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Original Article

Behavior of some higher order nonlinear rational partial difference equations

Tarek F. Ibrahim*

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt

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Abstract In this paper we give the closed form expressions of some higher order nonlinear rational partial difference equations in the form

$$X_{n,m}=\frac{X_{n-r,m-r}}{\Psi+\prod_{i=1}^{r}X_{n-i,m-i}}$$

where $n, m \in \mathbb{N}$ and the initial values $X_{n,t}$, $X_{t,m-r}$ are real numbers with $t \in \{0, -1, -2, \ldots, -r+1\}$ such that $\prod_{j=0}^{r-1} X_{j-r+1,i+j-r+1} \neq -\Psi$ and $\prod_{j=0}^{r-1} X_{i+j-r+2,j-r+1} \neq -\Psi$, $i \in \mathbb{N}_0$.

We will use a new method to prove the results by using what we call 'piecewise double mathematical induction' which we introduce here for the first time as a generalization of many types of mathematical induction. As a direct consequences, we investigate and conclude the explicit solutions of some higher order ordinary difference equations.

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1. Introduction

As we know, the examining of ordinary difference equations has been exceedingly remedied in the past. However, partial difference equations (PΔEs) have not happened on the same full attentiveness. Both of partial and ordinary difference equations might be found in the study of probability, dynamics and other branches of mathematical physics. Moreover, partial difference equations emerge in topics comprising population dynamics with spatial migrations, chemical reactions, and finite difference schemes. Indeed Lagrange and Laplace took into consideration the solution of partial difference equations in their studies of dynamics and probability.

As our first example (discrete heat equations) of modeling realistic problems by partial difference equations, consider the temperature distribution of a "very long" rod. Assume that the

* Tel.: +966 535561756.
E-mail addresses: tfibrahem@mans.edu.eg, tfoze@kku.edu.sa
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rod is so long that it can be laid on top of the set $\mathbb{Z}$ of integers. Let $V(s, t)$ be the temperature at the integral time t and integral position s of the rod. At time t, if the temperature $V(s-1, t)$ is higher than $V(s, t)$, heat will flow from the point $s-1$ to s. The amount of growing is $V(s, t+1) - V(s, t)$ and it is plausible to presume that the increase is proportional to the difference $V(s-1, t) - V(s, t)$, say $r(V(s-1, t) - V(s, t))$ where r is a positive diffusion rate constant, that is $V(s, t+1) - V(s, t) = r(V(s-1, t) - V(s, t))$, $r > 0$. Similarly, heat will outflow from the point $s+1$ to s if $V(s+1, t) > V(s, t)$. Thus, it is reasonable that the total effect is

$$\begin{aligned} &V(s, t+1) - V(s, t) \\ &\quad = r(V(s-1, t) - V(s, t) + r(V(s+1, t) - V(s, t)) \end{aligned}$$

Such a postulate can be regarded as a discrete Newton law of cooling.

An another example, the following partial difference equations:

$$s_k^{(n+1)} = s_{k-1}^{(n)} - n s_k^{(n)}, \quad 1 \le k < n.$$

$$S_k^{(n+1)} = S_{k-1}^{(n)} + k S_k^{(n)}, \quad 1 \le k < n.$$

The solutions of these partial difference equations are the Stirling numbers of the first kind $s_k^{(n)}$ and the Stirling numbers of the second kind $S_k^{(n)}$, respectively.

Some authors scrutinize the closed form solutions for nominated partial difference equations.

For instance, [1] Heins established the solution of the partial difference equation

$$T\{n+1, m\} + T\{n-1, m\} = 2T\{n, m+1\}$$

under some assumed conditions.

For more results about partial difference expressions we indicate to [2–13].

In this paper, we studied the closed form expressions of some higher order non-linear rational partial difference equations in the formularization

$$X_{n,m} = \frac{X_{n-r,m-r}}{\Psi + \prod_{i=1}^{r} X_{n-i,m-i}} \tag{1}$$

where $n, m \in \mathbb{N}$, $\Psi = \pm 1$ and the initial values $X_{n,t}$, $X_{t,m-r}$ are real numbers with $t \in \{0, -1, -2, \ldots, -r+1\}$ such that $\prod_{j=0}^{r-1} X_{j-r+1, i+j-r+1} \neq -\Psi$ and $\prod_{j=0}^{r-1} X_{i+j-r+2, j-r+1} \neq -\Psi$, $i \in \mathbb{N}_0$.

As a direct consequences, we investigate and conclude the explicit solutions of some higher order ordinary difference equations in the following form

$$X_n = \frac{X_{n-r}}{\pm 1 + \prod_{i=1}^{r} X_{n-i}} \tag{2}$$

where $n \in \mathbb{N}$, and the initial values X_p are real numbers with $p \in \{0, -1, -2, \ldots, -r+1\}$ such that $\prod_{j=1}^{r} X_{1-j} \neq \mp 1$.

In order to prove the main results we demand the following definition which we display here for the first time as a generalization of many types of mathematical induction.

Definition 1. (Piecewise Double Mathematical Induction of r-pieces). Let $H(n, m)$ be a statement involving two positive integer variables n and m. Besides, we suppose that the statement $H(n, m)$ is piecewise with r-pieces. Then the statement $H(n, m)$ holds if

1. $H(L_1 + \alpha, L_2 + \beta)$
2. If $H(n, L_2 + \beta)$, then $H(n + r, L_2 + \beta)$
3. If $H(n, m)$, then $H(n, m + r)$
   where $\alpha, \beta \in \{0, 1, 2, \ldots, r-1\}$ and L_1 and L_2 are the smallest values of n and m.

We briefly call this concept "r-double mathematical induction".

Remark 1. We can see that the previous concept generalize many types of mathematical induction. For instances,

1. If $r = 1$, we have $\alpha, \beta = 0$, thus we have the well known double mathematical induction.
2. If $r = 2$, we have $\alpha, \beta \in \{0, 1\}$, thus we have the odd–even double mathematical induction.
3. If $r = 3$, we have $\alpha, \beta \in \{0, 1, 2\}$, thus we have the 3-double mathematical induction.

Remark 2. If we put $n = m$ we have a special case of the above definition which introduce an another new concept. This type of mathematical induction called "Piecewise Mathematical Induction of r-pieces". In this case, if we put $r = 1$ with $n = m$ we easily get the basic mathematical induction. Also if we put $r = 2$ with $n = m$, we get easily the odd–even mathematical induction.

2. Forms of solutions

In this section we shall give explicit forms of solutions of the higher order partial difference Eq. (1).

2.1. Form of solutions for $P\Delta E$ (1) when $\Psi = 1$

In this section we study the following higher order partial difference equation

$$X_{n,m} = \frac{X_{n-r,m-r}}{1 + \prod_{i=1}^{r} X_{n-i,m-i}} \tag{3}$$

2.1.1. The case when $r = 2$

In this case we have a second order partial difference equation in the form

$$X_{n,m} = \frac{X_{n-2,m-2}}{1 + X_{n-1,m-1} X_{n-2,m-2}} \tag{4}$$

Here, we give the closed form solution of the partial difference Eq. (4).

Theorem 2. *Let $\{X_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of the partial difference Eq. (4), where $n, m \in \mathbb{N}$ and the initial values $X_{n,t}$, $X_{t,m-2}$ are real numbers with $t \in \{0, -1\}$. Suppose $\prod_{j=0}^{1} X_{j-1, i+j-1} \neq -1$ and $\prod_{j=0}^{1} X_{i+j, j-1} \neq -1$*

Then, the form of solutions of Eq. (4), for $n \ge m$ are as follows:

$$X_{m,n} = X_{-(2-q),n-m-(2-q)} \prod_{k=0}^{\frac{m-q}{2}} \frac{1+(2k+q-1)X_{0,n-m}X_{-1,n-m-1}}{1+(2k+q)X_{0,n-m}X_{-1,n-m-1}} \tag{5}$$

$$X_{n,m} = X_{n-m-(2-q),-(2-q)} \prod_{k=0}^{\frac{m-q}{2}} \frac{1+(2k+q-1)X_{n-m,0}X_{n-m-1,-1}}{1+(2k+q)X_{n-m,0}X_{n-m-1,-1}} \tag{6}$$

where $m = 2L + q$, $q = 1$ *or* 2 , *and* $L \geq 0$.

Proof. We shall use the principle of piecewise double mathematical induction defined in definition (1). Firstly, we shall prove that the relations (5) and (6) hold for $(n,m) = (1,1)$. From Eq. (4) we can see

$$X_{1,1} = \frac{X_{-1,-1}}{1+X_{0,0}X_{-1,-1}} = X_{-1,-1} \prod_{k=0}^{\frac{0}{2}} \frac{1+(2k)X_{0,0}X_{-1,-1}}{1+(2k+1)X_{0,0}X_{-1,-1}}$$

Now, we shall prove that the relations (5) and (6) hold for $(n,m) = (2,2)$.

$$X_{2,2} = \frac{X_{0,0}}{1+X_{0,0}X_{1,1}} = \frac{X_{0,0}}{1+X_{0,0}\left(\frac{X_{-1,-1}}{1+X_{0,0}X_{-1,-1}}\right)}$$
$$= X_{0,0}\left(\frac{1+X_{0,0}X_{-1,-1}}{1+2X_{0,0}X_{-1,-1}}\right)$$
$$= X_{0,0} \prod_{k=0}^{\frac{2-2}{2}} \frac{1+(2k+1)X_{0,0}X_{-1,-1}}{1+(2k+2)X_{0,0}X_{-1,-1}}$$

Moreover, we shall prove that the relations (5) and (6) hold for $(n,m) = (1,2)$ and $(n,m) = (2,1)$.

$$X_{1,2} = \frac{X_{-1,0}}{1+X_{-1,0}X_{0,1}} = X_{-1,0} \prod_{k=0}^{\frac{1-1}{2}} \frac{1+(2k)X_{-1,0}X_{0,1}}{1+(2k+1)X_{-1,0}X_{0,1}}$$

$$X_{2,1} = \frac{X_{0,-1}}{1+X_{0,-1}X_{1,0}} = X_{0,-1} \prod_{k=0}^{\frac{1-1}{2}} \frac{1+(2k)X_{0,-1}X_{1,0}}{1+(2k+1)X_{0,-1}X_{1,0}}$$

Now suppose that the relations (5) and (6) hold for $m = 1$ and $m = 2$ with $n \in \mathbb{N}$. So we have,

$$X_{n,1} = X_{n-2,-1} \prod_{k=0}^{0} \frac{1+(2k)X_{n-2,-1}X_{n-1,0}}{1+(2k+1)X_{n-2,-1}X_{n-1,0}}$$
$$= \frac{X_{n-2,-1}}{1+X_{n-2,-1}X_{n-1,0}}$$
$$X_{n,2} = X_{n-2,0}\left(\frac{1+X_{n-2,0}X_{n-3,-1}}{1+2X_{n-2,0}X_{n-3,-1}}\right)$$

Now we try to prove that relations (5) and (6) hold for $m = 1$ with $n + 2$.

$$X_{n+2,1} = \frac{X_{n,-1}}{1+X_{n,-1}X_{n+1,0}} = X_{n,-1} \prod_{k=0}^{\frac{0}{2}} \frac{1+(2k)X_{n,-1}X_{n+1,0}}{1+(2k+1)X_{n,-1}X_{n+1,0}}$$

Now we try to prove that relations (5) and (6) hold for $m = 2$ with $n + 2$.

$$X_{n+2,2} = \frac{X_{n,0}}{1+X_{n,0}X_{n+1,1}} = \frac{X_{n,0}}{1+X_{n,0}\left(\frac{X_{n-1,-1}}{1+X_{n-1,-1}X_{n,0}}\right)}$$

$$= \frac{X_{n,0}(1+X_{n-1,-1}X_{n,0})}{1+2X_{n-1,-1}X_{n,0}} = X_{n,0} \prod_{k=0}^{\frac{2-2}{2}} \frac{1+(2k+1)X_{n,0}X_{n-1,-1}}{1+(2k+2)X_{n,0}X_{n-1,-1}}$$

Finally, we suppose that relations (5) and (6) hold for $n, m \in \mathbb{N}$. We shall prove that relations (5) and (6) hold for $n, m+2 \in \mathbb{N}$.

From Eq. (4) we have

$$X_{n,m+2} = \frac{X_{n-2,m}}{1+X_{n-2,m}X_{n-1,m+1}}$$

There are four cases:

(1) If $n > m+2$ and m even.

$$X_{n,m+2} = \frac{X_{n-2,m}}{1+X_{n-2,m}X_{n-1,m+1}} = \frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1+(2k+1)X_{n-m-2,0}X_{n-m-3,-1}}{1+(2k+2)X_{n-m-2,0}X_{n-m-3,-1}}}{1+\left(X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1+(2k+1)X_{n-m-2,0}X_{n-m-3,-1}}{1+(2k+2)X_{n-m-2,0}X_{n-m-3,-1}}\right)\left(X_{n-m-3,-1}\prod_{k=0}^{\frac{m}{2}}\frac{1+(2k)X_{n-m-3,-1}X_{n-m-2,0}}{1+(2k+1)X_{n-m-3,-1}X_{n-m-2,0}}\right)}$$
$$= \frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1+(2k+1)X_{n-m-2,0}X_{n-m-3,-1}}{1+(2k+2)X_{n-m-2,0}X_{n-m-3,-1}}}{1+\frac{X_{n-m-2,0}X_{n-m-3,-1}}{1+(m+1)X_{n-m-2,0}X_{n-m-3,-1}}} = X_{n-m-2,0}\prod_{k=0}^{\frac{m}{2}}\frac{1+(2k+1)X_{n-m-2,0}X_{n-m-3,-1}}{1+(2k+2)X_{n-m-2,0}X_{n-m-3,-1}}$$

(2) If $n > m+2$ and m odd

$$X_{n,m+2} = \frac{X_{n-2,m}}{1+X_{n-2,m}X_{n-1,m+1}} = \frac{X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}}\frac{1+(2k)X_{n-m-3,-1}X_{n-m-2,0}}{1+(2k+1)X_{n-m-3,-1}X_{n-m-2,0}}}{1+\left(X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}}\frac{1+(2k)X_{n-m-3,-1}X_{n-m-2,0}}{1+(2k+1)X_{n-m-3,-1}X_{n-m-2,0}}\right)\left(X_{n-m-2,0}\prod_{k=0}^{\frac{m-1}{2}}\frac{1+(2k+1)X_{n-m-2,0}X_{n-m-3,-1}}{1+(2k+2)X_{n-m-2,0}X_{n-m-3,-1}}\right)}$$
$$= \frac{X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}}\frac{1+(2k)X_{n-m-3,-1}X_{n-m-2,0}}{1+(2k+1)X_{n-m-3,-1}X_{n-m-2,0}}}{1+\frac{X_{n-m-3,-1}X_{n-m-2,0}}{1+(m+1)X_{n-m-3,-1}X_{n-m-2,0}}} = X_{n-m-3,-1}\prod_{k=0}^{\frac{m+1}{2}}\frac{1+(2k)X_{n-m-3,-1}X_{n-m-2,0}}{1+(2k+1)X_{n-m-3,-1}X_{n-m-2,0}}$$

(3) If $n < m+2$ and m even
By symmetry, using relations (5) and (6), we can prove it like part (1).

(4) If $n < m+2$ and m odd
By symmetry, using relations (5) and (6), we can prove it like part (2).

Proposition 1. *We have the following properties for the solutions of Eq. (4):*

(i) *If m even and* $X_{n-m,0}=0$, *then* $X_{n,m}=0$.
(ii) *If m odd and* $X_{n-m,0}=0$, *then* $X_{n,m}=X_{n-m-1,-1}$.
(iii) *If m even and* $X_{n-m-1,-1}=0$, *then* $X_{n,m}=X_{n-m,0}$.
(iv) *If m odd and* $X_{n-m-1,-1}=0$, *then* $X_{n,m}=0$.
(v) *If m even and* $X_{0,n-m}=0$, *then* $X_{m,n}=0$.
(vi) *If m odd and* $X_{0,n-m}=0$, *then* $X_{m,n}=X_{-1,n-m-1}$.
(vii) *If m even and* $X_{-1,n-m-1}=0$, *then* $X_{m,n}=X_{0,n-m}$.
(viii) *If m odd and* $X_{-1,n-m-1}=0$, *then* $X_{m,n}=0$.

□

2.1.2. The case when $r=3$

In this case we have a third order partial difference equation in the form

$$X_{n,m}=\frac{X_{n-3,m-3}}{1+X_{n-1,m-1}X_{n-2,m-2}X_{n-3,m-3}} \tag{7}$$

Theorem 3. *Let* $\{X_{n,m}\}_{n,m=-k}^{\infty}$ *be a solution of the higher order partial difference Eq. (7), where* $n,m\in\mathbb{N}$ *,and the initial values* $X_{n,t}$, $X_{t,m-3}$ *are real numbers with* $t\in\{0,-1,-2\}$. *Suppose* $\prod_{j=0}^{2}X_{i+j-1,j-2}\neq-1$ *and* $\prod_{j=0}^{2}X_{j-2,i+j-2}\neq-1$. *Then, the form of solutions of Eq. (7),* $n\geq m$ *are as follows:*

$$X_{m,n}=X_{-(3-q),n-m-(3-q)}\prod_{k=0}^{\frac{m-q}{3}}\frac{1+(3k+q-1)\prod_{i=0}^{2}X_{-i,n-m-i}}{1+(3k+q)\prod_{i=0}^{2}X_{-i,n-m-i}}$$

$$X_{n,m}=X_{n-m-(3-q),-(3-q)}\prod_{k=0}^{\frac{m-q}{3}}\frac{1+(3k+q-1)\prod_{i=0}^{2}X_{n-m-i,-i}}{1+(3k+q)\prod_{i=0}^{2}X_{n-m-i,-i}}$$

where $m=3L+q$, $3\geq q\geq 1$, $q\in\mathbb{N}$ *and* $L\geq 0$.

Proof. We can prove the theorem by using 3-double mathematical induction (see definition (1)) as in Theorem (2). □

2.1.3. The general case for any value of r

Here, we give the generalized formulas of solutions for the higher order partial difference Eq. (3).

Theorem 4. *Let* $\{X_{n,m}\}_{n,m=-k}^{\infty}$ *be a solution of the higher order partial difference Eq. (3), where* $n,m\in\mathbb{N}$, *and the initial values* $X_{n,t}$, $X_{t,m-r}$ *are real numbers with* $t\in\{0,-1,-2,\ldots,-r+1\}$. *Suppose* $\prod_{j=0}^{r-1}X_{j-r+1,i+j-r+1}\neq-1$ *and* $\prod_{j=0}^{r-1}X_{i+j-r+2,j-r+1}\neq-1$, $i\in\mathbb{N}_0$. *Then, the form of solutions of Eq. (3), for* $n\geq m$ *are as follows:*

$$X_{m,n}=X_{-(r-q),n-m-(r-q)}\prod_{k=0}^{\frac{m-q}{r}}\frac{1+(rk+q-1)\prod_{i=0}^{r-1}X_{-i,n-m-i}}{1+(rk+q)\prod_{i=0}^{r-1}X_{-i,n-m-i}}$$

$$X_{n,m}=X_{n-m-(r-q),-(r-q)}\prod_{k=0}^{\frac{m-q}{r}}\frac{1+(rk+q-1)\prod_{i=0}^{r-1}X_{n-m-i,-i}}{1+(rk+q)\prod_{i=0}^{r-1}X_{n-m-i,-i}}$$

where $m=rL+q$, $r\geq q\geq 1$, $r\geq 2$,$r\in\mathbb{N}$ *and* $L\geq 0$.

Proposition 2. *Suppose* $L\geq 0$, $L\in\mathbb{N}$ *and* $p,q\geq 1$, *we have the following properties for the solutions of Eq. (3):*

(i) *If* $m=rL+p$ *and* $X_{-(r-q),n-m-(r-q)}=0$, *then*

$$X_{m,n}=\begin{cases}X_{-(r-p),n-m-(r-p)}, & p\neq q;\\ 0 & p=q\end{cases}$$

(ii) *If* $m=rL+p$ *and* $X_{n-m-(r-q),-(r-q)}=0$, *then*

$$X_{n,m}=\begin{cases}X_{n-m-(r-p),-(r-p)}, & p\neq q;\\ 0 & p=q\end{cases}$$

2.2. Form of solutions for P∆E (1) when $\Psi=-1$

In this section we study the following higher order partial difference equation

$$X_{n,m}=\frac{X_{n-r,m-r}}{-1+\prod_{i=1}^{r}X_{n-i,m-i}} \tag{8}$$

2.2.1. When r even

In this case we have the following higher order partial difference equation

$$X_{n,m}=\frac{X_{n-2\lambda,m-2\lambda}}{-1+\prod_{i=1}^{2\lambda}X_{n-i,m-i}} \tag{9}$$

where $\lambda\geq 1$, $\lambda\in\mathbb{N}$.

Theorem 5. *Let* $\{X_{n,m}\}_{n,m=-k}^{\infty}$ *be a solution of the higher order partial difference Eq. (9), where* $n,m\in\mathbb{N}$, *and the initial values* $X_{n,t}$, $X_{t,m-2\lambda}$ *are real numbers with* $t\in\{0,-1,-2,\ldots,-2\lambda+1\}$. *Suppose* $\prod_{j=0}^{2\lambda-1}X_{j-2\lambda+1,i+j-2\lambda+1}\neq 1$ *and* $\prod_{j=0}^{2\lambda-1}X_{i+j-2\lambda+2,j-2\lambda+1}\neq 1$, $i\in\mathbb{N}_0$. *Then, the form of solutions of Eq. (9) ,for* $n\geq m$ *are as follows:*

$$X_{m,n}=\begin{cases}\dfrac{X_{-(2\lambda-s),n-m-(2\lambda-s)}}{(-1+\prod_{i=0}^{2\lambda-1}X_{-i,n-m-i})^{\frac{m+2\lambda-s}{2\lambda}}}, & m=2\lambda K+s,\ s<2\lambda;\\ X_{-(2\lambda-t),n-m-(2\lambda-t)}\left(-1+\prod_{i=0}^{2\lambda-1}X_{-i,n-m-i}\right)^{\frac{m+2\lambda-t}{2\lambda}}, & m=2\lambda K+t,\ t\leq 2\lambda\end{cases}$$

$$X_{n,m}=\begin{cases}\dfrac{X_{n-m-(2\lambda-s),-(2\lambda-s)}}{(-1+\prod_{i=0}^{2\lambda-1}X_{n-m-i,-i})^{\frac{m+2\lambda-s}{2\lambda}}}, & m=2\lambda K+s,\ s<2\lambda;\\ X_{n-m-(2\lambda-t),-(2\lambda-t)}\left(-1+\prod_{i=0}^{2\lambda-1}X_{n-m-i,-i}\right)^{\frac{m+2\lambda-t}{2\lambda}}, & m=2\lambda K+t,\ t\leq 2\lambda\end{cases}$$

where $\lambda\geq 1$, $\lambda\in\mathbb{N}$, *s will be odd, t will be even and* $K\geq 0$.

2.2.2. When r odd

In this case we have the following higher order partial difference equation

$$X_{n,m}=\frac{X_{n-(2\mu+1),m-(2\mu+1)}}{-1+\prod_{i=1}^{(2\mu+1)}X_{n-i,m-i}} \tag{10}$$

where $\mu\geq 1$, $\mu\in\mathbb{N}$.

Theorem 6. *Let* $\{X_{n,m}\}_{n,m=-k}^{\infty}$ *be a solution of the higher order partial difference Eq. (10), where* $n,m\in\mathbb{N}$, *and the initial values* $X_{n,t}$, $X_{t,m-2\mu-1}$ *are real numbers with* $t\in\{0,-1,-2,\ldots,-2\mu\}$. *Suppose* $\prod_{j=0}^{2\mu}X_{j-2\mu,i+j-2\mu}\neq 1$ *and* $\prod_{j=0}^{2\mu}X_{i+j-2\mu+1,j-2\mu}\neq 1$, $i\in\mathbb{N}_0$. *Then, the form of solutions of Eq. (10), for* $n\geq m$ *are as follows:*

$$X_{m,n}=\begin{cases}\frac{X_{-(2\mu-s+1),n-m-(2\mu-s+1)}}{-1+\prod_{i=0}^{2\mu}X_{-i,n-m-i}}, m=(2\mu+1)K+s,\\ \quad s\leq 2\mu+1, s, odd;\\ X_{-(2\mu-t+1),n-m-(2\mu-t+1)}\left(-1+\prod_{i=0}^{2\mu}X_{-i,n-m-i}\right),\\ \quad m=(2\mu+1)K+t, t<2\mu+1, t, even\\ X_{-(2\mu-u+1),n-m-(2\mu-u+1)}, m=(2\mu+1)(K+1)+u,\\ \quad 1\leq u\leq 2\mu+1\end{cases}$$

$$X_{n,m}=\begin{cases}\frac{X_{n-m-(2\mu-s+1),-(2\mu-s+1)}}{-1+\prod_{i=0}^{2\mu}X_{n-m-i,-i}}, m=(2\mu+1)K+s,\\ \quad s\leq 2\mu+1, s, odd;\\ X_{n-m-(2\mu-t+1),-(2\mu-t+1)}\left(-1+\prod_{i=0}^{2\mu}X_{n-m-i,-i}\right),\\ \quad m=(2\mu+1)K+t, t<2\mu+1, t, even\\ X_{n-m-(2\mu-u+1),-(2\mu-u+1)}, m=(2\mu+1)(K+1)+u,\\ \quad 1\leq u\leq 2\mu+1\end{cases}$$

3. Applications

3.1. *When $n = m$ in Eq. (3)*

If we take into account that $n = m$ in Eq. (3), we have an ordinary difference equation of high order in the form

$$X_n=\frac{X_{n-r}}{1+\prod_{i=1}^{r}X_{n-i}} \tag{11}$$

We can drive the formulas of solutions for this difference equation from theorem (4).

Corollary 7. *Let $\{X_n\}_{n=-k}^{\infty}$ be a solution of the higher order difference Eq. (11), where $n\in\mathbb{N}$, and the initial values X_s are real numbers with $s\in\{0,-1,-2,\ldots,-r+1\}$. Suppose $\prod_{j=1}^{r}X_{1-j}\neq -1$ Then, the form of solutions of Eq. (11), is given by:*

$$X_n=X_{-(r-q)}\prod_{k=0}^{\frac{n-q}{r}}\frac{1+(rk+q-1)\prod_{i=0}^{r-1}X_{-i}}{1+(rk+q)\prod_{i=0}^{r-1}X_{-i}}$$

where $n=rL+q$, $r\geq q\geq 1$, $r\geq 2$, $r\in\mathbb{N}$ and $L\geq 0$.

Proposition 3. *Suppose $L\geq 0$, $L\in\mathbb{N}$ and $p, q\geq 1$, we have the following properties for the solutions of Eq. (11):*

If $n=rL+p$ and $X_{-(r-q)}=0$, then

$$X_n=\begin{cases}X_{-(r-p)}, & p\neq q;\\ 0 & p=q\end{cases}$$

3.2. *When $n = m$ in Eq. (8)*

We have following higher order ordinary difference equation

$$X_n=\frac{X_{n-r}}{-1+\prod_{i=1}^{r}X_{n-i}}$$

3.2.1. *When r even*

In this case we have the following higher order ordinary difference equation

$$X_n=\frac{X_{n-2\lambda}}{-1+\prod_{i=1}^{2\lambda}X_{n-i}} \tag{12}$$

where $\lambda\geq 1$, $\lambda\in\mathbb{N}$.

Corollary 8. *Let $\{X_n\}_{n=-k}^{\infty}$ be a solution of the higher order difference Eq. (12), where $n\in\mathbb{N}$, and the initial values X_p are real numbers with $p\in\{0,-1,-2,\ldots,-2\lambda+1\}$. Suppose $\prod_{j=1}^{2\lambda}X_{1-j}\neq 1$ Then, the form of solutions of Eq. (12) are as follows:*

$$X_n=\begin{cases}\frac{X_{-(2\lambda-s)}}{\left(-1+\prod_{i=0}^{2\lambda-1}X_{-i}\right)^{\frac{n+2\lambda-s}{2\lambda}}}, n=2\lambda K+s, s<2\lambda, s, odd;\\ X_{-(2\lambda-t)}\left(-1+\prod_{i=0}^{2\lambda-1}X_{-i}\right)^{\frac{n+2\lambda-t}{2\lambda}},\\ \quad n=2\lambda K+t, t\leq 2\lambda, t, even\end{cases}$$

where $\lambda\geq 1$, $\lambda\in\mathbb{N}$ and $K\geq 0$.

Remark 3. We can see that if $n\to\infty$, n odd, then $X_n\to 0$ and if $n\to -\infty$, n even, then $X_n\to 0$.

3.2.2. *When r odd*

In this case we have the following higher order difference equation

$$X_n=\frac{X_{n-(2\mu+1)}}{-1+\prod_{i=1}^{(2\mu+1)}X_{n-i}} \tag{13}$$

where $\mu\geq 1$, $\mu\in\mathbb{N}$.

Corollary 9. *Let $\{X_n\}_{n=-k}^{\infty}$ be a solution of the higher order difference Eq. (13), where $n\in\mathbb{N}$, and the initial values X_p are real numbers with $p\in\{0,-1,-2,\ldots,-(2\mu)\}$. Suppose $\prod_{j=1}^{2\mu+1}X_{1-j}\neq 1$ Then, the form of solutions of Eq. (13) are as follows:*

$$X_n=\begin{cases}\frac{X_{-(2\mu-s+1)}}{-1+\prod_{i=0}^{2\mu}X_{-i}}, & s\leq 2\mu+1, s, odd;\\ X_{-(2\mu-t+1)}\left(-1+\prod_{i=0}^{2\mu}X_{-i}\right), & t<2\mu+1, t, even\\ X_{-(2\mu-u+1)}, & 1\leq u\leq 2\mu+1\end{cases}$$

Remark 4. We can easy see that the solution of Eq. (13) is periodic of period $4\mu+2$.

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