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Original Article

Travelling wave solutions of the Schamel–Korteweg–de Vries and the Schamel equations

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Abstract In this paper, the extended (G′/G)-expansion method has been suggested for constructing travelling wave solutions of the Schamel–Korteweg–de Vries (s-KdV) and the Schamel equations with aid of computer systems like Maple or Mathematica. The hyperbolic function solutions and the trigonometric function solutions with free parameters of these equations have been obtained. Moreover, it has been shown that the suggested method is elementary, effective and has been used to solve nonlinear evolution equations in applied mathematics, engineering and mathematical physics.

2010 MATHEMATICAL SUBJECT CLASSIFICATION: 35B10; 35Q99; 35Q53

1. Introduction

Nonlinear evolution equations (NLEEs) are widely used to describe many phenomena and processes in various scientific fields, such as fluid mechanics, plasma physics, optical fibres, biology, solid state physics, chemical kinematics, chemical physics, geochemistry, etc. So, the investigation of exact solutions for nonlinear evolution equations (NLEEs) plays an important role for the understanding of most nonlinear physical phenomena or finding new phenomena. Moreover, the exact solutions of nonlinear evolution equations aid the numerical solvers to assess the correctness of their results and assist them in the stability analysis. Therefore, investigation of the exact solutions of NLEEs has become a major concern for many researchers. So, to obtain the travelling wave solutions, many powerful methods were attempted, such as inverse scattering method [1], Hirota's bilinear method [2], Backlund transformation [3], sine-cosine method [4,5], homogenous balance method [6], homotopy perturbation method [7], variational method [8], improved

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tanh function method [9,10], exp-function method [11,12]. Recently, the pioneer work Wang et al. [13] introduced the $(\frac{G'}{G})$-expansion method to find travelling wave solutions of nonlinear partial differential equations. The main idea of this method is that the travelling wave solutions of nonlinear equations can be expressed by polynomial in $(\frac{G'}{G})$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - ct$ and λ, μ and c are constants [14]. The degree of the polynomial can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in the NLPDEs and the coefficients of the polynomial can be obtained by solving a set of algebraic equations resulting from the process of using the proposed method. This method is widely used by the reference therein [13–23].

In this work we consider the Schamel–KdV (s-KdV) and Schamel equations as follows respectively,

$$u_t + (\alpha u^{\frac{1}{2}} + \beta u)u_x + \delta u_{xxx} = 0, \quad \delta\beta \neq 0 \tag{1}$$

$$u_t + u^{\frac{1}{2}} u_x + \delta u_{xxx} = 0 \tag{2}$$

where α, β and δ are constants which they are refer to the activation trapping, the convection and the dispersion coefficients respectively. Eq. (1) arises in number of scientific models, such as plasma physics and optical fibre. This equation describes the nonlinear interaction of ion-acoustic waves when electron trapping is present and also it governs the electrostatic potential for a certain electron distribution in velocity space [24]. We mention that, the Eq. (1) is a particular case of the generalised Gardner equations and also it contains the KdV equation when $\alpha = 0$ and the Schamel equation when $\beta = 0$ [24,25]. Eq. (2) is used to govern the propagation of ion-acoustic waves in a cold-ion plasma where some of the electrons do not behave isothermally during the passage of the wave but are trapped in it. The square root in the nonlinear term then translates to lowest order some of the kinetic effects, associated with electron trapping. Since these equations describe many phenomena in plasma physics and optical fibre, it is important to find exact wave solutions of Eqs. (1) and (2).

The extended $(\frac{G'}{G})$-expansion method, based on the $(\frac{G'}{G})$-expansion method, is used to obtain exact solutions of Eqs. (1) and (2). The main difference between this method and Wang's $(\frac{G'}{G})$-expansion method is that we assume a new symmetric form $U(\xi) = a_0 + \sum_{i=-m}^{m} [a_i (\frac{G'}{G})^i]$ for the solutions, instead of $U(\xi) = \sum_{i=0}^{m} [a_i (\frac{G'}{G})^i]$ in his method. This method has some pronounced merits over other methods like sine-cosine method, differential transform method, exp-function method, homotopy perturbation method: The solution procedure is direct and simple. The general solution has been obtained this method without approximation. The initial and boundary conditions haven't been required. The availability of computer systems like Maple, Mathematica or Matlab facilitates the tedious algebraic calculations. This method is elementary and effective. The obtained exact solutions are important to expose most complex physical phenomena or to find new phenomena. Moreover, these solutions aid the numerical solvers to assess the correctness of their results. We have noted that this method changes the given difficult problems which can be solved easily.

2. Description of the extended $(\frac{G'}{G})$-expansion method

Suppose that a non-linear partial equation is given by:

$$Q(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0 \tag{3}$$

where $u = u(x, t)$ is an unknown function, Q is a polynomial in $u = u(x, t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following lines, we give the main steps of the extended $(\frac{G'}{G})$-expansion method [14].

Step 1. The travelling wave variable as follow:

$$u = u(x, t) = U(\xi), \xi = x - ct.$$

where c is constant. After transform Eq. (3) is reduced to an ODE for $u = U(\xi)$ in the form:

$$Q'(U, U', -cU', -c^2U'', -cU'', U''', \ldots) = 0 \tag{4}$$

Step 2. Supposed that the solutions of Eq. (4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U(\xi) = a_0 + \sum_{i=1}^{m} \left[a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{-i} \right] \tag{5}$$

where $G = G(\xi)$ satisfies the second order LODE in the form:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{6}$$

where $a_0, a_i, b_i (i = 1, 2, \ldots, m)$, c, λ and μ are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and highest order non-linear terms appearing in Eq. (4).

Step 3. Substituting Eq. (5) into Eq. (4) and using Eq. (6), collecting all terms with the same power of $(\frac{G'}{G})$ together, and then equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for a_0, a_i, b_i $(i = 1, 2, \ldots, m)$, c, λ and μ.

Step 4. As a final step, since the general solutions of the second order LODE Eq. (6) have been well known for us, then a_0, a_i, b_i, λ, μ,c and the general solutions of Eq. (6) into Eq. (5) we have more travelling wave solutions of the Eq. (3). Solutions of Eq. (6) depending on whether $\lambda^2 - 4\mu > 0$, $\lambda^2 - 4\mu < 0$, $\lambda^2 - 4\mu = 0$,

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0 \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{C_1 \cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi - C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}{C_1 \sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi + C_2 \cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0 \\ \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right), & \lambda^2 - 4\mu = 0 \end{cases} \tag{7}$$

where C_1 and C_2 are arbitrary constants.

3. Applications to the Schamel–Korteweg–de Vries (s-KdV) and Schamel equations

In this section, we apply the extended $(\frac{G'}{G})$-expansion method to construct the travelling wave solutions for the s-KdV and the Schamel equations.

3.1. The Schamel–Korteweg–de Vries equation

Let us consider, the Schamel–Korteweg–de Vries equation [24,25]

$$u_t + (\alpha u^{\frac{1}{2}} + \beta u)u_x + \delta u_{xxx} = 0 \tag{8}$$

Let

$$u(x,t) = v^2(x,t), \quad v(x,t) = V(\xi), \quad \xi = x - ct, \tag{9}$$

where α, β, δ and c are constants. After the transform (9), we get:

$$-cVV' + (\alpha V^2 + \beta V^3)V' + \delta[VV''' + 3V'V''] = 0 \tag{10}$$

To determine the parameter m, we balance the linear terms of highest order in Eq. (10) with the highest order nonlinear terms. This in turn gives $m = 1$, so the solution of Eq. (8) is of the form:

$$V(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + b_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1}, \tag{11}$$

Substituting Eqs. (5) and (6) into Eq. (10), collection the coefficients of $(\frac{G'}{G})^m$, $(\frac{G'}{G})^{-m}(m = 0, 1, 2, 3, 4, 5)$ and set it zero we obtain the system (see Appendix A).

Solving this system by MAPLE gives

Case 1:

$$a_0 = \frac{2\alpha}{5\beta}\left(\pm\frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} - 1\right),$$

$$a_1 = 0, \quad b_1 = \pm\frac{4\mu\alpha}{5\beta}\frac{1}{\sqrt{\lambda^2 - 4\mu}}$$

$$c = -\frac{16\alpha^2}{75\beta}, \quad \delta = \frac{4\alpha^2}{75\beta(-\lambda^2 + 4\mu)} \tag{12}$$

Case 2:

$$a_0 = \frac{2\alpha}{5\beta}\left(\pm\frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} - 1\right), \quad a_1 = \pm\frac{4\alpha}{5\beta}\frac{1}{\sqrt{\lambda^2 - 4\mu}}$$

$$b_1 = 0, \quad \delta = \frac{4\alpha^2}{75\beta(-\lambda^2 + 4\mu)}, \quad c = -\frac{16\alpha^2}{75\beta} \tag{13}$$

Case 3:

$$a_0 = -\frac{4\alpha}{5\beta}, \quad a_1 = -\frac{4\alpha}{5\beta\lambda}, \quad b_1 = -\frac{4\alpha\mu}{5\beta\lambda},$$

$$c = \frac{16\alpha^2(-\lambda^2 + 4\mu)}{75\beta\lambda^2}, \quad \delta = \frac{-4\alpha^2}{75\beta\lambda^2} \tag{14}$$

Case 4:

$$a_0 = \frac{\delta}{2\alpha}(\pm 15(\sqrt{\lambda^2 - 4\mu}\lambda + (\lambda^2 - 4\mu))),$$

$$a_1 = 0, \quad b_1 = \pm\frac{15\mu\delta}{\alpha}\sqrt{\lambda^2 - 4\mu}$$

$$c = -4(-\lambda^2 + 4\mu)\delta, \quad \beta = \frac{4\alpha^2}{75\delta(-\lambda^2 + 4\mu)} \tag{15}$$

Case 5:

$$a_0 = -\frac{4\alpha}{5\beta}, \quad a_1 = -\frac{4\alpha}{5\beta\lambda}, \quad b_1 = -\frac{4\alpha\mu}{5\beta\lambda},$$

$$\delta = \frac{-4\alpha^2}{75\beta\lambda^2}, \quad c = -\frac{16\alpha^2(-\lambda^2 + 4\mu)}{75\beta\lambda^2} \tag{16}$$

Substituting the solution sets (12)–(16) and the corresponding solutions of Eq. (6) into Eq. (11), we have the solutions of Eq. (8) as follows:

Case 1: If $\lambda^2 - 4\mu > 0$, we have the hyperbolic type

$$u_1(x,t) = \left(\pm\frac{2\alpha}{5\beta}\frac{1}{\sqrt{\lambda^2 - 4\mu}} \times \left(\lambda + \frac{4\mu}{\sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}\right) - \lambda}\right) - \frac{2\alpha}{5\beta}\right)^2$$

where $\xi = x + \frac{16\alpha^2}{75\beta}t$.

If $\lambda^2 - 4\mu < 0$, we have the trigonometric type

$$u_2(x,t) = \left(\mp\frac{2\alpha}{5\beta}\frac{1}{\sqrt{4\mu - \lambda^2}}i \times \left(\lambda + \frac{4\mu}{\sqrt{4\mu - \lambda^2}\left(\frac{C_1 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi - C_2 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + C_2 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}\right) - \lambda}\right) - \frac{2\alpha}{5\beta}\right)^2$$

where $\xi = x + \frac{16\alpha^2}{75\beta}t$.

Case 2: If $\lambda^2 - 4\mu > 0$, we have the hyperbolic type

$$u_3(x,t) = \left(-\frac{2\alpha}{5\beta}\left(1 \mp \left(\frac{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi}{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi}\right)\right)\right)^2$$

where $\xi = x + \frac{16\alpha^2}{75\beta}t$.

If $\lambda^2 - 4\mu < 0$, we have the trigonometric type

$$u_4(x,t) = \left(-\frac{2\alpha}{5\beta}\left(1 \mp i\left(\frac{C_1 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi - C_2 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi}{C_1 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi + C_2 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi}\right)\right)\right)^2$$

where $\xi = x + \frac{16\alpha^2}{75\beta}t$.

Case 3: If $\lambda^2 - 4\mu > 0$, we have the hyperbolic type

$$u_5(x,t) = \left(-\frac{8\alpha}{5\beta\lambda}\left(\frac{1}{4}\sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right) + \frac{\mu}{\left(\sqrt{\lambda^2-4\mu}\frac{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi} - \lambda\right)}\right) - \frac{2\alpha}{5\beta}\right)^2$$

where $\xi = x + \frac{16\alpha^2(\lambda^2-4\mu)}{75\beta\lambda^2}t$.

If $\lambda^2 - 4\mu < 0$, we have the trigonometric type

$u_6(x,t)$

$$= \left(-\frac{8\alpha}{5\beta\lambda}\left(\frac{1}{4}\sqrt{4\mu-\lambda^2}\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi - C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} \right) + \frac{\mu}{\left(\sqrt{4\mu-\lambda^2}\,\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi - C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} - \lambda \right)} \right) - \frac{2\alpha}{5\beta} \right)^2$$

where $\xi = x + \frac{16\alpha^2(\lambda^2-4\mu)}{75\beta\lambda^2}t$.

Case 4: If $\lambda^2 - 4\mu > 0$, we have the hyperbolic type

$$u_7(x,t) = \left(\pm\frac{15\delta}{2\alpha}\sqrt{\lambda^2-4\mu}\left(\lambda + \frac{4\mu}{\sqrt{\lambda^2-4\mu}\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi} \right) - \lambda} \right) + \frac{15\delta}{2\alpha}(\lambda^2-4\mu) \right)^2$$

where $\xi = x - 4(\lambda^2 - 4\mu)t$.

If $\lambda^2 - 4\mu < 0$, we have the trigonometric type

$$u_8(x,t) = \left(\pm\frac{15\delta}{2\alpha}i\sqrt{4\mu-\lambda^2}\left(\lambda + \frac{4\mu}{\sqrt{4\mu-\lambda^2}\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi - C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} \right) - \lambda} \right) - \frac{15\delta}{2\alpha}(4\mu-\lambda^2) \right)^2$$

where $\xi = x - 4(\lambda^2 - 4\mu)t$.

Case 5: If $\lambda^2 - 4\mu > 0$, we have the hyperbolic type

$$u_9(x,t) = \left(\frac{2\alpha}{5\beta}\left(\pm\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi} \right) - 1 \right) \right)^2$$

where $\xi = x + \frac{16\alpha^2(\lambda^2-4\mu)}{75\beta\lambda^2}t$.

If $\lambda^2 - 4\mu < 0$, we have the trigonometric type

$$u_{10}(x,t) = \left(\frac{2\alpha}{5\beta}\left(\pm i\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi - C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} \right) - 1 \right) \right)^2$$

where $\xi = x + \frac{16\alpha^2(\lambda^2-4\mu)}{75\beta\lambda^2}t$.

3.2. The Schamel equation

Let us consider, the Schamel equation

$$u_t + u^{\frac{1}{2}}u_x + \delta u_{xxx} = 0 \tag{17}$$

Let

$$u(x,t) = v^2(x,t),\ v(x,t) = V(\xi),\ \xi = x - ct, \tag{18}$$

where δ and c are constants. After the transform (18), we get:

$$-cVV' + V^2V' + \delta[VV''' + 3V'V''] = 0 \tag{19}$$

Integrating Eq. (19) with respect to ξ and setting the integration constant equal to zero, we have

$$-\frac{c}{2}V^2 + \frac{1}{3}V^3 + \delta(V')^2 + \delta VV'' = 0 \tag{20}$$

Balancing (20) we get $m = 2$, so the solution of (8) is of the form:

$$V(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2}, \tag{21}$$

Substituting Eqs. (5) and (6) into (20), collection the coefficients of $(\frac{G'}{G})^m$, $(\frac{G'}{G})^{-m} (m = 0, 1, 2, 3, 4, 5, 6)$ and set it zero we obtain the system (see Appendix B).Solving this system by MAPLE gives

Case 1:

$$\begin{aligned} a_0 &= \pm\frac{3}{4}\sqrt{\lambda^2-4\mu}\lambda + \frac{3\lambda^2}{4} - 3\mu), \\ a_1 &= \pm\frac{3}{2}\sqrt{\lambda^2-4\mu},\ a_2 = 0,\ b_1 = 0,\ b_2 = 0 \\ c &= -4\mu + \lambda^2,\ \delta = -\frac{1}{2} \end{aligned} \tag{22}$$

Case 2:

$$\begin{aligned} a_0 &= 6\lambda^2, \quad a_1 = 24\lambda,\ a_2 = 24 \\ b_1 &= 0,\ b_2 = 0,\ \delta = -2, \quad c = 4\lambda^2 - 16, \end{aligned} \tag{23}$$

Case 3:

$$\begin{aligned} a_0 &= \frac{12(\frac{\lambda}{2} \pm \frac{\sqrt{\lambda^2-4\mu}}{2})\lambda}{5} - \frac{12\mu}{5}, \quad a_1 = \frac{12\lambda}{5} \pm \frac{12}{5}\sqrt{\lambda^2-4\mu} \\ a_2 &= \frac{12}{5}, \quad b_1 = 0, \quad b_2 = 0, \quad c = \frac{-32\mu}{5} + \frac{8\lambda^2}{5}, \quad \delta = -\frac{4}{5} \end{aligned} \tag{24}$$

Substituting the solution sets ([22,24]) and the corresponding solutions of Eq. (6) into Eq. (21), we have the solutions of Eq. (17) as follows:

Case 1: If $\lambda^2 - 4\mu > 0$, we have the hyperbolic type

$$u_{11}(x,t) = \left(\pm\frac{3}{4}\sqrt{\lambda^2-4\mu} \times \left[\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi} \right) \right] + \frac{3\lambda^2}{4} - 3\mu \right)^2$$

where $\xi = x - (\lambda^2 - 4\mu)t$.

If $\lambda^2 - 4\mu < 0$, we have the trigonometric type

$$u_{12}(x,t) = \left(\pm\frac{3}{4}i(4\mu-\lambda^2) \times \left[\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi - C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} \right) \right] + \frac{3\lambda^2}{4} - 3\mu \right)^2$$

where $\xi = x - (\lambda^2 - 4\mu)t$.

Case 2: If $\lambda^2-4\mu>0$, we have the hyperbolic type

$$u_{13}(x,t)=\left[6(\lambda^2-4\mu)\times\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)^2\right]^2$$

where $\xi=x-(4\lambda^2-16)t$.

If $\lambda^2-4\mu<0$, we have the trigonometric type

$$u_{14}(x,t)=\left[6(4\mu-\lambda^2)\times\left(\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi-C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}\right)^2\right]^2$$

where $\xi=x-(4\lambda^2-16)t$.

Case 3: When $\lambda^2-4\mu>0$, $K=\frac{C_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+C_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{C_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+C_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}$, we have the hyperbolic type

$$u_{15}(x,t)=\left[\pm\frac{6}{5}(\lambda^2-4\mu)K+\frac{3}{5}(\lambda^2-4\mu)K^2-\frac{12\mu}{5}+\frac{3}{5}\lambda^2\right]^2$$

where $\xi=x-(\frac{8\lambda^2}{5}-\frac{32\mu}{5})t$.

When $\lambda^2-4\mu<0$, $M=\frac{C_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi-C_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}{C_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi+C_2\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}$, we have the trigonometric type

$$u_{16}(x,t)=\left[\pm\frac{6}{5}i(4\mu-\lambda^2)M+\frac{3}{5}(4\mu-\lambda^2)M^2-\frac{12\mu}{5}+\frac{3}{5}\lambda^2\right]^2$$

where $\xi=x-(\frac{8\lambda^2}{5}-\frac{32\mu}{5})t$.

4. Conclusion

In this paper, travelling wave solutions in hyperbolic function form and trigonometric function form of the Schamel and Schamel–KdV equations are successfully found out by using the extended $(\frac{G'}{G})$-expansion method. The obtained solutions with free parameters may be important to expose most complex physical phenomena or to find new phenomena. It is shown in this paper that the extended $(\frac{G'}{G})$-expansion method, with the help of symbolic computation like Maple or Mathematica, is direct, concise, elementary and compared to other methods, like the sine-cosine method, exp-function method, homotopy perturbation method it is easier, effective and powerful in finding exact solutions of many other NLEEs in mathematical physics, applied mathematics and engineering. We checked the correctness of the obtained results by putting them back into the original equation with the aid of MAPLE. This provides an extra measure of confidence in the results.

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Appendix A

$$\left(\frac{G'}{G}\right)^0: -\alpha\mu a_0^2a_1-\beta\mu a_0^3a_1-\delta\lambda^2\mu a_0a_1+\alpha b_1a_0^2+2\delta\mu b_1a_0$$
$$-\alpha\mu a_1^2b_1-3\delta\lambda\mu^2a_1^2+3\beta a_0a_1b_1^2+\alpha a_1b_1^2+\beta a_0^3b_1$$
$$-2\delta\mu^2a_0a_1+\delta\lambda^2a_0b_1-ca_0b_1-3\beta\mu a_0a_1^2b_1+c\mu a_0a_1+3\delta\lambda b_1^2$$

$$\left(\frac{G'}{G}\right)^1: -\alpha\lambda a_0^2a_1+c\lambda a_0a_1-3\beta\mu a_0^2a_1^2-2\beta\mu a_1^3b_1-\alpha\lambda a_1^2b_1$$
$$-8\delta\lambda\mu a_0a_1+c\mu a_1^2-2\alpha\mu a_0a_1^2-\beta\lambda a_0^3a_1-3\beta\lambda a_0a_1^2b_1$$
$$-7\delta\lambda^2\mu a_1^2-8\delta\mu^2a_1^2-\delta\lambda^3a_0a_1$$

$$\left(\frac{G'}{G}\right)^2: -\alpha\mu a_1^3-\alpha a_1^2b_1-3\beta\lambda a_0^2a_1^2+c\lambda a_1^2-2\alpha\lambda a_0a_1^2+ca_0a_1$$
$$-3\beta a_0a_1^2b_1-2\beta\lambda a_1^3b_1-7\delta\lambda^2a_0a_1-8\delta\mu a_0a_1-\beta a_0^3a_1$$
$$-26\delta\lambda\mu a_1^2-3\beta\mu a_0a_1^3-4\delta\lambda^3a_1^2-\alpha a_0^2a_1$$

$$\left(\frac{G'}{G}\right)^3: -3\beta a_0^2a_1^2-20\delta\mu a_1^2-\alpha\lambda a_1^3-2a_1^3b_1-12\delta\lambda\mu a_0a_1$$
$$-2\alpha a_0a_1^2-3\beta\lambda a_1^3a_0-\beta\mu a_1^4+ca_1^2-19\delta\lambda^2a_1^2$$

$$\left(\frac{G'}{G}\right)^4: -6\delta a_0a_1-27\delta\lambda a_1^2-3\beta a_0a_1^3-\beta\lambda a_1^4-\alpha a_1^3$$

$$\left(\frac{G'}{G}\right)^5: -12\delta a_1^2-\beta a_1^4$$

$$\left(\frac{G'}{G}\right)^{-1}: \beta\lambda a_0^3b_1+2\alpha a_0b_1^2+\alpha\lambda a_0^2b_1+28\delta\mu b_1^2+\delta\lambda^3a_0b_1$$
$$+2\beta a_1b_1^3+3\beta\lambda a_0a_1b_1^2+3\beta a_0^2b_1^2+7\delta\lambda^2b_1^2$$
$$+8\delta\lambda\mu a_0b_1-c\lambda a_0b_1+\alpha\lambda a_1b_1^2-cb_1^2$$

$$\left(\frac{G'}{G}\right)^{-2}: 8\delta\mu b_1^2+26\delta\lambda\mu b_1^2+3\beta\lambda a_0^2b_1^2-c\mu a_0b_1+2\alpha\lambda a_0b_1^2$$
$$+3\beta a_0b_1^3+\alpha b_1^3+3\beta a_0a_1b_1^2+\alpha\mu a_1b_1^2+2\beta\lambda a_1b_1^3$$
$$-c\lambda b_1^2+4\delta\lambda^3b_1^3+\alpha\mu a_0^2b_1+\beta\mu a_0^3b_1+7\delta\lambda^2\mu a_0b_1$$

$$\left(\frac{G'}{G}\right)^{-3}: 3\beta\mu a_0^2b_1^2+3\beta\lambda a_0b_1^3+19\delta\mu\lambda^2b_1^2+20\delta\mu^2b_1^2+\beta b_1^4$$
$$+\alpha\lambda b_1^3+2\alpha\mu a_0b_1^2-c\mu b_1^2+2\beta\mu a_1b_1^3+12\delta\lambda\mu^2a_0b_1$$

$$\left(\frac{G'}{G}\right)^{-4}: 27\delta\lambda\mu^2b_1^2+\beta\lambda b_1^4+6\delta\mu^3a_0b_1+3\beta\mu a_0b_1^3+\alpha\mu b_1^3$$

$$\left(\frac{G'}{G}\right)^{-5}: \beta\mu b_1^4+12\delta\mu^3b_1^2$$

Appendix B

$$\left(\frac{G'}{G}\right)^0: \frac{a_0^3}{3}-\frac{ca_0^2}{2}+a_1^2\mu^2-2a_1b_1\lambda^2-8a_1b_2\lambda+a_1^2b_2$$
$$-ca_1b_1-ca_2b_2+2a_0a_1b_1+2a_0a_2b_2+b_1^2a_2-4a_1b_1\mu$$
$$-8a_2b_1\lambda\mu+b_1^2-8a_2b_2\lambda^2-16a_2b_2\mu+\delta a_1b_1\lambda^2+9\delta a_1b_2\lambda$$
$$+2\delta a_1b_1\mu+2\delta a_0a_2\mu^2-\delta a_0b_1\lambda+8\delta a_2b_2\lambda^2+16\delta a_2b_2\mu$$
$$+2\delta a_0b_2+7\delta a_2b_1\lambda\mu-\delta b_1^2+\delta a_0a_1\lambda\mu$$

$$\left(\frac{G'}{G}\right)^1: a_1^2b_1+2\delta a_1b_1\lambda+2\delta a_1a_2\mu^2-\delta a_0b_1+8\delta b_1a_2\mu-ca_0a_1$$
$$+\delta a_1^2\lambda\mu-4a_1b_2-cb_1a_2+\delta a_0a_1\lambda^2+4a_1a_2\mu^2+2a_1^2\lambda\mu$$

$$+4\delta b_1a_2\lambda^2+2a_1a_2b_2+6\delta a_0a_2\lambda\mu+4\delta a_1b_2+2\delta a_0a_1\mu$$
$$-4a_1b_1\lambda-4a_2b_1\lambda^2-8a_2b_1\mu-16a_2b_2\lambda+2a_0a_2b_1$$
$$+a_0^2a_1+16\delta a_2b_2\lambda$$

$$\left(\frac{G'}{G}\right)^2:\frac{-ca_1^2}{2}+8a_1a_2\lambda\mu-ca_0a_2+2a_1a_2b_1+\delta a_1^2\lambda^2+4a_2^2\mu^2$$
$$+7\delta a_1a_2\lambda\mu+2a_1^2\mu+9\delta a_2b_1\lambda++4\delta a_0a_2\lambda^2+\delta a_1b_1$$
$$+8\delta a_2b_2-8b_1a_2\lambda+b_2a_2^2+8\delta a_0a_2\mu+a_1^2\lambda^2+2\delta a_2^2\mu^2$$
$$-2a_1b_1+2\delta a_1^2\mu-8a_2b_2+3\delta a_0a_1\lambda+a_0a_1^2+a_0^2a_2$$

$$\left(\frac{G'}{G}\right)^3:a_2^2b_1+2a_0a_1a_2+2a_1^2\lambda+\frac{a_1^3}{3}+4a_1a_2\lambda^2+6\delta a_2^2\lambda\mu+5\delta b_1a_2$$
$$+10\delta a_0a_2\lambda+10\delta a_1a_2\mu+2\delta a_0a_1+8a_2^2\lambda\mu-ca_1a_2+8a_1a_2\mu$$
$$+5\delta a_1a_2\lambda^2-4b_1a_2+3\delta a_1^2\lambda+b_1^2\lambda^2+8b_1b_2\lambda+a_0^2b_2-\frac{cb_1^2}{2}$$
$$+2\delta b_2^2+2b_1^2\mu+8\delta a_0b_2\mu+5\delta b_1b_2\lambda+4\delta a_0b_2\lambda^2+2a_1b_1b_2$$

$$\left(\frac{G'}{G}\right)^4:8a_1a_2\lambda+a_1a_2^2+4a_2^2\lambda^2+13\delta a_1a_2\lambda+4\delta a_2^2\lambda^2+6\delta a_0a_2$$
$$-\frac{ca_2^2}{2}+a_1^2a_2+8a_2^2\mu+2\delta a_1^2+8\delta a_2^2\mu+a_1^2$$

$$\left(\frac{G'}{G}\right)^5:8\delta a_1a_2+a_1a_2^2+8a_2^2\lambda+4a_1a_2+10\delta a_2^2\lambda$$

$$\left(\frac{G'}{G}\right)^6:4a_2^2+6\delta a_2^2+\frac{1}{3}a_2^3$$

$$\left(\frac{G'}{G}\right)^{-1}:-4b_1a_2\mu^2-4a_1b_1\lambda\mu-cb_2a_1+\delta b_1b_2-\delta b_1^2\lambda+2a_1a_0b_1$$
$$+3\delta b_1a_2\mu^2+2b_1b_2a_2+10\delta a_1b_2\mu+16\delta a_2b_2\lambda\mu+2\delta a_1b_1\lambda\mu$$
$$+4b_1b_2-8a_1b_2\mu+6\delta a_0b_2\lambda-4a_1b_2\lambda^2+2b_1^2\lambda+a_0^2b_1-ca_0b_1$$
$$+a_1b_1^2-16a_2b_2\lambda\mu+5\delta a_1b_2\lambda^2$$

$$\left(\frac{G'}{G}\right)^{-2}:\delta a_0b_1\lambda\mu+4b_2^2+a_0b_1^2-8a_2b_2\mu^2-8a_1b_2\lambda\mu$$
$$+11\delta a_1b_2\lambda\mu-ca_0b_2+\delta a_1b_1\mu^2+8\delta a_2b_2\mu^2+b_1^2\lambda^2$$
$$+8b_1b_2\lambda+a_0^2b_2-\frac{cb_1^2}{2}+2\delta b_2^2+2b_1^2\mu$$
$$+8\delta a_0b_2\mu+5\delta b_1b_2\lambda+4\delta a_0b_2\lambda^2+2a_1b_1b_2-2a_1b_1\mu^2+a_2b_2^2$$

$$\left(\frac{G'}{G}\right)^{-3}:8\delta b_1b_2\mu+8b_1b_2\mu+\frac{b_1^3}{3}+\delta a_0b_1\mu^2+4b_1b_2\lambda^2$$
$$-4a_1b_2\mu^2+8b_2^2\lambda+6\delta a_1b_2\mu^2-cb_1b_2+a_1b_2^2+2a_0b_1b_2$$
$$+10\delta a_0b_2\lambda\mu+\delta b_1^2\lambda\mu+2b_1^2\lambda\mu+6\delta b_2^2\lambda+4\delta b_1b_2\lambda^2$$

$$\left(\frac{G'}{G}\right)^{-4}:-\frac{cb_2^2}{2}+b_1^2b_2+\delta b_1^2\mu^2+4\delta b_2^2\lambda^2+8\delta b_2^2\mu+8b_1b_2\lambda\mu$$
$$+4b_2^2\lambda^2+a_0b_2^2+8b_2^2\mu+b_1^2\mu^2$$

$$\left(\frac{G'}{G}\right)^{-5}:10\delta b_2^2\lambda\mu+4b_1b_2\mu^2+7\delta b_1b_2\mu^2+8b_2^2\lambda\mu+b_1b_2{}^2$$

$$\left(\frac{G'}{G}\right)^{-6}:4b_2^2\mu^2+\frac{1}{3}b_2^3+6\delta b_2^2\mu^2.$$

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