Journal of the Egyptian Mathematical Society (2016) 24, 367–372

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Review Paper

On the existence of solutions of two differential equations with a nonlocal condition

E.M.A. Hamd-Allah*

Faculty of Science, Alexandria University, Alexandria, Egypt

Received 2 August 2015; revised 17 September 2015; accepted 3 October 2015
Available online 23 December 2015

Keywords

Nonlocal conditions;
Existence of solution;
Fixed point theorem;
continuous dependence

Abstract In this paper we study the existence of solutions of two Cauchy problems of two nonlinear differential equations with nonlocal condition. The continuous dependence of the solutions on the coefficients of the nonlocal condition will be studied.

2010 Mathematics Subject Classifications: Primary 32A55; 11D09; Secondary 45D05; 60G22; 33E30,

1. Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1–5] and [6–13] and references therein.

Consider the two nonlinear differential equations

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right), \quad t \in (0, T], \tag{1}$$

and

$$\frac{dx(t)}{dt} = g\left(t, x(t), \frac{dx(t)}{dt}\right), \quad a.e. t \in (0, T], \tag{2}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_o, \quad \tau_k \in (0, T). \tag{3}$$

Our aim here is to study the existence of solutions for the two problems (1) with the nonlocal condition (3) and (2) with the nonlocal condition (3). Moreover, the continuous dependence of the solutions of the above two problems on x_0 and the nonlocal coefficients a_k will be studied.

2. Functional integral equations

Lemma 2.1. *Let* $\sum_{k=1}^{m} a_k \neq 0$. *The solution of the nonlocal problem (1) and (3) can be expressed by the integral equation*

* Tel.: +201224420145.
E-mail address: eman_hr2003@yahoo.com

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.10.002

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) + \int_0^t y(s)ds,$$
$$A = \left(\sum_{k=1}^{m} a_k\right)^{-1}, \tag{4}$$

where y is the solution of the functional integral equation

$$y(t) = f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds, y(t)\right),$$
$$t \in [0, T]. \tag{5}$$

Proof. Let $\frac{dx(t)}{dt} = y(t)$ in Eq. (1), then we obtain

$$y(t) = f(t, x(t), y(t))$$

where

$$x(t) = x(0) + \int_0^t y(s)ds. \tag{6}$$

Letting $t = \tau_k$in (6), we obtain

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k x(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds. \tag{7}$$

Then

$$x(0) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) \tag{8}$$

where $A = (\sum_{k=1}^{m} a_k)^{-1}$.

And we obtain

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) + \int_0^t y(s)ds,$$

where y is the solution of the functional integral equation

$$y(t) = f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds, y(t)\right),$$
$$t \in [0, T].$$

By similar way, the following lemma can be proved. □

Lemma 2.2. *Let* $\sum_{k=1}^{m} a_k \neq 0$. *The solution of the nonlocal problem (2) and (3) can be expressed by the integral equation*

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) + \int_0^t y(s)ds,$$
$$A = \left(\sum_{k=1}^{m} a_k\right)^{-1}, \tag{9}$$

where y is the solution of the functional integral equation

$$y(t) = g\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds, y(t)\right),$$
$$t \in [0, T]. \tag{10}$$

2.1. *Existence of solutions*

Consider the functional integral equations (5) and (10) with the following assumptions:

(i) f: $[0, T] \times R \times R \to R$ is continuous and satisfies Lipschitz condition

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le M_1(|u_1 - v_1| + |u_2 - v_2|),$$

(ii) g: $[0, T] \times R \times R \to R$ is measurable in $t \in [0, T]$ for any $(u_1, u_2) \in R \times R$ and satisfies Lipschitz condition

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \le M_2(|u_1 - v_1| + |u_2 - v_2|),$$

and

$$\int_0^t |g(t, 0, 0)|dt \le N$$

(iii) $M^* = M_1(2T + 1) < 1$
(iv) $M^{**} = M_2(2T + 1) < 1$.

Now we have the following theorem

Theorem 2.1. *Let the assumptions (i) and (iii) be satisfied. Then the functional integral equation (5) has a unique solution* $y \in C[0, T]$.

Proof. Define the operator H by

$$Hy(t) = f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^t y(s)\,ds,\ y(t)\right),$$
$$t \in [0, T]. \tag{11}$$

Let $y \in C[0, T]$, then

$$\begin{aligned}
&|Hy(t_2) - Hy(t_1)|\\
&= \left|f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_2} y(s)\,ds,\ y(t_2)\right)\right.\\
&\quad \left. - f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_2} y(s)\,ds,\ y(t_1)\right)\right|\\
&\quad + \left|f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_2} y(s)\,ds,\ y(t_1)\right)\right.\\
&\quad \left. - f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right|\\
&\quad + \left|f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right.\\
&\quad \left. - f\left(t_1, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right|\\
&\le M_1 \int_{t_1}^{t_2} |y(s)|ds + M_1|y(t_2) - y(t_1)|\\
&\quad + \left|f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right.\\
&\quad \left. - f\left(t_1, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds, y(t_1)\right)\right|
\end{aligned}$$

which implies that the operator H maps $C[0, T]$ into itself, i.e. H: $C[0, T] \to C[0, T]$.

Now, let $u, v \in C[0, T]$ then we have

$$Hu(t) - Hv(t)$$
$$= f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} u(s)\,ds + \int_0^t u(s)\,ds, u(t)\right)$$
$$-f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} v(s)\,ds + \int_0^t v(s)\,ds, v(t)\right)$$

and

$$|Hu(t) - Hv(t)| \le M_1|A|\left|\sum_{k=1}^{m} a_k\right| \int_0^{\tau_k} |u(s) - v(s)|ds + M_1 \int_0^t |u(s) - v(s)|ds + M_1|u(t) - v(t)|,$$

then

$$||Hu(t) - Hv(t)|| \le M_1\,(2T + 1)||u - v|| = M^*||u - v||.$$

And $M^* < 1$, which proves that the operator H: $C[0, T] \to C[0, T]$ is contraction.

Applying Banach contraction fixed point [14], then the functional equation (5) has a unique fixed point $y \in C[0, T]$.

Now, consider the nonlocal problem (1) and (3). □

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied. Then the nonlocal problem (1) and (3) has a unique solution $x \in C^1[0, T]$.*

Proof. From Lemma 2.1 and Theorem 2.1, the solution of the problem (1) and (3) is given by (4) where y is given by (5). To complete the proof, we prove that the integral equation (5) satisfies nonlocal problem (1) and (3).

Differentiating (4), we get

$$\frac{dx}{dt} = y(t) = f\left(t, x(t), \frac{dx}{dt}\right)$$

Let $t = \tau_k$ in (4), we get

$$\sum_{k=1}^{m} a_k\,x(\tau_k) = \sum_{k=1}^{m} a_k\,A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds$$
$$= x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds$$
$$= x_0.$$

This completes the proof. □

Now we have the following theorem:

Theorem 2.3. *Let the assumptions (ii) and (iv) be satisfied. Then the functional integral equation (10) has a unique solution $y \in L_1[0, T]$.*

Proof. Define the operator G by

$$Gy(t) = g\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^t y(s)\,ds, y(t)\right), \quad t \in [0, T]. \tag{12}$$

Let $y \in L_1[0, T]$, then

$$||Gy||_{L_1} = \int_0^T |(Gy)(t)|dt \le M_2 \int_0^T \left(\left|Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^t y(s)\,ds\right| + |y(t)|\right)dt + \int_0^T |g(t, 0, 0)|dt. \tag{13}$$

From condition (ii) we have

$$|g(t, u, v)| - |g(t, 0, 0)| \le |g(t, u, v) - g(t, 0, 0)| \le M_2(|u| + |v|),$$

then

$$|g(t, u, v)| \le M_2\,(|u| + |v|) + |g(t, 0, 0)|.$$

Substitute into (13), we get

$$||Gy||_{L_1} \le M_2\,T|A|x_0 + N\,T + M_2\,(2T + 1)||y||_{L_1}.$$

Then $||Gy||_{L_1} \in L_1$, which implies that the operator G maps $L_1[0, T]$ into itself, i.e. G: $L_1[0, T] \to L_1[0, T]$.

Now, let $u, v \in L_1[0, T]$ then we have

$$Gu(t) - Gv(t)$$
$$= g\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} u(s)\,ds + \int_0^t u(s)\,ds, u(t)\right)$$
$$-g\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} v(s)\,ds + \int_0^t v(s)\,ds, v(t)\right)$$

and

$$|Gu(t) - Gv(t)| \le M_2\left| - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} (u(s) - v(s))\,ds + \int_0^t (u(s) - v(s))\,ds\right| + M_2|u(t) - v(t)|$$
$$\le M_2|A|\left|\sum_{k=1}^{m} a_k\right| \int_0^{\tau_k} |u(s) - v(s)|ds + M_2 \int_0^t |u(s) - v(s)|ds + M_2|u(t) - v(t)|,$$

then

$$||Gu(t) - Gv(t)||_{L_1} \le (M_2\,T + M_2\,T + M_2)||u - v||_{L_1} \le M_2\,(2T + 1)||u - v||_{L_1} = M^{**}||u - v||_{L_1}.$$

And $M^{**} < 1$, which proves that the operator G: $L_1[0, T] \to L_1[0, T]$ is contraction.

Applying Banach contraction fixed point [14], then the functional equation (10) has a unique fixed point $y \in L_1[0, T]$ such that $y(t) = g(t, x(t), y(t))$, a.e.$t \in (0, T]$.

Now, consider the nonlocal problem (2) and (3). □

Theorem 2.4. *Let the assumptions of Theorem 2.3 be satisfied. Then the nonlocal problem (2) and (3) has a unique absolutely continuous solution $x \in AC[0, T]$.*

Proof. From Lemma 2.2 and Theorem 2.3, we deduce that there exists a unique absolutely continuous solution $x \in AC[0, T]$ of the integral equation (10). To complete the proof, we prove that the integral equation (10) satisfies the nonlocal problem (2) and (3).

Differentiating (9), we get

$$\frac{dx}{dt} = y(t) = g\left(t, x(t), \frac{dx}{dt}\right)$$

Let $t = \tau_k$ in (9) ,we get

$$\sum_{k=1}^{m} a_k\, x(\tau_k) = x_0.$$

This completes the proof.

This implies that there exists a unique absolutely continuous solution $x \in AC[0, T]$ of the non local problem (2) and (3). □

3. Continuous dependence

Here we study the continuous dependence of the solution of the problem (1) and (3)

Definition 1. The solution $x \in C^1[0, T]$ of the problem (1) and (3) is called continuously dependent on x_0 if, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|x_0 - \widetilde{x}_0| < \delta$ implies $||x - \widetilde{x}|| < \epsilon$ where $x, \widetilde{x}$ are the solutions of the problems (1) with the condition (3) and (1) with the condition

$$\sum_{k=1}^{m} a_k\, x(\tau_k) = \widetilde{x}_o, \quad \tau_k \in (0, T). \tag{14}$$

Theorem 3.1. *Let the assumptions of Theorem 2.1 be satisfied. Then the solution of the problem (1) and (3) depends continuously on x_0.*

Proof. Since the solution of the problem (1) and (3) is the solution of the integral equation

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds\right) + \int_0^t y(s)\, ds$$

and the solution of the problem (1) and (14) is the solution of the integral equation

$$\widetilde{x}(t) = A\left(\widetilde{x}_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \widetilde{y}(s)\, ds\right) + \int_0^t \widetilde{y}(s) ds.$$

Then, for the two corresponding solutions x and $\widetilde{x}$ we have

$$|x(t) - \widetilde{x}(t)| \le |A||x_0 - \widetilde{x}_0| + |A|\left|\sum_{k=1}^{m} a_k\right| \int_0^{\tau_k} |y(s) - \widetilde{y}(s)| ds + \int_0^t |y(s) - \widetilde{y}(s)| ds.$$

Hence

$$\begin{aligned} ||x(t) - \widetilde{x}(t)|| &\le |A||x_0 - \widetilde{x}_0| + T|A|\left|\sum_{k=1}^{m} a_k\right| ||y - \widetilde{y}|| \\ &\quad + T||y - \widetilde{y}|| \\ &\le |A||x_0 - \widetilde{x}_0| + 2\,T||y - \widetilde{y}||. \end{aligned} \tag{15}$$

But

$$\begin{aligned} ||y(t) - \widetilde{y}(t)|| &\le M_1|A||x_0 - \widetilde{x}_0| + M_1\,(1 + 2\,T)||y - \widetilde{y}|| \\ &\le \frac{M_1|A|}{1 - (1 + 2\,T)\,M_1}|x_0 - \widetilde{x}_0|. \end{aligned}$$

Substitute in (15), we get

$$\begin{aligned} ||x(t) - \widetilde{x}(t)|| &\le |A||x_0 - \widetilde{x}_0| + \frac{2\,T\,M_1|A|}{1 - (1 + 2\,T)\,M_1}|x_0 - \widetilde{x}_0| \\ &\le \frac{1 - M_1}{1 - (1 + 2\,T)\,M_1}|A||x_0 - \widetilde{x}_0| \end{aligned}$$

Hence

$$||x(t) - \widetilde{x}(t)|| \le (1 - M_1)(1 - M^*)^{-1}|A|\delta = \epsilon.$$

which completes the proof of the theorem. □

Definition 2. The solution $x \in C^1[0, T]$ of the problem (1) and (3) is called continuously dependent on the coefficients $\sum_{k=1}^{m} a_k$ if, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\sum_{k=1}^{m} |a_k - \widetilde{a}_k| < \delta$ implies $||x - \widetilde{x}|| < \epsilon$ where x and $\widetilde{x}$ are the solutions of the problems (1) with the condition (3) and (1) with the condition

$$\sum_{k=1}^{m} \widetilde{a}_k x(\tau_k) = x_o, \quad \tau_k \in (0, T). \tag{16}$$

Theorem 3.2. *Let the assumptions of Theorem 2.1 be satisfied. Then the solution of the problem (1) and (3) depends continuously on the coefficients $\sum_{k=1}^{m} a_k$.*

Proof. Since the solution of the problem (1) and (3) is the solution of the integral equation

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds\right) + \int_0^t y(s)ds$$

and the solution of the problem (1) and (16) is the solution of the integral equation

$$\widetilde{x}(t) = \widetilde{A}\,(x_0 - \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds) + \int_0^t \widetilde{y}(s)\, ds$$

Then, for the two corresponding solution x and $\widetilde{x}$ we have

$$\begin{aligned} x(t) - \widetilde{x}(t) &= (A - \widetilde{A})\,x_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds \\ &\quad + \widetilde{A}\sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds + \int_0^t (y(s) - \widetilde{y}(s))ds. \end{aligned} \tag{17}$$

Since

$$\begin{aligned} &A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds \\ &\quad - \widetilde{A}\sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds = A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds \\ &\quad - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} \widetilde{y}(s)\, ds + A\sum_{k=1}^{m} a_k \int_0^{\tau_k} \widetilde{y}(s)\, ds \end{aligned}$$

$$
\begin{aligned}
& - A \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds + A \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds \\
& - \widetilde{A} \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds \\
& = A \sum_{k=1}^{m} a_k \int_0^{\tau_k} (y(s) - \widetilde{y}(s))\, ds \\
& + A \sum_{k=1}^{m} (a_k - \widetilde{a}_k) \int_0^{\tau_k} \widetilde{y}(s)\, ds \\
& + (A - \widetilde{A}) \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds. \qquad (18)
\end{aligned}
$$

Substitute from (18) in (17), we obtain

$$
\begin{aligned}
x(t) - \widetilde{x}(t) = {} & (A - \widetilde{A})\, x_0 - A \sum_{k=1}^{m} a_k \int_0^{\tau_k} (y(s) - \widetilde{y}(s))\, ds \\
& - A \sum_{k=1}^{m} (a_k - \widetilde{a}_k) \int_0^{\tau_k} \widetilde{y}(s)\, ds \\
& - (A - \widetilde{A}) \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds \\
& + \int_0^{t} (y(s) - \widetilde{y}(s)) ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
& ||x(t) - \widetilde{x}(t)|| \\
& \le |A - \widetilde{A}||x_0| + T||y - \widetilde{y}|| + T|A| \left| \sum_{k=1}^{m} \right| a_k - \widetilde{a}_k|||\widetilde{y}|| \\
& \quad + T|A - \widetilde{A}| \left| \sum_{k=1}^{m} \widetilde{a}_k \right| ||\widetilde{y}|| + T||y - \widetilde{y}|| \\
& \le |A - \widetilde{A}||x_0| + T||\widetilde{y}|| \left(|A| \left| \sum_{k=1}^{m} \right| a_k - \widetilde{a}_k| + |A - \widetilde{A}| \left| \sum_{k=1}^{m} \widetilde{a}_k \right| \right) \\
& \quad + 2\, T||y - \widetilde{y}|| \\
& \le |A|.|\widetilde{A}| \left| \sum_{k=1}^{m} \right| a_k - \widetilde{a}_k||x_0| + T|A| \left| \sum_{k=1}^{m} \right| a_k - \widetilde{a}_k|||\widetilde{y}|| \\
& \quad + T|A|.|\widetilde{A}| \left| \sum_{k=1}^{m} \widetilde{a}_k \right| \sum_{k=1}^{m} |a_k - \widetilde{a}_k|||\widetilde{y}|| + 2\, T||y - \widetilde{y}|| \\
& \le |A| \left| \sum_{k=1}^{m} \right| a_k - \widetilde{a}_k|(|\widetilde{A}||x_0| + 2\, T||\widetilde{y}||) + 2\, T||y - \widetilde{y}||. \qquad (19)
\end{aligned}
$$

Now

$$
\begin{aligned}
y(t) - \widetilde{y}(t) = {} & f\Bigg(t, A x_0 - A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds \\
& + \int_0^{t} y(s)\, ds, y(t)\Bigg) \\
& - f\Bigg(t, \widetilde{A}\, x_0 - \widetilde{A} \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds \\
& + \int_0^{t} \widetilde{y}(s)\, ds, \widetilde{y}(t)\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
|y(t) - \widetilde{y}(t)| \le {} & M_1 |A - \widetilde{A}||x_0| + M_1 \left| A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds \right. \\
& \left. - \widetilde{A} \sum_{k=1}^{m} \widetilde{a}_k \int_0^{\tau_k} \widetilde{y}(s)\, ds \right| \\
& + M_1\, T|y(s) - \widetilde{y}(s)| + M_1 |y(s) - \widetilde{y}(s)|. \qquad (20)
\end{aligned}
$$

Substitute from (18) into (20), we get

$$
\begin{aligned}
|y(t) - \widetilde{y}(t)| \le {} & M_1 |A - \widetilde{A}||x_0| \\
& + M_1 |A| \left| \sum_{k=1}^{m} a_k \right| \int_0^{\tau_k} |y(s) - \widetilde{y}(s)| ds \\
& + M_1 |A| \sum_{k=1}^{m} |a_k - \widetilde{a}_k| \int_0^{\tau_k} |\widetilde{y}(s)| ds \\
& + M_1 |A - \widetilde{A}| \left| \sum_{k=1}^{m} \widetilde{a}_k \right| \int_0^{\tau_k} |\widetilde{y}(s)| ds \\
& + M_1\, T|y(s) - \widetilde{y}(s)| + M_1 |y(s) - \widetilde{y}(s)|.
\end{aligned}
$$

Then

$$
\begin{aligned}
||y(t) - \widetilde{y}(t)|| \le {} & M_1 |A - \widetilde{A}||x_0| + M_1\, T||\widetilde{y}|| \\
& \times \left(|A| \sum_{k=1}^{m} |a_k - \widetilde{a}_k| + |A - \widetilde{A}| \left| \sum_{k=1}^{m} \widetilde{a}_k \right| \right) \\
& + 2\, M_1\, T||y - \widetilde{y}|| + M_1 ||y - \widetilde{y}||.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||y(t) - \widetilde{y}(t)|| \le {} & \frac{M_1 |A|}{1 - (1 + 2T)\, M_1} \\
& \times \sum_{k=1}^{m} |a_k - \widetilde{a}_k|(|\widetilde{A}||x_0| + 2\, T||\widetilde{y}||). \qquad (21)
\end{aligned}
$$

Substitute from (21) into (19),we get

$$
\begin{aligned}
||x(t) - \widetilde{x}(t)|| \le {} & |A| \sum_{k=1}^{m} |a_k - \widetilde{a}_k|(|\widetilde{A}||x_0| + 2\, T||\widetilde{y}||) \\
& + \frac{2\, T\, M_1 |A|}{1 - (1 + 2T)\, M_1} \sum_{k=1}^{m} |a_k - \widetilde{a}_k|(|\widetilde{A}||x_0| + 2\, T||\widetilde{y}||) \\
\le {} & \frac{(1 - M_1)|A|}{1 - (1 + 2T)\, M_1} (|\widetilde{A}||x_0| + 2\, T||\widetilde{y}||) \sum_{k=1}^{m} |a_k - \widetilde{a}_k|
\end{aligned}
$$

Hence

$$
\begin{aligned}
||x(t) - \widetilde{x}(t)|| \le {} & (1 - M_1)(1 - M^*)^{-1} |A| \\
& \times (|\widetilde{A}||x_0| + 2T||\widetilde{y}||)\delta = \epsilon.
\end{aligned}
$$

which completes the proof of the theorem. □

Now for the continuous dependence of the solution of the problem (2) and (3) with respect to x_0 and the coefficients $\sum_{k=1}^{m} a_k$ we have the following

Theorem 3.3. *Let the assumptions of Theorem 2.3 be satisfied. Then the solution of the problem (2) and (3) depends continuously on x_0.*

Proof. Since the solution of the problem (2) and (3) is the solution of the integral equation

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds\right) + \int_0^t y(s)\,ds$$

and the solution of the problem (2) and (14) is the solution of the integral equation

$$\widetilde{x}(t) = A\left(\widetilde{x}_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \widetilde{y}(s)\,ds\right) + \int_0^t \widetilde{y}(s)ds.$$

Then

$$\begin{aligned}||x(t) - \widetilde{x}(t)|| &\le |A||x_0 - \widetilde{x}_0| \\ &\quad + |A|\sum_{k=1}^{m} |a_k|||y - \widetilde{y}||_{L_1} + ||y - \widetilde{y}||_{L_1} \\ &\le |A||x_0 - \widetilde{x}_0| + 2||y - \widetilde{y}||_{L_1}. \end{aligned} \tag{22}$$

and

$$\begin{aligned}||y(t) - \widetilde{y}(t)||_{L_1} &\le M_2\,T|A||x_0 - \widetilde{x}_0| + M_2\,(1 + 2\,T)||y - \widetilde{y}||_{L_1} \\ &\le \frac{M_2\,T|A|}{1 - (1 + 2\,T)\,M_2}|x_0 - \widetilde{x}_0|.\end{aligned}$$

Substitute in (22), we get

$$\begin{aligned}||x(t) - \widetilde{x}(t)|| &\le |A||x_0 - \widetilde{x}_0| + \frac{2\,T\,M_2|A|}{1 - (1 + 2\,T)\,M_2}|x_0 - \widetilde{x}_0| \\ &\le \frac{1 - M_2}{1 - (1 + 2\,T)\,M_2}|A||x_0 - \widetilde{x}_0|.\end{aligned}$$

Hence

$$||x(t) - \widetilde{x}(t)|| \le (1 - M_2)(1 - M^{**})^{-1}|A|\delta = \epsilon.$$

which completes the proof of the theorem. □

Theorem 3.4. *Let the assumptions of Theorem 2.3 be satisfied. Then the solution of the problem (2) and (3) depends continuously on the coefficients* $\sum_{k=1}^{m} a_k$.

Proof. The proof straights forward as the proof of Theorem 3.2. □

Acknowledgment

The authors would like to thank the referee for the time taken to review this paper and for his/her suggestions and valuable remarks that helped improve the quality of this paper

References

[1] M. Benchohra, E.P. Gatsori, S.K. Ntouyas, Existence results for seme-linear integrodifferential inclusions with nonlocal conditions, Rocky Mt. J. Mat. 34 (3) (2004) 1–10.
[2] M. Benchohra, S. Hamani, S. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Anal. 71 (2009) 2391–2396.
[3] A. Boucherif, First-order differential inclusions with nonlocal initial conditions, Appl. Math. Lett. 15 (2002) 409–414.
[4] A. Boucherif, Nonlocal Cauchy problems for first-order multivalued differential equations, Electron. J. Differ. Equ. 2002 (47) (2002) 1–9.
[5] A. Boucherif, R. Precup, On the nonlocal initial value problem for first order differential equations, Fixed Point Theory 4 (2) (2003) 205–212.
[6] R. Ma, Existence and uniqueness of solutions to first - order three - point boundary value problems, Appl. Math. Lett. 15 (2002) 211–216.
[7] A.M.A. El-Sayed, Sh. A. Abd El-Salam, On the stability of a fractional order differential equation with nonlocal initial condition, EJQTDE 29 (2008) 1–8.
[8] A.M.A. El-Sayed, E.M. Hamdallah, Kh. W. El-Kadeky, Solutions of a class of nonlocal problems for the differential inclusion $x'(t) \in F(t, x(t))$, Appl. Math. Inf. Sci.-An Int. J. 5 (3) (2011) 413–421.
[9] A.M.A. El-Sayed, E.M. Hamdallah, Kh. W. El-Kadeky, Solutions of a class of deviated-advanced nonlocal problems for the differential inclusion $x'(t) \in F(t, x(t))$, Abstr. Appl. Anal. 2011 (2011). Article ID 476392.
[10] A.M.A. El-Sayed, E.M. Hamdallah, Kh. W. El-Kadeky, Monotonic positive solutions of nonlocal boundary value problems for a second-order functional differential equation, Abstr. Appl. Anal. 2012 (2012). Article ID 489353.
[11] A.M.A. El-Sayed, E.M. Hamdallah, Kh. W. El-Kadeky, Uniformly stable positive monotonic solution of a nonlocal Cauchy problem, Adv. Pure Math. 2 (2) (2012). Article ID 5300149.
[12] A.M.A. El-Sayed, E.M. Hamdallah, Kh. W. El-Kadeky, Solutions of a class of nonlocal problems for the differential inclusion $x'(t) \in F(t, x(t))$, Egypt.-Chin. J. Comput. Appl. Math. 1 (1) (2012) 1–8.
[13] E. Gatsori, S. Ntouyas, Y.G. Sficas, On a nonlocal cauchy problem for differential inclusions, Abstr. Appl. Anal. (2004) 425–434.
[14] R.F. Curtain, A.J. Pritchard, Functional Analysis in modern Applied Mathematics, Academic Press, 1977.