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively.

For an ordinary mapping $f : X \longrightarrow Y$, one can define the mappings

$$f_L^{\rightarrow} : L^X \rightarrow L^Y \text{and } f_L^{\leftarrow} : L^Y \rightarrow L^X$$

by

$$f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : x \in X, f(x) = y\} \text{and } f_L^{\leftarrow}(B) = B \circ f$$

respectively.

Theorem 2.6 ([19]). *Let $L \in |\mathbf{SQuant}|$, X, Y be a nonempty ordinary sets and $f : X \longrightarrow Y$ be an ordinary mapping, then we have:*

(1) *$f_L^{\rightarrow}$ preserves arbitrary $\bigvee$;*
(2) *$f_L^{\leftarrow}$ preserves arbitrary $\bigvee$, $\otimes$, and all constant maps;*
(4) *$f_L^{\leftarrow}$ preserves the unit if $L \in |\mathbf{USQuant}|$.*

For a fixed $L \in |\mathbf{SQuant}|$ and a set X, an L-quasi-topology on X [19] is a subs-quantale τ of $L^X = (L^X, \leq, \otimes)$, i.e., the following axioms are satisfied:

(T_1) For all $A, B \in L^X$, $A, B \in \tau \Rightarrow A \otimes B \in \tau$.

(T_2) For all $\{A_j : j \in J\} \subseteq L^X$, $\{A_j : j \in J\} \subseteq \tau \Rightarrow \bigvee_j A_j \in \tau$.

An L-quasi-topology τ is said to be strong [20] iff it is strong as a subs-quantale of L^X, i.e., τ satisfies the additional axiom:

(T_3) $\underline{\top} \in \tau$.

If L is a us-quantale with unit e, a subus-quantale τ of L^X is called an L-topology on X [19]; so,τ satisfies (T_1), (T_2) and the following:

(T_4) $\underline{e} \in \tau$.

If $\tau \subseteq L^X$ is an L-quasi-topology (resp. L-topology), then the pair (X, τ) is said to be an L-quasi-topological (resp. L-topological) space. A mapping f: $(X, \tau) \to (Y, \sigma)$ is said to be *L-continuous* (resp., *L-open*) [22] if $(f_L^{\leftarrow})_{|\rho} : \tau \leftarrow \sigma$ (resp., $(f_L^{\rightarrow})_{|\tau} : \tau \to \sigma$). An L-continuous bijection f: $(X, \tau) \to (Y, \sigma)$ is an *L-homeomorphism* [22] if f^{-1} is L-continuous.

In an obvious way L-quasi-topological (resp.strong L-quasi-topological and L-topological) spaces and L-continuous maps form a category denoted by ***L*-QTop** (resp. ***L*-SQTop** and ***L*-Top**).

One can easily prove that each of ***L*-QTop** ***L*-SQTop** and ***L*-Top**)) is topological category over the category **Set** of sets and set-morphisms.

3. Quantic spectrum adjunction

In this section we will introduce and study a more general adjunction between the category of semi-quantales and the category of lattice-valued quasi-topological spaces. Also we will generalize the concept of L-sober topological spaces of ([26]-[28]) for $L \in |\mathbf{SFrm}|$ to the more general case for $L \in |\mathbf{SQuant}|$.

For $L \in |\mathbf{SQuant}|$ and $(X, \tau) \in |\,L\text{-}\mathbf{QTop}\,|$. The functor

$$\Omega_L : L - \mathbf{QTop} \to \mathbf{SQuant}^{op}$$

is defined as follows.

$\Omega_L(X, \tau)$ is the L-quasi-topology of a space (X, τ), i.e., the semi-quantale $\tau \subseteq L^X$, and $\Omega_L(f$: $(X, \tau) \to (Y, \sigma))$, for an L-continuous map f, is $[f_L^{\leftarrow}\,|_\sigma]^{op} : \tau \to \sigma$.

The standard spectrum construction for a semi-quantale Q may be summarized as follows:

$$Lpt(Q) = \{p : Q \to L : p \in |\mathbf{SQuant}|\}$$

$$\Phi_L : Q \to L^{Lpt(Q)} \text{ by } \Phi_L(q)(p) = p(q)$$

Then it can be shown that Φ_L preserves $\otimes$ and arbitrary $\bigvee$, where these are inherited by the codomain of Φ_L from L. It can now be shown that $\Phi_L^{\rightarrow}(Q)$ is closed under these operations and hence is an L-quasi-topology on $Lpt(Q)$. Thus we have

$$Q \to (Lpt(Q), \Phi_L^{\rightarrow}(Q))$$

where the latter is an L-quasi-topological space; so we put

$$LPT(Q) \equiv (Lpt(Q), \Phi_L^{\rightarrow}(Q)) \in |L - \mathbf{QTop}|$$

and given f: $Q_1 \to Q_2$ in **SQuant**, i.e. f^{op}: $Q_1 \leftarrow Q_2$ in $\mathbf{SQuant}^{op}$. We define

$$Lpt(f) : Lpt(Q_1) \to Lpt(Q_2)$$

by

$$Lpt(f)(p) = p \circ f^{op}.$$

Lemma 3.1. *For a fixed $L \in |\mathbf{SQuant}|$ and $Q_1, Q_2 \in |\mathbf{SQuant}|$, the mapping*

$$LPT(f) : (Lpt(Q_1), \Phi_L^{\rightarrow}(Q_1)) \to (Lpt(Q_2), \Phi_L^{\rightarrow}(Q_2))$$

is L-continuous.

Proof. For all $q_2 \in Q_2$, $p \in Lpt(Q_1)$, we have

$$\begin{aligned} Lpt(f)^{\leftarrow}(\Phi_L(q_2)(p)) &= \Phi_L(q_2)(Lpt(f)(p)) \\ &= \Phi_L(q_2)(p \circ f^{op}) \\ &= \Phi_L(f^{op}(q_2))(p). \end{aligned}$$

hence $Lpt(f)^{\leftarrow}(\Phi_L(q_2)(p)) = \Phi_L(f^{op}(q_2))(p)$. Now the function $LPT(f)$ is L-continuous iff $\forall \mu \in \Phi_L^{\rightarrow}(Q_2)$, $\exists \nu \in \Phi_L^{\rightarrow}(Q_1)$ such that $Lpt(f)^{\leftarrow}(\nu) = \mu$. □

Then we have the spectrum or point functor

$$LPT : \mathbf{SQuant}^{op} \to L - \mathbf{QTop}.$$

Now, we turn to study the adjunction between the functors

$$LPT : \mathbf{SQuant}^{op} \to L - \mathbf{QTop}.$$

and

$$\Omega_L : L - \mathbf{QTop} \to \mathbf{SQuant}^{op}$$

To this aim we give the following definitions

For $(X, \tau) \in |L - \mathbf{QTop}|$ and $L, Q \in |\mathbf{SQuant}|$ define the maps:

- $\eta_X : (X, \tau) \longrightarrow (Lpt(\tau), \Phi_L^{\rightarrow}(\tau))$, by setting, $\forall x \in X$ and $\mu \in \tau$, $\eta_X(x)(\mu) = \mu(x)$;
- $\varepsilon_Q^{op} : Q \longrightarrow \Omega_L(LPT(Q))$ by setting $\varepsilon_Q^{op} = \Phi_L\,|_{\Phi_L^{\rightarrow}(Q)}$.

It is clear that by definition ε_Q^{op} always surjective.

As given in [27,28], we have the following easily established results:

Lemma 3.2. *For $(X, \tau) \in |L - \mathbf{QTop}|$ and $L, Q \in |\mathbf{SQuant}|$,*

1. *The map $\eta_X : (X, \tau) \longrightarrow (Lpt(\tau), \Phi_L^{\rightarrow}(\tau))$ is L-continuous, and L-open w.r.t. its range in $(Lpt(\tau), \Phi_L^{\rightarrow}(\tau))$ and*
2. *The map $\varepsilon_Q^{op} : Q \longrightarrow \Omega_L(LPT(Q))$ is an s-quantale morphism.*

From the definition of ε_Q^{op} one can easily have the following result:

Lemma 3.3. *For every $Q \in |\mathbf{SQuant}|$, ε_Q^{op} is injective if and only if for any $a, b \in Q$ with $a \neq b$ there exists $p \in Lpt(Q)$ with $p(a) \neq p(b)$.*

As a consequence of the above lemma, we have the following result

Corollary 3.4. *Given* $(X,\tau) \in |L-\mathbf{QTop}|$, *The map* $\varepsilon^{op}_{\Omega_L(x,\tau)}$ *is injective.*

Lemma 3.5. *For* $(X,\tau) \in |L-\mathbf{QTop}|$, *we have* $(\eta_X)^{\leftarrow}_L \circ \Phi_L^{\Omega(X,\tau)} = 1_{\Omega(X,\tau)}$, *where* $(\eta_X)^{\leftarrow}_L : L^{Lpt(\Omega(X,\tau))} \longrightarrow \Omega(X,\tau)$ *and* $\Phi_L^{\Omega(X,\tau)} : \Omega(X,\tau) \to L^{Lpt(\Omega(X,\tau))}$.

Proof. The proof is straightforward. □

As a consequence of the above, we have that:

$$LPT : \mathbf{SQuant}^{op} \to L-\mathbf{QTop}$$

is a right adjoint to

$$\Omega_L : L-\mathbf{QTop} \to \mathbf{SQuant}^{op}.$$

This adjunction given in the form $L-\mathbf{QTop} \dashv \mathbf{SQuant}^{op}$.

For the case of the category **SSQuant** (resp., **USQuant**) of strong (resp., unital) semi-quantales and the category $L-$**SQTop** (resp., $L-$**Top**) of strong L-quasi-topological spaces (resp., L-topological spaces) one can similarly have the following dual adjunctions:

$$L-\mathbf{SQTop} \dashv \mathbf{SSQuant}^{op}.$$

and

$$L-\mathbf{Top} \dashv \mathbf{USQuant}^{op}.$$

Definition 3.6. For $L, Q \in |\mathbf{SQuant}|$. A semi-quantale Q is said to be L-Qspatial iff the map ε^{op}_Q is injective.

Lemma 3.7. *For fixed* $L \in |\mathbf{SQuant}|$. *An* $Q \in |\mathbf{SQuant}|$ *is* L*-Qspatial if and only if* ε^{op}_Q *is isomorphism*

Proof. The proof is straightforward. □

Corollary 3.8. *For* $(X,\tau) \in |L-\mathbf{QTop}|$, *the* L*-quasi-topology* $\Omega_L(X,\tau)$ *is* L*-Qspatial.*

Proof. Let $\mu, \nu \in \Omega(X,\tau)$ with $\mu \neq \nu$, then there exists an $x_0 \in X$ such that $\mu(x_0) \neq \nu(x_0)$. Putting $p = \eta_X(x_0) \in Lpt(\tau)$. Then $\varepsilon^{op}_{\Omega_L(X,\tau)}(\mu)(p) = p(\mu) = \eta_X(x_0)(\mu) = \mu(x_0) \neq \nu(x_0) = \eta_X(x_0)(\nu) = p(\nu) = \varepsilon^{op}_{\Omega_L(X,\tau)}(\nu)(p)$

Thus $\varepsilon^{op}_{\Omega_L(X,\tau)}(\mu) \neq \varepsilon^{op}_{\Omega_L(X,\tau)}(\nu)$ on $Lpt(\tau)$, which means that

$$\varepsilon^{op}_{\Omega_L(X,\tau)} : \Omega_L(X,\tau) \to \Omega_L(LPT(\Omega_L(X,\tau))$$

is injective on $\Omega_L(X,\tau)$. So $\Omega_L(X,\tau)$ is L-Qspatial. □

Definition 3.9. An $(X,\tau) \in |L-\mathbf{QTop}|$ is called

(1) $L-QT_0$ if for every $x, y \in X$ with $x \neq y$ there exists $\mu \in \Omega_L(X,\tau)$ with $\mu(x) \neq \mu(y)$.
(2) L-Qsober iff $\eta_X : (X,\tau) \longrightarrow (Lpt(\tau), \Phi^{\rightarrow}_L(\tau))$ is bijective.

The next two lemmas show a characterization of $L-QT_0$ as well as L-Qsober spaces.

Lemma 3.10. *An* $(X,\tau) \in |L-\mathbf{QTop}|$ *is* $L-QT_0$ *iff* η_X *is injective.*

Proof. Recall the definition η_X. □

Also, from the definition of η_X, we have the following result:

Lemma 3.11. An $(X,\tau) \in |L-\mathbf{QTop}|$ L-Qsober iff

$$\eta_X : (X,\tau) \longrightarrow (Lpt(\tau), \Phi^{\rightarrow}_L(\tau))$$

is L-homeomorphism.

Proof. Let $(X,\tau) \in |L-\mathbf{QTop}|$ be an L-Qsober, then $\eta_X : (X,\tau) \longrightarrow (Lpt(\tau), \Phi^{\rightarrow}_L(\tau))$ is bijective. Since $\eta_X : (X,\tau) \longrightarrow (Lpt(\tau), \Phi^{\rightarrow}_L(\tau))$ is continuous, then it remain to prove the continuity of $\eta_X^{-1} : (Lpt(\tau), \Phi^{\rightarrow}_L(\tau)) \longrightarrow (X,\tau)$. To this end, let $\mu \in \Omega(X,\tau)$, then by Lemma 3.5 we get $(\eta_X^{-1})^{\leftarrow}_L(\mu) = \mu \circ \eta_X^{-1} = (\eta_X)^{\leftarrow}_L \circ \Phi_L^{\Omega(X,\tau)}(\mu) \circ \eta_X^{-1} = \Phi_L^{\Omega(X,\tau)}(\mu) \circ \eta_X \circ \eta_X^{-1} = \Phi_L^{\Omega(X,\tau)}(\mu)$.

The converse is clear. □

Lemma 3.12. *For all* $Q \in |\mathbf{SQuant}|$, $LPT(Q)$ *is* L*-Qsober.*

Proof. Show bijectivity of the map

$$\eta_{Lpt(Q)} : (Lpt(Q), \Phi^{\rightarrow}_L(Q)) \longrightarrow LPT(\Phi^{\rightarrow}_L(Q)).$$

For injectivity, let $p_1, p_2 \in Lpt(Q)$ with $p_1 \neq p_2$. Then there is $a \in Q$ with $p_1(a) \neq p_2(a)$ i.e., there is $\Phi_L(a) \in \Phi^{\rightarrow}_L(Q)$ such that

$$\begin{aligned}\eta_{Lpt(Q)}(p_1)(\Phi_L(a)) &= \Phi_L(a)(p_1) = p_1(a) \neq p_2(a)\\ &= \Phi_L(a)(p_2) = \eta_{Lpt(Q)}(p_2)(\Phi_L(a))\end{aligned}$$

which shows that $\eta_{Lpt(Q)}(p_1) \neq \eta_{Lpt(Q)}(p_2)$. Thus $\eta_{Lpt(Q)}$ is injective.

To show the surjectivity of $\eta_{Lpt(Q)}$, let $q \in LPT(\Phi^{\rightarrow}_L(Q)) = (Lpt(\Phi^{\rightarrow}_L(Q), \Phi^{\rightarrow}_L(\Phi^{\rightarrow}_L(Q)))$ and put $p = q \circ \Phi_L$. Clearly $p \in Lpt(Q)$. Furthermore, for all $a \in Q$, we have

$$\eta_{Lpt(Q)}(p)(a) = \Phi_L(a)(p) = p(a) = q \circ \Phi_L(a) = q(\Phi_L(a)).$$

So $\eta_{Lpt(Q)}(p) = q$, which means that $\eta_{Lpt(Q)}$ is surjective. □

Let L-**Qsob** (resp. L-**Qspat**) be the full subcategory of L-**QTop** (resp. **SQuant**) consisting of all L-Qsober spaces (resp. L-Qspatial semi-quantales).

By analogy with [22,23,27,28]), we prove the following theorem

Theorem 3.13. *The categories* L-***Qsob*** *and* L-***Qspat*** *are equivalent.*

Proof. By Corollary 3.8 and Corollary 3.12 the adjunction

$$\Omega_L \dashv LPT : \mathbf{SQuant}^{op} \to L-\mathbf{QTop}$$

restricts to the categories L-**Qsob** and L-**Qspat**. By Lemma 3.7 and Lemma 3.11 the restrictions of the unit η and counit ε to the aforesaid categories give natural isomorphisms. □

4. Quantic separation axioms

In this section we will discuss the counterparts of the separation axioms quantic regularity and normality of objects in the category **SQuant** with applications to objects in the category $L-\mathbf{QTop}$.

Definition 4.1. Let $Q \in |\mathbf{SQuant}|$, $M \subseteq Q$, and $a, b \in M$. An element a is said to be well-inside of b (w.r.t M), denoted $a \preceq b$, if

$$\exists c \in M \text{ with } a \otimes c = \bot \text{ and } c \vee b = \top.$$

Equivalently $a \preceq b \equiv a^* \vee b = \top$ where $a^* = \vee\{c \in Q : a \otimes c = \bot\}$.

Some time we say that $a \preceq b$ via c.

Lemma 4.2. (*see [23]*) *For $Q \in |\mathbf{SQuant}|$ and $a, b, c, d \in Q$, the following holds*

(1) *$a \preceq b$ implies $a \leq b$, and*
(2) *$a \leq b \preceq c \leq d$ implies $a \preceq d$.*

Definition 4.3. An $Q \in |\mathbf{SQuant}|$ is said to be T_2 if for any $a \in Pr(Q)$, we have $a = \bigvee\{x \in Q : x \preceq a\}$.

Definition 4.4. Let $(X, \tau) \in |L-\mathbf{QTop}|$ and $L, Q \in |\mathbf{SQuant}|$.

(1) Q is said to be regular, iff

$$\forall a \in Q, \exists D \subseteq \{b \in Q : b \preceq a\}, a = \bigvee D$$

If $Q \in |\mathbf{Quant}|$, then $Q \in |\mathbf{Frm}|$ ([29], Theorem 2.5).

(2) (X, τ) is quantic regular, or regular, iff τ is a regular subsemi-quantale of L^X.

Proposition 4.5. *An $Q \in |\mathbf{DSQuant}|$ is regular if and only if*

$$\forall a \in Q, a = \bigvee\{b \in Q : b \preceq a\}$$

Proof. Let $Q \in |\mathbf{DSQuant}|$. Distributivity and $b \preceq a$ imply $a \leq b$. Let $D \subseteq \{b \in Q : b \preceq a\}$, such that $a = \bigvee D$. Then,

$$\bigvee D \leq \bigvee\{b \in Q : b \preceq a\} \leq \bigvee\{b \in Q : b \leq a\} = a = \bigvee D$$

This shows

$$a = \bigvee D = \bigvee\{b \in Q :, b \preceq a\}$$

and from this follows the claims. □

By the definition of T_2, one can easily have the following result:

Corollary 4.6. *Every quantic regular semi-quantale is T_2.*

As a consequence of the above proposition, we have the following result:

Proposition 4.7. *Let $L \in |\mathbf{DSQuant}|$. An $(X, \tau) \in |L-\mathbf{QTop}|$ is regular if and only if*

$$\forall \mu \in \tau, \mu = \bigvee\{\nu \in \tau : \nu \preceq \mu\}$$

Proposition 4.8. *A quantic quotient $S \subseteq Q$ of a regular semi-quantale Q is regular.*

Proof. Let $j : Q \longrightarrow Q$ be a quantic nucleus on Q and let $b \in S$ be an arbitrary element. For $a, b \in Q$ with $a \preceq b$, there is $c \in Q$ with $a \otimes c = \bot$ and $b \vee c = \top$. With the quantic nucleus $j : Q \longrightarrow Q$, we have $j(a) \otimes j(c) \leq j(a \otimes c) = j(\bot)$ and $b \vee j(c) = j(b) \vee j(c) = j(b \vee c) = j(\top)$ which implies that $j(a) \preceq b$ in Q_j.

Since Q is regular, then for all $b \in Q$, we have

$$\begin{aligned} b &= \bigvee\{a \in Q : a \preceq b \text{ w.r.t } Q\} \\ &\leq \bigvee\{j(a) \in Q_j : a \preceq b \text{ w.r.t } Q\} \\ &\leq \bigvee\{a' \in Q_j : a' \preceq b' \text{ w.r.t } Q_j\} \end{aligned}$$

So quantic quotient $S \subseteq Q$ is regular. □

Definition 4.9. Let $Q \in |\mathbf{SQuant}|$, $S \subseteq Q$ and D be the dyadic rationales in [0, 1]. For $a, b \in Q$, a is said to be **really-inside** b (with respect to S), denoted $a \overset{=}{<} b$, iff $\exists \{a_q : q \in D\} \subseteq S$ such that

(1) $a \leq a_0 \preceq a_1 \leq b$ and
(2) $p < q \Rightarrow a_p \preceq a_q$.

This definition comes from [23].

Definition 4.10. Let $(X, \tau) \in |L-\mathbf{QTop}|$ and $L, Q \in |\mathbf{SQuant}|$.

(1) Q is said to be quantic completely regular, iff

$$\forall a \in Q, \exists D \subseteq \{b \in Q : b \overset{=}{<} a\}, a = \bigvee D.$$

(2) (X, τ) is quantic completely regular, iff τ is a quantic completely regular subsemi-quantale of L^X.

Proposition 4.11. *An $Q \in |\mathbf{DSQuant}|$ is quantic completely regular if and only if*

$$\forall a \in Q, a = \bigvee\{b \in Q : b \overset{=}{<} a\}$$

Proof. The proof is analogous to those of Proposition 4.5. □

As a consequence of the above proposition, we have the following result:

Proposition 4.12. *Let $L \in |\mathbf{DSQuant}|$. An $(X, \tau) \in |L-\mathbf{QTop}|$ is quantic completely regular if and only if*

$$\forall \mu \in \tau, \mu = \bigvee\{\nu \in \tau : \nu \overset{=}{<} \mu\}$$

Definition 4.13. Let $(X, \tau) \in |L-\mathbf{QTop}|$ and $L, Q \in |\mathbf{SQuant}|$.

(1) Q is said to be quantic normal, iff

$$\forall a, b \in Q \text{ with } a \vee b = \top, \exists c, d \in Q \text{ with } c \otimes d = \bot, c \vee b = \top = a \vee d.$$

This comes from [29]. (Equivalently, if $a \vee b = \top$, $\exists c \in Q$ with $c^* \vee b = \top = a \vee c$.)

(2) (X, τ) is quantic normal, iff τ is a quantic normal subsemi-quantale of L^X.

Proposition 4.14. *Let* $(X, \tau) \in |L - \mathbf{QTop}|$ *and* $L, Q \in |\mathbf{CSQuant}|$.

(1) *A commutative semi-quantale* Q *is quantic normal, iff*

$$\forall a, b \in Q \text{ with } a \vee b = \top, \exists c, d \in Q \text{ with } c \preceq a \text{ via } d,\ d \preceq b \text{ via } c.$$

If $Q \in |\mathbf{SFrm}|$, Q *is localic normal iff* Q *is a normal semilocale in the sense of [27, 28].*

(2) (X, τ) *is quantic normal, iff*

$$\forall \mu, \nu \in \tau \text{ with } \mu \vee \nu = \underline{\top}, \exists \lambda, \upsilon \in \tau \text{ with } \lambda \preceq \mu \text{ via } \upsilon,\ \upsilon \preceq \nu \text{ via } \lambda.$$

Proof. (1) Let $a, b \in Q$ with $a \vee b = \top$. Quantic normality of the commutative semi-quantale $Q \Leftrightarrow \exists c, d \in Q$ with $c \otimes d = \bot = d \otimes c$, $c \vee b = \top$ and $a \vee d = \top$. Then

(i) $c \otimes d = \bot$ and $a \vee d = \top \Rightarrow c \preceq a$ via d.
(ii) $d \otimes c = \bot$ and $b \vee c = \top \Rightarrow d \preceq b$ via c.

(2) Follows from (1). □

Proposition 4.15. *In any normal semi-quantale the relation* $\preceq$ *implies* $\overset{=}{<}$ *and quantic regularity implies quantic complete regularity.*

Proof. Let Q be a quantic normal semi-quantale and $a \preceq b$. Since $a \preceq b \Leftrightarrow a^* \vee b = \top$, then there are $c, d \in Q$ such that $c \otimes d = \bot \Leftrightarrow c \leq d^*$ and $a^* \vee d = \top = b \vee c$. Then one have $a \preceq d$ and $d^* \vee b \geq c \vee b = \top$ so that $d \preceq b$ which implies $a \overset{=}{<} b$. □

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