Journal of the Egyptian Mathematical Society (2016) **24**, 508–514

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

On classifications of rational sextic curves

Mohammed A. Saleem

Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

Received 2 July 2014; revised 5 May 2015; accepted 19 May 2015
Available online 16 June 2015

Keywords
Rational curves;
Sextic curves;
Singularities

Abstract In this paper, we extend the Yang's list of reduced sextic plane curves to rational irreducible projective plane curves of type $(6, 3, 1)$.

Mathematics Subject Classification: 14H45; 14R20; 14H30; 14H50

1. Introduction

The determination of all possible configurations of plane algebraic curves for a given degree d is one of the classical and interesting problems in algebraic geometry. Throughout this paper, we work over the field of the complex numbers $\mathbb{C}$. We denote by $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ the projective plane over the field of the complex numbers. The genus is a geometric invariant associated with the curve C, and in the case of $C \subset \mathbb{P}^2$ by a Theorem of Noether (see for instant [1] page 614 or [2] page 222) can be computed as

$$g = \frac{(d-1)(d-2)}{2} - \sum_{P \in \mathrm{Sing}(C)} \frac{m_P(m_P - 1)}{2},$$

where $\mathrm{Sing}(C)$ is the singular points of the curve C and m_P denotes the multiplicity of the singularities of $P \in C$ (including the infinitely near points of P). This invariant plays a very important role in algebraic geometry. For instance, plane curves C with $g = 0$ are called rational curves. In case $g = 1, 2$, C are called elliptic and hyperelliptic curves, respectively. Also, by the genus formula, we easily see that, the lines and the conics have no singular points and an irreducible cubic has at most one double point.

Yoshihara in [3–5] , classified plane curves with small degrees whose singular points are only cusps. Curves of degrees $d = 4, 5$ and 6 are called quartic, quintic and sextic curves, respectively. We focus in this paper on a very important type of curves that is irreducible rational projective plane sextic curves.

Let $P \in C$ be a singular point, and let r_P be the number of the branches of C at P. Put $\iota(C) = \sum_{P \in \mathrm{Sing}(C)} (r_P - 1)$. The notation (d, ν, ι) is used for curves of degree d, maximal multiplicity of the singularities ν and $\iota = \iota(C)$. For $r_P = 1$, P is called a cusp. In case $r_P \geq 2$, Saleem in [7], introduced the notion of the system of the multiplicity sequences of the branches of the curve C at P which explains after how many times of blowing ups of C at P the branches separate from each other.

Yang in [6], gave a list of reduced sextic curves. In his list, he showed the existence of the configurations of these curves. Here, we extend to give a list of irreducible rational projective sextic curves of type $(6, 3, 1)$.

E-mail address: abuelhassan@yahoo.com
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2015 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. This is an open access article under the CC BY-NC-ND license. (http://creativecommons.org/licenses/by-nc-nd/4.0/)
http://dx.doi.org/10.1016/j.joems.2015.05.001

In [8], Sakai and Saleem classified all rational plane curves of type $(d, d-2)$, possibly with multibranched singularities. They generalized the results with Tono in [9] to plane curves of type $(d, d-2)$ with any genus. It turns out that still the answer of Matsuka and Sakai's conjectured in [10], is affirmative. As a generalization of these results, we have the following question: Is any rational plane curve of type $(d, d-3)$ is transformable into a line by a Cremona transformation? The cuspidal case has been already discussed and answered affirmatively by Flenner-Zaidenberg in [11,12] and Fenske in [13]. In this paper, we answer the question for some classes of rational plane curves of types $(6, 3, 1)$.

For constructing curve germs, Sakai and Tono in [14] used the quadratic Cremona transformations $\varphi_c : (x, y, z) \to (xy, y^2, x(z-cx))$ for $c \in \mathbb{C}$, where (x, y, z) are homogeneous coordinates on $\mathbb{P}^2$.

2. Preliminaries

In this section, we investigate a tool for constructing plane curves which is the Cremona transformations $\varphi_c : (x, y, z) \to (xy, y^2, x(z-cx))$ for $c \in \mathbb{C}$.

2.1. Singularities on plane curves

Let $(C, P) \subset (\mathbb{C}^2, P)$ be a plane curve germ, where $P \in C$ is a singular point. We obtain the minimal embedded resolution of the singularity (C, P), by means of a sequence of blowing-ups $X_i \overset{\pi_i}{\to} X_{i-1}$, $i = 1, 2, \ldots, k$, over P. Let $C^{(i)} \subset X_i$ be the strict (also called proper) transform of C in X_i and E is the exceptional divisor of the whole resolution. Hence, the total transform of C in X_k is a simple normal crossing (SNC) divisor $D = E + C^{(k)}$ as in the following diagram:

$$C^{(k)} \xrightarrow{\pi_k} C^{(k-1)} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} C^{(1)} \xrightarrow{\pi_1} C = C^{(0)},$$

where k is a finite positive integer. We recall the properties of the multiplicity sequence $\underline{m}_P(C) = (m_0, m_1, \ldots, m_k)$ of (C, P). Let m_i be the multiplicity of $C^{(i)}$ at P_i, where P_i is the infinitely near point of P on $C^{(i)}$. We define the *multiplicity sequence* of (C, P) to be $\underline{m}_P(C) = (m_0, m_1, \ldots, m_k)$, where $m_0 \geq m_1 \geq \cdots \geq m_k = 1$. For the sequence $(\overbrace{m, \ldots, m}^{a-\text{times}}, 1, 1)$, we write (m_a).

Here, we recall the definition of *the system of the multiplicity sequences of* $P \in C$ in case the number of the branches of C at P equals 2, (see [7,8]for more details).

Definition 1. The systems of the multiplicity sequences of two branches are defined as follows:

$$\underline{m}_P(\zeta_1, \zeta_2) = \left\{\begin{pmatrix} m_{1,0} \\ m_{2,0} \end{pmatrix} \cdots \begin{pmatrix} m_{1,\rho} \\ m_{2,\rho} \end{pmatrix} \begin{matrix} m_{1,\rho+1}, m_{1,\rho+2}, \ldots, m_{1,s_1} \\ m_{2,\rho+1}, m_{2,\rho+2}, \ldots, m_{2,s_2} \end{matrix}\right\},$$

where the brackets mean that the germs go through the same infinitely near points of P and $\underline{m}_P(\zeta_i) = (m_{i,0}, m_{i,1}, \ldots, m_{i,s_i})$ are the multiplicity sequences of the branches (ζ_i, P), $i = 1, 2$, of the germ (C, P).

For a classification of a bibranched singular point Q with multiplicity $d-3$, we give the following proposition.

Proposition 1 ([7]). *Let C be a rational plane curve of type $(d, d-3)$. Let $Q \in C$be a bibranched singular point with multiplicity $d-3$. Then, the system of the multiplicity sequences of Q are divided into the following two types $(r, s, v, k > 0$, $i, j \geq 0)$:*

(1) *Branches with the same tangent line*	(2) *Branches with different tangent lines*	
$\left\{\begin{pmatrix} k \\ k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{k+j}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 2r \end{pmatrix}\begin{matrix} 2_{k-1} \\ 2_{r+i} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 2k \\ 3r+2 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r, 2 \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ k \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2_i}_{k+j}\right\}$	$\left\{\begin{pmatrix} 2k \\ 2r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 2_{r+i} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r \end{pmatrix}\begin{matrix} 3_{k+v} \\ 3_{r+s} \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ k \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{k+j}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r \end{pmatrix}\begin{matrix} 2_{k-1} \\ 3_{r+s} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_{r+s} \end{matrix}\right\}$
$\left\{\begin{pmatrix} k \\ k+r \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{k}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_{r+s}, 2 \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_{r+s}, 2 \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ k+r \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2j}_{k}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r+1 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+1 \end{pmatrix}\begin{matrix} 3_{k+v} \\ 3_r \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ r \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2_{k+j-r}}_{r}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r+2 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r, 2 \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+1 \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_r \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k-1 \\ r \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2_{k-r-1}}_{r}\right\}$	$\left\{\begin{pmatrix} 2k \\ 3r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_{r+s} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+2 \end{pmatrix}\begin{matrix} 3_{k+v} \\ 3_{r+s}, 2 \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k-1 \\ k+j \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{k-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 2k \\ 3r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_{r+s}, 2 \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+2 \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_{r+s}, 2 \end{matrix}\right\}$
	$\left\{\begin{pmatrix} 2k \\ 3r+1 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k+1 \\ 3r+2 \end{pmatrix}\begin{matrix} 3_k \\ 3_r, 2 \end{matrix}\right\}$

Since we deal with curves of types $(6, 3, 1)$, then by using the above proposition and the other results in [7] we have the following Lemma.

Lemma 1 ([7]). *Let P be a unibranched or a bibranched singular point with multiplicity 3. Then, the system of the multiplicity sequences of P are divided into the following types* $(k > 0, i \geq 0)$:

Number of branches	System of the multiplicity sequences
1	(3_k), $(3_k, 2)$
2	$\left\{\begin{pmatrix}2\\1\end{pmatrix}_k \begin{pmatrix}1\\1\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}2\\1\end{pmatrix}_k^{2_i}\right\}$

2.1.1. *Quadratic Cremona transformation*

In this section, we give a tool to construct curve germs with one branch and two branches which we will use in this paper.

Let $(x, y, z) \in \mathbb{P}^2$ be homogeneous coordinates. Sakai and Tono in [14] defined the (degenerate) quadratic Cremona transformation $\varphi_c : (x, y, z) \to (xy, y^2, x(z - cx))$ for $c \in \mathbb{C}$. The inverse of this transformation is $\varphi_c^{-1}(x, y, z) = (x^2, xy, yz + cx^2)$. By a suitable change of coordinates, we can set the two lines l and t such that $l : x = 0, t : y = 0$ and the points O, A and B have the coordinates $O = (0, 0, 1), A = (1, 0, c)$ and $B = (0, 1, 0)$. We remark that the base points of φ_c are O, A and the infinitely near point of O which corresponds to the direction of l and the base points of φ_c^{-1} are O, B and the infinitely near point of O which corresponds to the direction of t (see also [11]).

Now, successive compositions of the quadratic Cremona transformations $\varphi = \varphi_{c_k} \circ \cdots \circ \varphi_{c_1}$ for $c_1, \ldots, c_k \in \mathbb{C}$ can be written as

$$\varphi^{-1}(x, y, z) = \left(x^{k+1}, x^k y, y^k z + \sum_{i=2}^{k+1} c_{k+2-i} x^i y^{k+1-i}\right).$$

Definition 2. Let $\mathrm{Sing}(C) = \{P_1, P_2, \ldots, P_s\}$ be the set of all the singular points on the rational plane curve C. The collection of the systems of the multiplicity sequences of C at the points P_i is called *the numerical data of C* and is written as $\mathrm{Data}(C) = [\underline{m}_{p_1}(C), \underline{m}_{p_2}(C), \ldots, \underline{m}_{p_s}(C)]$.

3. Main results

In this section, we construct some classes of rational plane curves of type $(6, 3)$ with a bibranched singular point and we show that these curves are transformable into a line by using suitable Cremona transformations.

Yang in [6], gave a list of sextic curves. He showed the existence of the configurations of these curves. Here, we extend to give a list of irreducible sextic curves of type $(6, 3, 1)$. Our result is written in the following theorem.

Theorem 3. *Let C be a rational plane curve of type* $(6, 3, 1)$. *Then,* Data(C) *are classified as follows (up to projective equivalent):*

ClassI (the maximal multiplicity is a unibranched singular point (cusp))				ClassII (the maximal multiplicity is a bibranched singular point with two coincident tangent lines)	
No.	Data(C)	No.	Data(C)	No.	Data(C)
1	$\left[(3,2), (2), (2), (2_3), \begin{pmatrix}1\\1\end{pmatrix}\right]$	29	$\left[(3), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_5\right]$	1	$\left[(2), (2), (2_2), (2_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
2	$\left[(3,2), (2_2), (2_3), \begin{pmatrix}1\\1\end{pmatrix}\right]$	30	$\left[(3), (2), \begin{pmatrix}1\\1\end{pmatrix}_6\right]$	2	$\left[(2), (2_2), (2_3), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
3	$\left[(3,2), (2), (2_4), \begin{pmatrix}1\\1\end{pmatrix}\right]$	31	$\left[(3), \begin{pmatrix}1\\1\end{pmatrix}_7\right]$	3	$\left[(2_3), (2_3), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
4	$\left[(3,2), (2_5), \begin{pmatrix}1\\1\end{pmatrix}\right]$	32	$\left[(3_2), (2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}\right]$	4	$\left[(2_2), (2_4), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
5	$\left[(3_2,2), (2), (2), \begin{pmatrix}1\\1\end{pmatrix}\right]$	33	$\left[(3_2), (2_3), \begin{pmatrix}1\\1\end{pmatrix}\right]$	5	$\left[(2), (2_5), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
6	$\left[(3_2,2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}\right]$	34	$\left[(3_2), (2), (2), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	6	$\left[(2_6), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
7	$\left[(3,2), (2), (2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	35	$\left[(3_2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	7	$\left[(3), (2), (2_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
8	$\left[(3,2), (2), (2_3), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	36	$\left[(3_2), (2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$	8	$\left[(3), (2_3), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
9	$\left[(3,2), (2_4), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	37	$\left[(3_2), \begin{pmatrix}1\\1\end{pmatrix}_4\right]$	9	$\left[(3), (3), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
10	$\left[(3,2), (2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$	38	$\left[(3_3), \begin{pmatrix}1\\1\end{pmatrix}\right]$	10	$\left[(3_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
11	$\left[(3,2), (2_3), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$	39	$\left[(3), (3,2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}\right]$	11	$\left[(3,2), (2), (2), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$

(*continued on next page*)

(*continued*)

ClassI (the maximal multiplicity is a unibranched singular point (cusp))				ClassII (the maximal multiplicity is a bibranched singular point with two coincident tangent lines)	
No.	Data(C)	No.	Data(C)	No.	Data(C)
12	$\left[(3_2, 2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$	40	$\left[(3), (3, 2), (2), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	12	$\left[(3, 2), (2_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
13	$\left[(3, 2), (2), (2), \begin{pmatrix}1\\1\end{pmatrix}_4\right]$	41	$\left[(3), (3, 2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$	13	$\left[(2), (2), (2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2, \begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
14	$\left[(3, 2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_4\right]$	42	$\left[(3, 2), (3, 2), (2), \begin{pmatrix}1\\1\end{pmatrix}\right]$	14	$\left[(2), (2_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2, \begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
15	$\left[(3, 2), (2), \begin{pmatrix}1\\1\end{pmatrix}_5\right]$	43	$\left[(3, 2), (3, 2), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	15	$\left[(2_3), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2, \begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
16	$\left[(3, 2), \begin{pmatrix}1\\1\end{pmatrix}_6\right]$	44	$\left[(3), (3), (2_3), \begin{pmatrix}1\\1\end{pmatrix}\right]$	16	$\left[(3), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2, \begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$
17	$\left[(3), (2), (2_2), (2_3), \begin{pmatrix}1\\1\end{pmatrix}\right]$	45	$\left[(3), (3), (2), (2), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	17	$\left\{\begin{pmatrix}2\\1\end{pmatrix}_3, \begin{pmatrix}1\\1\end{pmatrix}\right\}$
18	$\left[(3), (2), (2), (2_4), \begin{pmatrix}1\\1\end{pmatrix}\right]$	46	$\left[(3), (3), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$	18	$\left[(2_2), (2_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2\right\}\right]$
19	$\left[(3), (2_2), (2_4), \begin{pmatrix}1\\1\end{pmatrix}\right]$	47	$\left[(3), (3), (2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$	19	$\left[(2_4), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2\right\}\right]$
20	$\left[(3), (2), (2_5), \begin{pmatrix}1\\1\end{pmatrix}\right]$	48	$\left[(3), (3), (3), \begin{pmatrix}1\\1\end{pmatrix}\right]$	20	$\left[(3), (2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2\right\}\right]$
21	$\left[(3), (2_6), \begin{pmatrix}1\\1\end{pmatrix}\right]$	49	$\left[(3), (3), \begin{pmatrix}1\\1\end{pmatrix}_4\right]$	21	$\left[(3, 2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2\right\}\right]$
22	$\left[(3), (2_2), (2_3), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$			22	$\left[(2), (2), (2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^2\right\}\right]$
23	$\left[(3), (2), (2_4), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$			23	$\left[(2_3), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^2\right\}\right]$
24	$\left[(3), (2_5), \begin{pmatrix}1\\1\end{pmatrix}_2\right]$			24	$\left[(3), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^2\right\}\right]$
25	$\left[(3), (2), (2), (2), (2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$			25	$\left[(2_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^{2_2}\right\}\right]$
26	$\left[(3), (2_2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$			26	$\left\{\begin{pmatrix}2\\1\end{pmatrix}_2^{2_4}\right\}$
27	$\left[(3), (2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_4\right]$			27	$\left[(2), \left\{\begin{pmatrix}2\\1\end{pmatrix}_3\right\}\right]$
28	$\left[(3), (2_3), \begin{pmatrix}1\\1\end{pmatrix}_4\right]$			28	$\left\{\begin{pmatrix}2\\1\end{pmatrix}_3^2\right\}$

ClassIII (the maximal multiplicity is a bibranched singular with two different tangent lines)					
No.	Data(C)	No.	Data(C)	No.	Data(C)
1	$\left[(2), (2), (2_2), (2_3), \begin{pmatrix}2\\1\end{pmatrix}\right]$	15	$\left[(3_2, 2), \begin{pmatrix}2\\1\end{pmatrix}\right]$	29	$\left[(3), (2_2), \left\{\begin{pmatrix}2\\1\end{pmatrix}^{2_2}\right\}\right]$
2	$\left[(2_2), (2_2), (2_3), \begin{pmatrix}2\\1\end{pmatrix}\right]$	16	$\left[(3_2), (2), \begin{pmatrix}2\\1\end{pmatrix}\right]$	30	$\left[(3, 2), (2), \left\{\begin{pmatrix}2\\1\end{pmatrix}^{2_2}\right\}\right]$

(*continued on next page*)

(*continued*)

ClassIII (the maximal multiplicity is a bibranched singular with two different tangent lines)

No.	Data(C)	No.	Data(C)	No.	Data(C)
3	$\left[(2), (2), (2), (2_4), \binom{2}{1}\right]$	17	$\left[(3), (3, 2), \binom{2}{1}\right]$	31	$\left[(2), (2), (2_2), \left\{\binom{2}{1}^{2_3}\right\}\right]$
4	$\left[(2), (2_2), (2_4), \binom{2}{1}\right]$	18	$\left[(2), (2_2), (2_3), \left\{\binom{2}{1}^{2}\right\}\right]$	32	$\left[(2), (2_3), \left\{\binom{2}{1}^{2_3}\right\}\right]$
5	$\left[(2_3), (2_4), \binom{2}{1}\right]$	19	$\left[(2_3), (2_3), \left\{\binom{2}{1}^{2}\right\}\right]$	33	$\left[(2_4), \left\{\binom{2}{1}^{2_3}\right\}\right]$
6	$\left[(2), (2), (2_5), \binom{2}{1}\right]$	20	$\left[(2_2), (2_4), \left\{\binom{2}{1}^{2}\right\}\right]$	34	$\left[(3), (2), \left\{\binom{2}{1}^{2_3}\right\}\right]$
7	$\left[(2_2), (2_5), \binom{2}{1}\right]$	21	$\left[(2), (2_5), \left\{\binom{2}{1}^{2}\right\}\right]$	35	$\left[(3, 2), \left\{\binom{2}{1}^{2_3}\right\}\right]$
8	$\left[(2), (2_6), \binom{2}{1}\right]$	22	$\left[(2_6), \left\{\binom{2}{1}^{2}\right\}\right]$	36	$\left[\left\{(2), (2_2), \binom{2}{1}^{2_4}\right\}\right]$
9	$\left[(2_7), \binom{2}{1}\right]$	23	$\left[(3), (2_3), \left\{\binom{2}{1}^{2}\right\}\right]$	37	$\left[(2_3), \left\{\binom{2}{1}^{2_4}\right\}\right]$
10	$\left[(3), (2_2), (2_2), \binom{2}{1}\right]$	24	$\left[(3, 2), (2_2), \left\{\binom{2}{1}^{2}\right\}\right]$	38	$\left[(3), \left\{\binom{2}{1}^{2_4}\right\}\right]$
11	$\left[(3), (2), (2_3), \binom{2}{1}\right]$	25	$\left[(3_2), \left\{\binom{2}{1}^{2}\right\}\right]$	39	$\left[(2_2), \left\{\binom{2}{1}^{2_5}\right\}\right]$
12	$\left[(3), (2_4), \binom{2}{1}\right]$	26	$\left[(2), (2_2), (2_2), \left\{\binom{2}{1}^{2_2}\right\}\right]$	40	$\left[(2), \left\{\binom{2}{1}^{2_6}\right\}\right]$
13	$\left[(3, 2), (2), (2_2), \binom{2}{1}\right]$	27	$\left[(2_2), (2_3), \left\{\binom{2}{1}^{2_2}\right\}\right]$	41	$\left\{\binom{2}{1}^{2_7}\right\}$
14	$\left[(3, 2), (2_3), \binom{2}{1}\right]$	28	$\left[(2_5), \left\{\binom{2}{1}^{2_2}\right\}\right]$		

Remark 1. Rational plane curves of type (6, 3, 0) are classified in [13] as follows:

Class	Data(C)
(S1)	$(3_3, 2)$
($S2$)	$[(3_3), (2)]$
($S3$)	$[(3_2), (3, 2)]$
($S4$)	$[(3), (3_2, 2)]$

In [7] Appenthe singular points of thedix A, we give a complete comparison of all Fenske's curves with their defining equations. By applying a suitable quadratic Cremona transformations, we give a construction of cuspidal rational plane sextic curves. By a suitable change of coordinates, we set the two lines l and t and the points O, A, and B as follows: $l: x = 0, t: y = 0, O = (0, 0, 1),\ A = (1, 0, c)$ and $B = (0, 1, 0)$. In what follows, Applying φ_c, we construct the curve C' from the curve C, where C' is the strict transform of C via φ_c.

As a technique for choosing the initial curves C with a specific Data(C), we apply the inverse of a suitable quadratic Cremona transformations. These initial curves with given data are not unique (see [7], Section 4.2 for more details).

($S1$): We begin with the quintic curve C with Data(C) = $[(3, 2), (2_2)]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 3P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (3_3, 2)$ and $O' = S$ with $\underline{m}_{O'} = (1)$.

($S2$): We start with the quintic curve C with Data(C) = $\left[(3, 2), (2), \binom{1}{1}\right]$. We choose the lines l and t such that $l \cdot C = 2O + 3R$ and $t \cdot C = 3O + 2A$. We see that $R' = B$ with $\underline{m}_{R'} = (3_2, 2)$ and $O' = O$ with $\underline{m}_{O'} = (3)$.

($S3$): In this case we begin with the cuspidal cubic curve C. We choose two lines l and t such that $l \cdot C = 3R$ and $t \cdot C = 3P$, where P is a flex point. We choose c_1 such that

$A_1 = (1, 0, c_1) \neq P$. We find that $R' = B$ with multiplicity sequence $\underline{m}_{R'} = (3, 2)$ and $O' = O$ with $\underline{m}_{O'} = (3_2)$.

($S4$): We start with the quintic curve C with $\mathrm{Data}(C) = [(3), (2_3)]$. We choose two lines l and t as in figure (3.4). Choosing c_1 such that $A_1 = (1, 0, c_1) \neq P$. We find that $R' = B$, $O' = S$ with $\underline{m}_{O'} = (2)$ and $P' = O$ with $\underline{m}_{P'} = (3_3)$.

4. Construction

In this section, we construct some of the curves in Theorem 3 by using suitable Cremona transformations. We deal with some of the cases which were not included in the Yang's list. The other curves can be constructed in the same manner. By changing coordinates suitably, we may assume that $l : x = 0$, $t : y = 0$, $O = (0, 0, 1)$, $A = (1, 0, c)$ and $B = (0, 1, 0)$. In what follows, Applying $\varphi_c : (x, y, z) \to (xy, y^2, x(z - cx))$ for $c \in \mathbb{C}$, we construct the curve C' from the curve C, where C' is the strict transform of C via φ_c.

(I, 5): We begin with the quartic curve C with $\mathrm{Data}(C) = [(2), (2), (2)]$. We choose two lines l and t such that $l \cdot C = 2O + R + S$ and $t \cdot C = O + 3P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (3_2, 2)$, and $R' = S' = B$ with $\underline{m}_{R'} = \begin{pmatrix}1\\1\end{pmatrix}$.

(I, 6): We start with the quartic curve C with $\mathrm{Data}(C) = [(2), (2_2)]$. We choose two lines l and t such that $l \cdot C = 2O + R + S$ and $t \cdot C = O + 3P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (3_2, 2)$, and $S = R' = B$ with $\underline{m}_{O'} = \begin{pmatrix}1\\1\end{pmatrix}$.

(I, 12): We begin with the quintic curve C with $\mathrm{Data}(C) = \left[(2), (2_2), \begin{pmatrix}1\\1\end{pmatrix}_3\right]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 3P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (3_2, 2)$, and $O' = S$ with $\underline{m}_{O'} = (1)$.

(I, 32): We start with the quartic curve C with $\mathrm{Data}(C) = [(2), (2_2)]$. We choose two lines l and t such that $l \cdot C = 2O + R + S$ and $t \cdot C = O + 3P$. By applying quadratic Cremona transformation,we get $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (3_2)$, and $R' = S' = B$ with $\underline{m}_{R'} = \begin{pmatrix}1\\1\end{pmatrix}$.

(I, 37): We start with the quartic curve C with $\mathrm{Data}(C) = \left[\begin{pmatrix}1\\1\end{pmatrix}_3\right]$. We choose two lines l and t such that $l \cdot C = 2O + 2R$ and $t \cdot C = O + 3P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (3_2)$, and $R' = B$ with $\underline{m}_{R'} = \begin{pmatrix}1\\1\end{pmatrix}_4$.

(II, 13): We start with the quartic curve C with $\mathrm{Data}(C) = \left[(2), (2), \begin{pmatrix}1\\1\end{pmatrix}\right]$. We choose two lines l and t such that $l \cdot C = 2O + 2R$ and $t \cdot C = O + 3P$. We find that $P' = O' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\begin{pmatrix}2\\1\end{pmatrix}_2 \begin{pmatrix}1\\1\end{pmatrix}\right\}$, and $R' = B$ with $\underline{m}_{R'} = (2)$.

(II, 15): In this case,we begin with the quartic curve C with $\mathrm{Data}(C) = \left[(2_2), \begin{pmatrix}1\\1\end{pmatrix}\right]$. We choose two lines l and t such that $l \cdot C = 2O + 2R$ and $t \cdot C = O + 3P$. We find that $P' = O' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\begin{pmatrix}2\\1\end{pmatrix}_2 \begin{pmatrix}1\\1\end{pmatrix}\right\}$, and $R' = B$ with $\underline{m}_{R'} = (2_3)$.

(II, 18): We begin with the quintic curve C with $\mathrm{Data}(C) = [(2_2), (2_4)]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 2P + S$. We find that $P' = S' = O$ with multiplicity sequence $\underline{m}_{P'} = \begin{pmatrix}2\\1\end{pmatrix}_2$, and $O' = N$ with $\underline{m}_{O'} = (2_2)$.

(II, 19): We start with the quintic curve C with $\mathrm{Data}(C) = [(2_2), (2_4)]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 2P + S$. We find that $P' = S' = O$ with multiplicity sequence $\underline{m}_{P'} = \begin{pmatrix}2\\1\end{pmatrix}_2$, and $O' = N$ with $\underline{m}_{O'} = (1)$.

(II, 22): In this case,we start with the quintic curve C with $\mathrm{Data}(C) = [(2), (2), (2), (2_3)]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 2P + S$. We find that $P' = S' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^2\right\}$, and $O' = N$ with $\underline{m}_{O'} = (2)$.

(II, 25): We start with the quintic curve C with $\mathrm{Data}(C) = [(2_2), (2_4)]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 2P + S$. We find that $P' = S' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^{2_2}\right\}$, and $O' = N$ with $\underline{m}_{O'} = (2_2)$.

(II, 26): We begin with the quintic curve C with $\mathrm{Data}(C) = [(2_2)(2_4)]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 2P + S$. We find that $P' = S' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^{2_4}\right\}$, and $O' = N$ with $\underline{m}_{O'} = (1)$.

(II, 24): We start with the cubic curve C with $\mathrm{Data}(C) = \{(2)\}$. We choose two lines l and t such that $l \cdot C = 3R$ and $t \cdot C = 2P + S$. We find that $P' = S' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\begin{pmatrix}2\\1\end{pmatrix}_2^2\right\}$, and $R' = B$ with $\underline{m}_{R'} = (3)$.

(II, 21): In this case, we begin with the cubic curve C with $\mathrm{Data}(C) = \{(2)\}$. We choose two lines l and t such that $l \cdot C = 3R$ and $t \cdot C = 2P + S$. We find that $P' = S' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\begin{pmatrix}2\\1\end{pmatrix}_2\right\}$, and $R' = B$ with $\underline{m}_{R'} = (3, 2)$.

(II, 27): We begin with the quintic curve C with $\mathrm{Data}(C) = \left[\begin{pmatrix}2\\1\end{pmatrix}, (2_3)\right]$. We choose two lines $l$and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 3P$. We find that P'

$= O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\binom{2}{1}_3\right\}$, and $O' = N$ with $\underline{m}_{O'} = (2)$.

(II, 28): We start with the quintic curve C with $\mathrm{Data}(C) = \left[\left\{\binom{2}{1}^2\right\}, (2_2)\right]$. We choose two lines l and t such that $l \cdot C = 4O + R$ and $t \cdot C = 2O + 3P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\binom{2}{1}^2_3\right\}$, and $O' = N$ with $\underline{m}_{O'} = (1)$.

(III, 10): We begin with the quintic curve C with $\mathrm{Data}(C) = \left[(2_2), (2_3), \binom{1}{1}\right]$. We choose two lines l and t such that $l \cdot C = 2O + 3R$ and $t \cdot C = 3O + 2A$. We find that $R' = B$ with multiplicity sequence $\underline{m}_{P'} = (3)$, and $O' = O$ with $\underline{m}_{O'} = \binom{2}{1}$.

(III, 16): In this case, we begin with the cubic curve C with $\mathrm{Data}(C) = \{(2)\}$. We choose two lines l and t such that $l \cdot C = 2R + S$ and $t \cdot C = 3P$. We find that $R' = S' = B$ with multiplicity sequence $\underline{m}_{R'} = \left\{\binom{2}{1}\right\}$, and $P' = O$ with $\underline{m}_{R'} = (3_2)$.

(III, 25): We begin with the cubic curve C with $\mathrm{Data}(C) = \{(2)\}$. We choose two lines l and t such that $l \cdot C = 2R + S$ and $t \cdot C = 3P$. We find that $R' = S' = B$ with multiplicity sequence $\underline{m}_{R'} = \left\{\binom{2}{1}^2\right\}$, and $P' = O$ with $\underline{m}_{R'} = (3_2 0)$.

(III, 38): We begin with the quartic curve C with $\mathrm{Data}(C) = \{(2_3)\}$. We choose two lines l and t such that $l \cdot C = O + 3R$ and $t \cdot C = O + 2P + A$. We have that $P' = O' = O$ with multiplicity sequence $\underline{m}_{P'} = \left\{\binom{2}{1}^{2_4}\right\}$, and $R' = B$ with $\underline{m}_{R'} = (3)$.

Acknowledgement

The author would like to thank the referee for his valuable and helpful comments.

References

[1] E. Brieskorn, H. Knörrer, Plane Algebraic Curves, Birkhäuser, Basel, 1986.
[2] Theo de Jong, Gerhard Pfister, Local Analytic Geometry, Vieweg, 2000.
[3] H. Yoshihara, On plane rational curves, Proc. Jpn. Acad. 55 (A) (1979) 152–155.
[4] H. Yoshihara, Rational curves with one cusps, Proc. Am. Math. Soc. 89 (1983) 24–26; H. Yoshihara, Rational curves with one cusps, Proc. Am. Math. Soc. 100 (1987) 405–406.
[5] H. Yoshihara, Plane curves whose singular points are cusps, Proc. Am. Math. Soc. 103 (1988) 737–740.
[6] J.-G. Yang, Sextic curves with simple singularities, Tohoko Math. J. 48 (1996) 203–227.
[7] M. Saleem, On the Classification of Rational Plane Curves of Types (d, m), Lambert Acad. Publishing, 2011 ISBN: 978-3-8443-9988-2.
[8] F. Sakai, M. Saleem, Rational plane curves of type $(d, d-2)$, Saitama Math. J. 22 (2004) 11–34.
[9] F. Sakai, M. Saleem, K. Tono, Hyperelliptic plane curves of type $(d, d-2)$, Contribut. Algebra Geometry 51 (1) (2010) 31–44.
[10] T. Matsuoka, F. Sakai, The degree of rational cuspidal plane curves, Math. Ann. 285 (1989) 233–247 657–673.
[11] H. Flenner, M. Zaidenberg, On a class of rational cuspidal plane curves, Manuscripta Math. 89 (1996) 439–460.
[12] H. Flenner, M. Zaidenberg, Rational cuspidal plane curves of type $(d, d-3)$, Math. Nach. 210 (2000) 93–110.
[13] T. Fenske, Rational 1- and 2-cuspidal plane curves, Beiträge Alg. Geometrie 40 (2) (1999) 309–329.
[14] F. Sakai, K. Tono, Rational cuspidal curves of type $(d, d-2)$ with one or two cusps, Osaka J. Math. 37 (2000) 415–504.