Journal of the Egyptian Mathematical Society (2016) **24**, 538–544

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

# Closed form expression of tractable semi-max-type two-dimensional system of difference equations with variable coefficients

**Tarek F. Ibrahim***

*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt*

Received 7 January 2016; revised 25 February 2016; accepted 1 March 2016
Available online 5 April 2016

**Keywords**

Eventually periodic solutions;
Difference equations;
Semi-max-type system

**Abstract** In this paper we shall examine the periodicity and formularization of the solutions for a system of semi-max-type difference equations of second order in the form

$$x_{n+1} = \max\left\{\frac{A_n}{y_{n-1}}, x_{n-1}\right\},$$
$$y_{n+1} = \min\left\{\frac{B_n}{x_{n-1}}, y_{n-1}\right\},$$

$n \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are two-periodic positive sequences, and initial values $x_0, x_{-1}, y_0, y_{-1} \in (0, +\infty)$.

**2010 Mathematics Subject Classification:** Primary 39A10; 40A05



## 1. Introduction

In recent years, the studying of nonlinear difference equations has been a considerable solicitude where there exist abundant models portray sociology, economics, real life situations in population biology, genetics, probability theory, psychology etc. whose exemplified by these kinds of equations (see, e.g. [1–8]). Also some papers are devoted to the implementing of max-type difference equations, see, for proverb [9] and references cited therein. In particular many experts have been focused on the investigation of the behavior of the following difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, x_{n-s}\right\}, \quad n \in \mathbb{N}_0, \tag{1}$$

* Tel.: +966535561756.
E-mail addresses: tfibrahem@mans.edu.eg, tfoze@kku.edu.sa
Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier


http://dx.doi.org/10.1016/j.joems.2016.03.002

where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$ is a sequence of real numbers (see, for example, [10,11]). Positive solutions of Eq. (1) are usually related with the periodicity. If solutions are not of constant sign then it is known that Eq. (1) can have non-periodic solutions which could be even unbounded [12]. For more papers about max-type difference equations we refer the reader to the references [13–16]. Motivated by above mentioned papers, here we will study solutions of the system of difference equations

$$x_{n+1} = \max\left\{\frac{A_n}{y_{n-1}}, x_{n-1}\right\},$$

$$y_{n+1} = \min\left\{\frac{B_n}{x_{n-1}}, y_{n-1}\right\} \tag{2}$$

where $n \in \mathbb{N}_0$, $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are positive periodic sequences with period two, and initial values $x_0$, $x_{-1}$, $y_0$, $y_{-1} \in (0, +\infty)$.

## 2. Closed form expression of system (2)

In the section, we study the behavior of solutions of system (2). In order to achieve this target , we sub-edit and establish four theorems depending on the relationships between the quantities $\frac{A_0}{y_{-1}}$ and $x_{-1}$, and, $\frac{B_0}{x_{-1}}$ and $y_{-1}$.

**Remark 1.** From (2), we can note that every even (respectively odd) term of the sequences $(x_n)$, $(y_n)$ depend only on $A_0$, $B_0$ (respectively $A_1$, $B_1$) and the previous even (respectively odd) terms of both $(x_n)$, $(y_n)$.

**Definition 1.** A solution $(x_n, y_n)_{n=-1}^{\infty}$ of system (2), is said to be *eventually periodic with period* $p \in \mathbb{N}$ if there is an $n_0 \geq -1$, such that $x_{n+p} = x_n$, $y_{n+p} = y_n$ for $n \geq n_0$. If $n_0 = -1$, then we say that the sequence $(x_n, y_n)_{n=-1}^{\infty}$ is *periodical and with period p*. Period $p$ is said to be a *minimal* one if there is no $p_1 < p$ which is a period to the sequence $(x_n, y_n)_{n=-1}^{\infty}$.

**Theorem 2.** *Suppose that $(x_n, y_n)$ is a solution of system (2) such that $\frac{A_0}{y_{-1}} \leq x_{-1}$, $\frac{B_0}{x_{-1}} \leq y_{-1}$. Then the following statements hold:*

1. *If $y_0 \leq \frac{B_1}{x_0}$, $x_0 \geq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then*
   $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$;$x_{4n+2} = x_{4n+4} = x_0$; $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$;
   $y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$; $y_{4n+2} = y_{4n+4} = y_0$; $y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$
   (b) *If $1 \geq \frac{A_0}{B_0}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$;$x_{4n+2} = x_{4n+4} = x_0$;
   $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$; $y_{4n+2} = y_{4n+4} = y_0$
2. *If $y_0 \geq \frac{B_1}{x_0}$, $x_0 \leq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$; $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$; $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$; $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$; $y_{4n+1} = y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$;
   $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$; $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$ ;
   (b) *If $1 \geq \frac{A_0}{B_0}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$,
   $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$,
   $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
3. *If $y_0 \leq \frac{B_1}{x_0}$, $x_0 \leq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then*
      i. *If $1 \leq \frac{B_1}{A_1}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2} = x_{4n+4} = \frac{A_1}{y_0}$,
      $x_{4n+3}=\frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$, $y_{4n+1}=y_{4n+3}=\frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2}=y_{4n+4}=y_0$,
      ii. *If $1 \geq \frac{B_1}{A_1}$, then*
      $x_{4n+1}=\frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2}=x_{4n+4} = \frac{A_1^{n+1}}{B_1^n y_0}$, $x_{4n+3}=\frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$,
      $y_{4n+1}=y_{4n+3}=\frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = \frac{B_1^n}{A_1^n}y_0$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
   (b) *If $1 \geq \frac{A_0}{B_0}$, then*
      i. *If $1 \leq \frac{B_1}{A_1}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4}$
      $= \frac{A_1}{y_0}$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = y_{4n+4} = y_0$,
      ii. *If $1 \geq \frac{B_1}{A_1}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4}$
      $= \frac{A_1^{n+1}}{B_1^n y_0}$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = \frac{B_1^n}{A_1^n}y_0$, $y_{4n+4}$
      $= \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
4. *If $y_0 \geq \frac{B_1}{x_0}$, $x_0 \geq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then*
      i. *If $1 \leq \frac{A_1}{B_1}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2} = \frac{A_1^n}{B_1^n}x_0$, $x_{4n+3}$
      $= \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$, $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$,
      $y_{4n+2} = y_{4n+4} = \frac{B_1^{n+1}}{A_1^n x_0}$,
      ii. *If $1 \geq \frac{A_1}{B_1}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2} = x_{4n+4} = x_0$,
      $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$,$y_{4n+1} = y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = y_{4n+4}$
      $= \frac{B_1}{x_0}$
   (b) *If $1 \geq \frac{A_0}{B_0}$, then*
      i. *If $1 \leq \frac{A_1}{B_1}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = \frac{A_1^n}{B_1^n}x_0$,
      $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = y_{4n+4} =$
      $\frac{B_1^{n+1}}{A_1^n x_0}$
      ii. *If $1 \geq \frac{A_1}{B_1}$, then*
      $x_{4n+1} = x_{4n+3} = x_{-1}$,$x_{4n+2} = x_{4n+4} = x_0$,
      $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$,$y_{4n+2} = y_{4n+4} = \frac{B_1}{x_0}$

**Proof.** By mathematical induction. For $n = 0$, the result holds. Now suppose that $k > 0$ and that all the relations in the theorem hold for $n = k$. Now we shall prove that the relations hold for $n = k + 1$ .

1. If $y_0 \leq \frac{B_1}{x_0}$, $x_0 \geq \frac{A_1}{y_0}$, then
   (a) If $1 \leq \frac{A_0}{B_0}$, then
   $x_{4(k+1)+1} = \max\{\frac{A_{4(k+1)}}{y_{4(k+1)-1}}, x_{4(k+1)-1}\} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\}$
   $= \max\{\frac{A_0}{\frac{B_0^{k+1}}{A_0^k x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{(k+1)}}{B_0^{(k+1)}}x_{-1}$
   $x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1}{y_0}, x_0\} = x_0$
   $y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\} =$
   $\min\{\frac{B_0}{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}}, \frac{B_0^{k+1}}{A_0^k x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$
   $y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \min\{\frac{B_1}{x_0}, y_0\} = y_0$

$$x_{4(k+1)+3}$$
$$= \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\} = \max\{\frac{A_0}{\frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{k+2}}{B_0^{k+2}}x_{-1}$$
$$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \max\{\frac{A_1}{y_0}, x_0\} = x_0$$
$$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \min\{\frac{B_0B_0^{k+1}}{A_0^{k+1}x_{-1}}, \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}\}$$
$$= \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$$
$$y_{4(k+1)+4} = \min\{\frac{B_1}{x_{4k+6}}, y_{4k+6}\} = \min\{\frac{B_1}{x_0}, y_0\} = y_0$$

(b) If $1 \ge \frac{A_0}{B_0}$, then

$$x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} = \max\{\frac{A_0x_{-1}}{B_0}, x_{-1}\} = x_{-1}$$
$$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1}{y_0}, x_0\} = x_0$$
$$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\}$$
$$= \frac{B_0}{x_{-1}}$$
$$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \min\{\frac{B_1}{x_0}, y_0\} = y_0$$
$$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\} = \max\{\frac{A_0x_{-1}}{B_0}, x_{-1}\} = x_{-1}$$
$$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \max\{\frac{A_1}{y_0}, x_0\} = x_0$$
$$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\} = \frac{B_0}{x_{-1}}$$
$$y_{4(k+1)+4} = \min\{\frac{B_1}{x_{4k+6}}, y_{4k+6}\} = \min\{\frac{B_1}{x_0}, y_0\} = y_0$$

2. If $y_0 \ge \frac{B_1}{x_0}$, $x_0 \le \frac{A_1}{y_0}$, then

(a) If $1 \le \frac{A_0}{B_0}$, then

$$x_{4(k+1)+1} = \max\{\frac{A_{4(k+1)}}{y_{4(k+1)-1}}, x_{4(k+1)-1}\}$$
$$= \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} = \max\{\frac{A_0}{\frac{B_0^{k+1}}{A_0^{k}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{(k+1)}}{B_0^{(k+1)}}x_{-1}$$
$$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1^{k+2}}{B_1^{k+1}y_0}, \frac{A_1^{k+1}x_0}{B_1^{k+1}}\}$$
$$= \frac{A_1^{k+2}}{B_1^{k+1}y_0}$$
$$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\}$$
$$= \min\{\frac{B_0}{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}}, \frac{B_0^{k+1}}{A_0^{k}x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$$
$$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \min\{\frac{B_1}{\frac{A_1^{k+1}}{B_1^{k+1}}x_0}, \frac{B_1^{k+1}y_0}{A_1^{k+1}}\}$$
$$= \frac{B_1^{k+2}}{A_1^{k+1}x_0}$$
$$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\} = \max\{\frac{A_0}{\frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\}$$
$$= \frac{A_0^{k+2}}{B_0^{k+2}}x_{-1}$$
$$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \max\{\frac{A_1}{\frac{B_1^{k+2}}{A_1^{k+1}x_0}}, \frac{A_1^{k+2}}{B_1^{k+1}y_0}\}$$
$$= \frac{A_1^{k+2}}{B_1^{k+2}}x_0$$
$$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \min\{\frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}, \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}\}$$
$$= \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$$
$$y_{4(k+1)+4} = \min\{\frac{B_1}{\frac{A_1^{k+2}}{B_1^{k+1}y_0}}, \frac{B_1^{k+2}}{A_1^{k+1}x_0}\} = \frac{B_1^{k+2}}{A_1^{k+2}}y_0$$

(b) If $1 \ge \frac{A_0}{B_0}$, then

$$x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} = \max\{\frac{A_0}{\frac{B_0}{x_{-1}}}, x_{-1}\} = x_{-1}$$
$$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1^{k+2}}{B_1^{k+1}y_0}, \frac{A_1^{k+1}x_0}{B_1^{k+1}}\}$$
$$= \frac{A_1^{k+2}}{B_1^{k+1}y_0}$$
$$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\}$$
$$= \frac{B_0}{x_{-1}}$$
$$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \min\{\frac{B_1}{\frac{A_1^{k+1}}{B_1^{k+1}}x_0}, \frac{B_1^{k+1}y_0}{A_1^{k+1}}\}$$
$$= \frac{B_1^{k+2}}{A_1^{k+1}x_0}$$
$$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\} = \max\{\frac{A_0x_{-1}}{B_0}, x_{-1}\} = x_{-1}$$
$$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \max\{\frac{A_1}{\frac{B_1^{k+2}}{A_1^{k+1}x_0}}, \frac{A_1^{k+2}}{B_1^{k+1}y_0}\}$$
$$= \frac{A_1^{k+2}}{B_1^{k+2}}x_0$$
$$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\} = \frac{B_0}{x_{-1}}$$
$$y_{4(k+1)+4} = \min\{\frac{B_1}{\frac{A_1^{k+2}}{B_1^{k+1}y_0}}, \frac{B_1^{k+2}}{A_1^{k+1}x_0}\} = \frac{B_1^{k+2}}{A_1^{k+2}}y_0$$

3. If $y_0 \le \frac{B_1}{x_0}$, $x_0 \le \frac{A_1}{y_0}$, then

(a) If $1 \le \frac{A_0}{B_0}$, then

i. If $1 \le \frac{B_1}{A_1}$, then

$$x_{4(k+1)+1} = \max\{\frac{A_{4(k+1)}}{y_{4(k+1)-1}}, x_{4(k+1)-1}\}$$
$$= \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\}$$
$$= \max\{\frac{A_0}{\frac{B_0^{k+1}}{A_0^{k}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{(k+1)}}{B_0^{(k+1)}}x_{-1}$$
$$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1}{y_0}, \frac{A_1}{y_0}\} = \frac{A_1}{y_0}$$
$$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\}$$
$$= \min\{\frac{B_0}{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}}, \frac{B_0^{k+1}}{A_0^{k}x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$$
$$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \min\{\frac{B_1}{A_1}y_0, y_0\} = y_0$$
$$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\}$$
$$= \max\{\frac{A_0}{\frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{k+2}}{B_0^{k+2}}x_{-1}$$
$$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \max\{\frac{A_1}{y_0}, \frac{A_1}{y_0}\} = \frac{A_1}{y_0}$$
$$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\}$$
$$= \min\{\frac{B_0B_0^{k+1}}{A_0^{k+1}x_{-1}}, \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$$
$$y_{4(k+1)+4} = \min\{\frac{B_1}{x_{4k+6}}, y_{4k+6}\} = \min\{\frac{B_1}{A_1}y_0, y_0\} = y_0$$

ii. If $1 \ge \frac{B_1}{A_1}$, then

$$x_{4(k+1)+1}$$
$$= \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} = \max\{\frac{A_0}{\frac{B_0^{k+1}}{A_0^{k}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\}$$
$$= \frac{A_0^{(k+1)}}{B_0^{(k+1)}}x_{-1}$$
$$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \frac{A_1^{(k+2)}}{B_1^{(k+1)}y_0}$$
$$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\}$$
$$= \min\{\frac{B_0}{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}}, \frac{B_0^{k+1}}{A_0^{k}x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$$
$$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \frac{B_1^{k+1}}{A_1^{k+1}}y_0$$

$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\}$
$= \max\{\frac{A_0}{\frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{k+2}}{B_0^{k+2}}x_{-1}$
$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \frac{A_1^{k+2}}{B_1^{k+1}y_0}$
$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$
$y_{4(k+1)+4} = \min\{\frac{B_1}{x_{4k+6}}, y_{4k+6}\} = \frac{B_1^{k+2}}{A_1^{k+2}}y_0$

(b) If $1 \ge \frac{A_0}{B_0}$, then

i. If $1 \le \frac{B_1}{A_1}$, then
$x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} = \max\{\frac{A_0}{\frac{B_0}{x_{-1}}}, x_{-1}\} = x_{-1}$
$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1}{y_0}, \frac{A_1}{y_0}\} = \frac{A_1}{y_0}$
$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\}$
$= \frac{B_0}{x_{-1}}$
$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \min\{\frac{B_1 y_0}{A_1}, y_0\} = y_0$
$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\} = \max\{\frac{A_0 x_{-1}}{B_0}, x_{-1}\}$
$= x_{-1}$
$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \max\{\frac{A_1}{y_0}, \frac{A_1}{y_0}\} = \frac{A_1}{y_0}$
$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\} = \frac{B_0}{x_{-1}}$
$y_{4(k+1)+4} = \min\{\frac{B_1}{A_1}y_0, y_0\} = y_0$

ii. If $1 \ge \frac{B_1}{A_1}$, then
$x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} = \max\{\frac{A_0 x_{-1}}{B_0}, x_{-1}\}$
$= x_{-1}$
$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \frac{A_1^{(k+2)}}{B_1^{(k+1)}y_0}$
$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\}$
$= \frac{B_0}{x_{-1}}$
$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \frac{B_1^{k+1}}{A_1^{k+1}}y_0$
$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\} = \max\{\frac{A_0 x_{-1}}{B_0}, x_{-1}\}$
$= x_{-1}$
$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \frac{A_1^{k+2}}{B_1^{k+1}y_0}$
$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\} = \frac{B_0}{x_{-1}}$
$y_{4(k+1)+4} = \min\{\frac{B_1}{x_{4k+6}}, y_{4k+6}\} = \frac{B_1^{k+2}}{A_1^{k+2}}y_0$

4. If $y_0 \ge \frac{B_1}{x_0}$, $x_0 \ge \frac{A_1}{y_0}$, then

(a) If $1 \le \frac{A_0}{B_0}$, then

i. If $1 \le \frac{A_1}{B_1}$, then
$x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\}$
$= \max\{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}, \frac{A_0}{\frac{B_0^{k+1}}{A_0^{k}x_{-1}}}\} = \frac{A_0^{(k+1)}}{B_0^{(k+1)}}x_{-1}$,
$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \frac{A_1^{(k+1)}}{B_1^{(k+1)}}x_0$,
$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\}$
$= \min\{\frac{B_0}{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}}, \frac{B_0^{k+1}}{A_0^{k}x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$,
$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \frac{B_1^{k+2}}{A_1^{k+1}x_0}$,
$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\}$
$= \max\{\frac{A_0}{\frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{k+2}}{B_0^{k+2}}x_{-1}$,
$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \frac{A_1^{k+2}}{B_1^{k+2}}x_0$,
$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$,
$y_{4(k+1)+4} = \min\{\frac{B_1}{x_{4k+6}}, y_{4k+6}\} = \frac{B_1^{k+2}}{A_1^{k+1}x_0}$,

ii. If $1 \ge \frac{A_1}{B_1}$, then
$x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\}$
$= \max\{\frac{A_0}{\frac{B_0^{k+1}}{A_0^{k}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{(k+1)}}{B_0^{(k+1)}}x_{-1}$,
$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1 x_0}{B_1}, x_0\} = x_0$,
$y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\}$
$= \min\{\frac{B_0}{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}}, \frac{B_0^{k+1}}{A_0^{k}x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$,
$y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\}$
$= \min\{\frac{B_1}{x_0}, \frac{B_1}{x_0}\} = \frac{B_1}{x_0}$, $x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\}$
$= \max\{\frac{A_0}{\frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{k+2}}{B_0^{k+2}}x_{-1}$,
$x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \max\{\frac{A_1 x_0}{B_1}, x_0\} = x_0$,
$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\}$
$= \min\{\frac{B_0 B_0^{k+1}}{A_0^{k+1}x_{-1}}, \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$,
$y_{4(k+1)+4} = \min\{\frac{B_1}{x_0}, \frac{B_1}{x_0}\} = \frac{B_1}{x_0}$

(b) If $1 \ge \frac{A_0}{B_0}$, then

i. If $1 \le \frac{A_1}{B_1}$, then
$x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} = \max\{\frac{A_0}{\frac{B_0}{x_{-1}}}, x_{-1}\} = x_{-1}$,
$x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1}{\frac{B_1^{k+1}}{A_1^{k}x_0}}, \frac{A_1^{k+1}}{B_1^{k+1}}x_0\}$
$= \frac{A_1^{(k+1)}}{B_1^{(k+1)}}x_0$, $y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\}$
$= \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\} = \frac{B_0}{x_{-1}}$, $y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\}$
$= \min\{\frac{B_1}{\frac{A_1^{k+1}}{B_1^{k+1}}x_0}, \frac{B_1^{k+1}}{A_1^{k}x_0}\} = \frac{B_1^{k+2}}{A_1^{k+1}x_0}$,
$x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\} = \max\{\frac{A_0 x_{-1}}{B_0}, x_{-1}\}$
$= x_{-1} x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\} = \frac{A_1^{(k+2)}}{B_1^{(k+2)}}x_0$,
$y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\}$
$= \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\} = \frac{B_0}{x_{-1}}$,
$y_{4(k+1)+4} = \frac{B_1^{k+2}}{A_1^{k+1}x_0}$

ii. If $1 \ge \frac{A_1}{B_1}$, then $x_{4(k+1)+1} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\} =$
$\max\{\frac{A_0}{\frac{B_0}{x_{-1}}}, x_{-1}\} = x_{-1}$, $x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\}$
$= \max\{\frac{A_1}{\frac{B_1}{x_0}}, x_0\} = x_0$, $y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\}$
$= \min\{\frac{B_0}{x_{-1}}, \frac{B_0}{x_{-1}}\} = \frac{B_0}{x_{-1}} y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\}$
$= \min\{\frac{B_1}{x_0}, \frac{B_1}{x_0}\} = \frac{B_1}{x_0}$, $x_{4(k+1)+3} = \max\{\frac{A_0}{y_{4k+5}}, x_{4k+5}\}$
$= \max\{\frac{A_0 x_{-1}}{B_0}, x_{-1}\} = x_{-1}$, $x_{4(k+1)+4} = \max\{\frac{A_1}{y_{4k+6}}, x_{4k+6}\}$
$= x_0 y_{4(k+1)+3} = \min\{\frac{B_0}{x_{4k+5}}, y_{4k+5}\} =$
$\min\{B_0/x_{-1}, B_0/x_{-1}\} = \frac{B_0}{x_{-1}}$
$y_{4(k+1)+4} = \frac{B_1}{x_0}$

∎

Similarly we can prove the following theorems.

**Theorem 3.** *If we suppose that $(x_n, y_n)$ is a solution for system (2) provided that $\frac{A_0}{y_{-1}} \le x_{-1}$, $\frac{B_0}{x_{-1}} \ge y_{-1}$, then the following statements hold:*

1. *If* $x_0 \ge \frac{A_1}{y_0}$, $y_0 \le \frac{B_1}{x_0}$, *then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4} = x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = y_0$
2. *If* $x_0 \ge \frac{A_1}{y_0}$, $y_0 \ge \frac{B_1}{x_0}$, *then*
   (a) *If* $1 \le \frac{B1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4} = x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = \frac{B_1}{x_0}$
   (b) *If* $1 \ge \frac{B_1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = \frac{A_1^n}{B_1^n}x_0$, $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = \frac{B_1^{n+1}}{A_1^n x_0}$
3. *If* $x_0 \le \frac{A_1}{y_0}$, $y_0 \le \frac{B_1}{x_0}$, *then*
   (a) *If* $1 \ge \frac{B_1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4} = \frac{A_1^{n+1}}{B_1^n y_0}$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = \frac{B_1^n}{A_1^n}y_0$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
   (b) *If* $1 \le \frac{B_1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4} = \frac{A_1}{y_0}$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = y_0$,
4. *If* $x_0 \le \frac{A_1}{y_0}$, $y_0 \ge \frac{B_1}{x_0}$, *then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$, $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$

*where* $n \in \mathbb{N}_0$

**Proof.** By mathematical induction as in Theorem 2. □

**Theorem 4.** *If we suppose that $(x_n, y_n)$ is a solution for system (2) provided that $\frac{A_0}{y_{-1}} \ge x_{-1}$, $\frac{B_0}{x_{-1}} \ge y_{-1}$, then the following statements hold:*

1. *If* $x_0 \ge \frac{A_1}{y_0}$, $y_0 \le \frac{B_1}{x_0}$, *then*
   (a) *If* $1 \le \frac{B_0}{A_0}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0}{y_{-1}}$, $x_{4n+2} = x_{4n+4} = x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = y_0$,
   (b) *If* $1 \ge \frac{B_0}{A_0}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = x_{4n+4} = x_0$, $y_{4n+1} = \frac{B_0^n}{A_0^n}y_{-1}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$, $y_{4n+2} = y_{4n+4} = y_0$,
2. *If* $x_0 \le \frac{A_1}{y_0}$, $y_0 \ge \frac{B_1}{x_0}$, *then*
   (a) *If* $1 \le \frac{B_0}{A_0}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_1}{y_{-1}}$, $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$, $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
   (b) *If* $1 \ge \frac{B_0}{A_0}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$, $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = \frac{B_0^n}{A_0^n}y_{-1}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$, $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
3. *If* $x_0 \ge \frac{A_1}{y_0}$, $y_0 \ge \frac{B_1}{x_0}$, *then*
   (a) *If* $1 \le \frac{B_0}{A_0}$, *then*
      i. *If* $1 \le \frac{A_1}{B_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0}{y_{-1}}$, $x_{4n+2} = x_{4n+4} = \frac{A_1^n}{B_1^n}x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = \frac{B_1^{n+1}}{A_1^n x_0}$,
      ii. *If* $1 \ge \frac{A_1}{B_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0}{y_{-1}}$, $x_{4n+2} = x_{4n+4} = x_0$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = \frac{B_1}{x_0}$,
   (b) *If* $1 \ge \frac{B_0}{A_0}$, *then*
      i. *If* $1 \le \frac{A_1}{B_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = x_{4n+4} = \frac{A_1^n}{B_1^n}x_0$, $y_{4n+1} = \frac{B_0^n}{A_0^n}y_{-1}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$, $y_{4n+2} = y_{4n+4} = \frac{B_1^{n+1}}{A_1^n x_0}$,
      ii. *If* $1 \ge \frac{A_1}{B_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = x_{4n+4} = x_0$, $y_{4n+1} = \frac{B_0^n}{A_0^n}y_{-1}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$, $y_{4n+2} = y_{4n+4} = \frac{B_1}{x_0}$,
4. *If* $x_0 \le \frac{A_1}{y_0}$, $y_0 \le \frac{B_1}{x_0}$, *then*
   (a) *If* $1 \le \frac{B_0}{A_0}$, *then*
      i. *If* $1 \le \frac{B_1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0}{y_{-1}}$, $x_{4n+2} = x_{4n+4} = \frac{A_1}{y_0}$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = y_{4n+4} = y_0$,
      ii. *If* $1 \ge \frac{B_1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0}{y_{-1}}$, $x_{4n+2} = x_{4n+4} = \frac{A_1^{n+1}}{B_1^n y_0}$, $y_{4n+1} = y_{4n+3} = y_{-1}$, $y_{4n+2} = \frac{B_1^n}{A_1^n}y_0$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$,
   (b) *If* $1 \ge \frac{B_0}{A_0}$, *then*
      i. *If* $1 \le \frac{B_1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = x_{4n+4} = \frac{A_1}{y_0}$, $y_{4n+1} = \frac{B_0^n}{A_0^n}y_{-1}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$, $y_{4n+2} = y_{4n+4} = y_0$,
      ii. *If* $1 \ge \frac{B_1}{A_1}$, *then* $x_{4n+1} = x_{4n+3} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = x_{4n+4} = \frac{A_1^{n+1}}{B_1^n y_0}$, $y_{4n+1} = \frac{B_0^n}{A_0^n}y_{-1}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$, $y_{4n+2} = \frac{B_1^n}{A_1^n}y_0$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$,

**Proof.** By mathematical induction as in Theorem (2). □

**Theorem 5.** *If we suppose that $(x_n, y_n)$ is a solution for system (2) provided that $\frac{A_0}{y_{-1}} \ge x_{-1}$, $\frac{B_0}{x_{-1}} \le y_{-1}$, then the following statements hold:*

1. *If* $x_0 \ge \frac{A_1}{y_0}$, $y_0 \le \frac{B_1}{x_0}$, *then* $x_{4n+1} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = x_{4n+4} = x_0$, $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$, $y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = y_{4n+4} = y_0$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$
2. *If* $x_0 \le \frac{A_1}{y_0}$, $y_0 \ge \frac{B_1}{x_0}$, *then* $x_{4n+1} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$, $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1} x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1} y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$,
3. *If* $x_0 \ge \frac{A_1}{y_0}$, $y_0 \ge \frac{B_1}{x_0}$, *then*
   (a) *If* $1 \le \frac{A_1}{B_1}$, *then* $x_{4n+1} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = \frac{A_1^n}{B_1^n}x_0$, $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1} x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1} y_{4n+4} = \frac{B_1^{n+1}}{A_1^n x_0}$,
   (b) *If* $1 \ge \frac{A_1}{B_1}$, *then* $x_{4n+1} = \frac{A_0^{n+1}}{B_0^n y_{-1}}$, $x_{4n+2} = x_{4n+4} = x_0$, $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$, $y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = y_{4n+4} = \frac{B_1}{x_0}$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}}y_{-1}$
4. *If* $x_0 \le \frac{A_1}{y_0}$, $y_0 \le \frac{B_1}{x_0}$, *then*

(a) *If* $1 \leq \frac{B_1}{A_1}$, *then* $x_{4n+1} = \frac{A_0^{n+1}}{B_0^n y_{-1}}, x_{4n+2} = x_{4n+4} = \frac{A_1}{y_0}$, $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}} x_{-1}, y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}, y_{4n+2} = y_{4n+4} = y_0$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}} y_{-1}$

(b) *If* $1 \geq \frac{B_1}{A_1}$, *then* $x_{4n+1} = \frac{A_0^{n+1}}{B_0^n y_{-1}}, x_{4n+2} = x_{4n+4} = \frac{A_1^{n+1}}{B_1^n y_0}$, $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}} x_{-1} y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}, y_{4n+2} = \frac{B_1^n}{A_1^n} y_0$, $y_{4n+3} = \frac{B_0^{n+1}}{A_0^{n+1}} y_{-1} y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}} y_0$,

**Proof.** By mathematical induction as in Theorems (2). □

**Corollary 6.** *All solutions of system (2) are periodical of period either two or four when the two-periodic sequences $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are the same .*

**Proof.** It follows from Theorems (2), (3), (4) and (5), as a direct consequence. □

Now we shall study the behavior of solutions as n approaches infinity.

**Corollary 7.** *Suppose that $(x_n, y_n)$ is a solution of system (2) such that $x_{-1}y_{-1} \geq A_0$, $B_0$ and $n \to \infty$. Then the following statements hold:*

1. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 1, 2, 3, 4$ *in each one of the following cases :*
   (a) $x_0y_0 \leq B_1 < A_1$ *and* $B_0 < A_0$
   (b) $x_0y_0 \geq A_1 > B_1$ *and* $B_0 < A_0$
   (c) $B_1 < x_0y_0 \leq A_1$ *and* $B_0 < A_0$
2. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 1, 3$ *in each one of the following cases :*
   (a) $A_1 \leq x_0y_0 \leq B_1$ *and* $B_0 < A_0$
   (b) $x_0y_0 \leq A_1 < B_1$ *and* $B_0 < A_0$
   (c) $x_0y_0 \geq B_1 > A_1$ *and* $B_0 < A_0$
3. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 2, 4$ *in each one of the following cases :*
   (a) $x_0y_0 \leq B_1 < A_1$ *and* $B_0 < A_0$
   (b) $x_0y_0 \geq A_1 > B_1$ *and* $B_0 > A_0$
   (c) $B_1 \leq x_0y_0 \leq A_1$ *and* $B_0 > A_0$

**Proof.** We can easy prove this corollary by using Theorem 2. □

**Corollary 8.** *Suppose that $(x_n, y_n)$ is a solution of system (2) such that $B_0 \geq x_{-1}y_{-1} \geq A_0$ and $n \to \infty$. Then $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 2, 4$ in each one of the following cases :*

1. $x_0y_0 \geq A_1 > B_1$
2. $x_0y_0 \leq B_1 < A_1$
3. $B_1 < x_0y_0 \leq A_1$

**Proof.** We can easy prove this corollary by using Theorem 3. □

**Corollary 9.** *Suppose that $(x_n, y_n)$ is a solution of system (2) such that $x_{-1}y_{-1} \leq A_0$, $B_0$ and $n \to \infty$. Then the following statements hold:*

1. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 1, 2, 3, 4$ *in each one of the following cases :*
   (a) $x_0y_0 \geq A_1 > B_1$ *and* $B_0 < A_0$
   (b) $x_0y_0 \leq B_1 < A_1$ *and* $B_0 < A_0$
   (c) $B_1 \leq x_0y_0 < A_1$ *and* $B_0 < A_0$
2. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 1, 3$ *in each one of the following cases :*
   (a) $A_1 \leq x_0y_0 \leq B_1$ *and* $B_0 < A_0$
   (b) $x_0y_0 \leq A_1 \leq B_1$ *and* $B_0 < A_0$
   (c) $x_0y_0 \geq B_1 \geq A_1$ *and* $B_0 < A_0$
3. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 2, 4$ *in each one of the following cases :*
   (a) $x_0y_0 \leq B_1 < A_1$ *and* $B_0 \geq A_0$
   (b) $x_0y_0 \geq A_1 > B_1$ *and* $B_0 \geq A_0$
   (c) $B_1 < x_0y_0 \leq A_1$ *and* $B_0 > A_0$
4. $\lim_{n\to\infty} \frac{1}{y_{4n+i}} = \lim_{n\to\infty} x_{4n+i} = 0$, $i = 2, 4$ *if* $x_0y_0 \geq B_1 > A_1$ , $A_0 \leq B_0$ .

**Proof.** We can easy prove this corollary by using Theorem 4. □

**Corollary 10.** *Suppose that $(x_n, y_n)$ is a solution of system (2) such that $B_0 < x_{-1}y_{-1} \leq A_0$ and $n \to \infty$. Then the following statements hold:*

1. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 1, 2, 3, 4$ *in each one of the following cases :*
   (a) $x_0y_0 \geq A_1 > B_1$
   (b) $x_0y_0 \leq B_1 < A_1$
   (c) $B_1 < x_0y_0 \leq A_1$
2. $\lim_{n\to\infty} \frac{1}{x_{4n+i}} = \lim_{n\to\infty} y_{4n+i} = 0$, $i = 1, 3$ *in each one of the following cases :*
   (a) $A_1 \leq x_0y_0 \leq B_1$
   (b) $x_0y_0 \leq A_1 \leq B_1$
   (c) $x_0y_0 \geq B_1 \geq A_1$

**Proof.** We can easy prove this corollary by using Theorem 5. □

**Conjecture 1.** Consider the following high order semi-max-type two-dimensional system of difference equations

$$x_{n+1} = \max\left\{\frac{A_n}{y_{n-h}}, x_{n-h}\right\},$$

$$y_{n+1} = \min\left\{\frac{A_n}{x_{n-h}}, y_{n-h}\right\}, n \in \mathbb{N}_0, \tag{3}$$

where $(A_n)_{n\in\mathbb{N}_0}$ is a positive sequence which is periodical with period 2 and initial values $x_0$, $x_{-1}$,.....,$x_{-h}$,$y_0$, $y_{-1}$,.....,$y_{-h}$ $\in (0, +\infty)$. Show that every well-defined solution of this system is periodic with period either h+1 or 2h+2 .

**Remark 2.** Note that the Conjecture 1 is obviously true when$(A_n)_{n\in\mathbb{N}_0} = \{0, 0, 0, ....\}$. Hence the case $(A_n)_{n\in\mathbb{N}_0} \neq \{0, 0, 0, \ldots\}$ is of some interest.

**Remark 3.** It is extremely contemplated that for h small, Conjecture 1 can be confirmed by some calculations look like to those in our paper. However, the most important thing is to find a comparatively straightforward process to confirm this conjecture.

**Acknowledgment**

The conclusions in this paper were obtained whereas the author had worked in the University of King Khalid, Abha, Saudi Arabi. The author would like to express his heartfelt thanks to the referees for their suggestions and prosperous comments.

## References


[1] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+x_n x_{n-1}}$, Appl. Math. Comp 150 (2004) 21–24.

[2] Q. Din, T.F. Ibrahim, K.A. Khan, Behavior of a competitive system of second-order difference equations, Sci. World J. 2014 (2014) 9. Article ID 283982.

[3] Q. Din, T.F. Ibrahim, Global behavior of a neural networks system, Indian J. Comput. Appl. Math. 01 (1) (2013) 79–92.

[4] M.A. El-Moneam, E.M.E. Zayed, Dynamics of the rational difference equation, Inf. Sci. Lett. 3 (2) (2014) 1–9.

[5] E.M. Elsayed, T.F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, Hacet. J. Math. Stat. 44 (6) (2015) 1361–1390.

[6] T.F. Ibrahim, Q. Zhang, Stability of an anti-competitive system of rational difference equations, Arch. Des Sci. 66 (5) (2013).

[7] T.F. Ibrahim, Oscillation, non-oscillation, and asymptotic behavior for third order nonlinear difference equations, Dyn. Contin. Discret. Impuls. Syst. Ser. A: Math. Anal. 20 (4) (2013) 523–532.

[8] T.F. Ibrahim, Periodicity and global attractivity of difference equation of higher order, J. Comput. Anal. Appl. 16 (3) (2014) 552–564.

[9] J. Feuer, On the eventual periodicity of $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ with a period-four parameter, J. Differ. Equ. Appl. 12 (2006) 467–486.

[10] E.M. Elsayed, B.D. Iričanin, On the max-type difference equation $x_{n+1} = \max\{A/x_n, x_{n-3}\}$, Discret. Dyn. Nat. Soc. 2009 (2009) 10. Article ID 675413.

[11] E.M. Elsayed, S. Stević, On the max-type equation $x_{n+1} = \max\{A/x_n, x_{n-2}\}$, Nonlinear Anal. TMA 71 (2009) 910–922.

[12] E.M. Elsayed, B.D. Iričanin, S. Stević, On the max-type equation $x_{n+1} = \max\{A_n/x_n, x_{n-1}\}$, Ars Combin. 95 (2010) 187–192.

[13] B.D. Iričǎnin, N. Touafek, On a second order max-type system of difference equations, Indian J. Math. 54 (2012) 119–142.

[14] T.F. Ibrahim, Three-dimensional max-type cyclic system of difference equations, Int. J. Phys. Sci. 8 (15) (2013) 629–634.

[15] T.F. Ibrahim, N. Touafek, Max-type system of difference equations with positive two-periodis sequences, Math. Methods Appl. Sci. 37 (16) (2014) 2562–2569.

[16] T.F. Ibrahim, Solving a class of three-order max-type difference equations, Dyn. Contin. Discret. Impuls. Syst. Ser. A: Math. Anal. 21 (2014) 333–342.