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Nonuniform biorthogonal wavelets on positive half line via Walsh Fourier transform

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Abstract

In this article, we introduce the notion of nonuniform biorthogonal wavelets on positive half line. We first establish the characterizations for the translates of a single function to form the Riesz bases for their closed linear span. We provide the complete characterization for the biorthogonality of the translates of scaling functions of two nonuniform multiresolution analysis and the associated biorthogonal wavelet families in $L^2(\mathbb{R}^+)$. Furthermore, under the mild assumptions on the scaling functions and the corresponding wavelets associated with nonuniform multiresolution analysis, we show that the wavelets can generate Reisz bases.

Keywords: Nonuniform biorthogonal wavelet, Nonuniform multiresolution analysis, Walsh-Fourier transform

Mathematics Subject Classification: 42C40, 42C15, 43A70, 11S85

Introduction

The theory of wavelet transforms have emanated as a broadly used tool in various disciplines of science and engineering including image processing, spectrometry, turbulence, computer graphics, optics and electromagnetism, telecommunications, DNA sequence analysis, quantum physics, solution of differential equations. In context of signal processing, it has been assumed that orthogonality is the key property for synthesis and analysing signals. In order to study a higher-level signal processing, biorthogonality plays a vital role in which two sets are incorporated: one serves for the analysis and the other one for synthesis. Towards the culminating years of 1990's, biorthogonal wavelets are considered as cornerstone technique in image compression due to their natural feature of concentrating energy in a few transform coefficients and advantageous over orthogonal wavelets, by relaxing orthonormal to biorthogonal, additional degrees of freedom are added to design problems. Biorthogonal wavelets in $L^2(\mathbb{R})$ were investigated by Bownik and Garrigos [1], Cohen et al. [2], Chui and Wang [3]. The numerical aspect of biorthogonal wavelets were studied by Karoui and Vaillancourt [4].

Multiresolution analysis is the heart of wavelet analysis as it gives a general framework for analysing wavelet systems. All the signals in real life applications are not obtained from the uniform shifts. For the analysis and decomposition of these signals by means of stable mathematical technique, Gabardo and Nashed [5] introduced a notion of nonuniform

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MRA where the translation set acting on the scaling function associated with the MRA to generate the subspace $\mathcal{V}_0$ is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Shah and Abdullah [6] established NUMRA on non-Archimedean local fields.

In the recent years, the development of wavelet theory in the context of Walsh analysis have been extensively studied by many authors including Farkov [7], Meenakshi [8] but still more concepts need to be studied for its enhancement. Recently, Ahmad and his collaborators investigated wavelet frames [9], nonuniform wavelet frames [10–14], Nonuniform p-tight wavelet frames [15], Tight Framelets [16, 17], wavepacket systems [18, 19], Frames associated with shift invariant spaces [20], Gabor frames [21], numerical study of wavelets [22–30] and obtained many interested results. Continuing our research on wavelet and wavelet frames, we in this paper introduce the notion of nonuniform biorthogonal wavelets in $L^2(\mathbb{R}^+)$. We obtain the characterization for the translates of a single function to form the Riesz bases for their closed linear span. We also provide a complete characterization for the biorthogonality of the translates of scaling functions of two NUMRA's and the associated biorthogonal wavelet families. Moreover, under mild assumptions on the scaling functions and the corresponding wavelets, we show that the nonuniform wavelets can generate Reisz bases for $L^2(\mathbb{R}^+)$.

The article is structured as follows. In Section "Methods", we recall methods of Fourier analysis on positive half line including basic definitions of MRA and NUMRA . In Section "Results and discussion", we establish necessary and sufficient conditions for the translates of a function to form a Riesz basis for its closed linear span. Furthermore, we show that the wavelets associated with dual MRA's are biorthogonal and generate Riesz bases for $L^2(\mathbb{R}^+)$.

Methods

Let $\mathbb{R}^+$, $\mathbb{Z}^+$ and $\mathbb{N}$ respectively denotes the set of nonnegative real numbers, set of non-negative integers and the set of natural numbers. By the symbols $[x]$ and $\{x\}$, we mean the integer and fractional part of x respectively. Let $p > 1$ be a fixed natural number. For $\xi \in \mathbb{R}^+$ and any integer $j > 0$, we set

$$\xi_j = [p^j \xi](\mathrm{mod}\, p), \qquad \xi_{-j} = [p^{1-j}\xi](\mathrm{mod}\, p), \tag{2.1}$$

where $\xi_j, \xi_{-j} \in \{0, 1, \ldots, p-1\}$. Clearly, ξ_j and ξ_{-j} are the digits in the p-ary expansion of ξ:

$$\xi = \sum_{j<0} \xi_{-j} p^{-j-1} + \sum_{j>0} \xi_j p^{-j}.$$

The first sum on the right is always finite and

$$[\xi] = \sum_{j<0} \xi_{-j} p^{-j-1}, \qquad \{\xi\} = \sum_{j>0} x_j p^{-j}.$$

On $\mathbb{R}^+$ the addition is defined in the following manner:

$$\xi \oplus \eta = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = \xi_j + \eta_j (\mathrm{mod}\, p)$, $j \in \mathbb{Z} \setminus \{0\}$, where $\zeta_j \in \{0, 1, \ldots, p-1\}$ and ξ_j, η_j are given by (2.1). Clearly, $[\xi \oplus \eta] = [\xi] \oplus [\eta]$ and $\{\xi \oplus \eta\} = \{\xi\} \oplus \{\eta\}$. We write $\delta = \xi \ominus \eta$ if $\delta \oplus \eta = \xi$, where $\ominus$ denotes subtraction modulo p on $\mathbb{R}^+$.

Let $\varepsilon_p = \exp(2\pi i/p)$, we define a function $s_0(x)$ on [0, 1) by

$$s_0(\xi) = \begin{cases} 1, & \text{if } \xi \in [0, 1/p) \\ \varepsilon_p^{\ell}, & \text{if } \xi \in \left[\ell p^{-1}, (\ell+1)p^{-1}\right), \quad \ell \in \{1, 2, \ldots, p-1\}. \end{cases}$$

The system of *generalized Walsh functions* $\left\{w_m(\xi) : m \in \mathbb{Z}^+\right\}$ on [0, 1) is defined in the following way:

$$w_0(\xi) \equiv 1 \quad \text{and} \quad w_m(\xi) = \prod_{j=0}^{k} \left(s_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \ldots, p-1\}$, $\mu_k \neq 0$. These functions form a complete orthogonal system. A finite linear combination of Walsh functions is known as Walsh polynomial. For $\xi, \eta \in \mathbb{R}^+$, let

$$\chi(\xi, \eta) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (\xi_j \eta_{-j} + \xi_{-j} \eta_j)\right), \tag{2.2}$$

where ξ_j, η_j are defined as in (2.1).

It is easy to see that

$$\chi\left(\xi, \frac{m}{p^n}\right) = \chi\left(\frac{\xi}{p^n}, m\right) = w_m\left(\frac{\xi}{p^n}\right), \qquad \forall\, \xi \in [0, p^n),\ m, n \in \mathbb{Z}_+,$$

and

$$\chi(\xi \oplus \eta, \delta) = \chi(\xi, \delta)\, \chi(\eta, \delta), \quad \chi(\xi \ominus \eta, \delta) = \chi(\xi, \delta)\, \overline{\chi(\eta, \delta)},$$

where $\xi, \eta, \delta \in \mathbb{R}_+$ and $\xi \oplus \eta$ is p-adic irrational. It is well known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ form an orthonormal bases in $L^2[0,1]$ (See [31, 32]).

For a function $\phi \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, the *Walsh-Fourier transform* is defined as

$$\widehat{\phi}(\zeta) = \int_{\mathbb{R}_+} \phi(\xi)\, \overline{\chi(\xi, \zeta)}\, \mathrm{d}\xi, \tag{2.3}$$

where $\chi(\xi, \zeta)$ is given by (2.2). The Walsh-Fourier operator $\mathfrak{F} : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $\mathfrak{F}f = \widehat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The Walsh-Fourier transform enjoys similar properties to those of the classic Fourier transform [31–34]. In particular, if $\phi \in L^2(\mathbb{R}^+)$, then $\widehat{\phi} \in L^2(\mathbb{R}^+)$ and

$$\left\|\widehat{\phi}\right\|_{L^2(\mathbb{R}^+)} = \|\phi\|_{L^2(\mathbb{R}^+)}. \tag{2.4}$$

Furthermore, if $\phi \in L^2[0, 1]$, then we can define the Walsh-Fourier coefficients of ϕ as

$$\widehat{\phi}(n) = \int_0^1 \phi(\xi)\overline{w_n(\xi)}\, dx. \tag{2.5}$$

The series $\sum_{n\in\mathbb{Z}_+} \widehat{\phi}(n)w_n(\xi)$ is called the *Walsh-Fourier series* of ϕ. From the standard L^2 -theory, we can observe that the Walsh-Fourier series of ϕ converges to ϕ in $L^2[0,1]$ and Parseval's identity holds:

$$\|\phi\|_2^2 = \int_0^1 \left|\phi(\xi)\right|^2 \mathrm{d}\xi = \sum_{\mathrm{n}\in\mathbb{Z}_+} \left|\widehat{\phi}(\mathrm{n})\right|^2. \tag{2.6}$$

By p-adic interval $I \subset \mathbb{R}_+$ of range n, we mean intervals of the form

$$I = I_n^\ell = \left[\ell p^{-n}, (\ell+1)p^{-n}\right), \ \ \ell \in \mathbb{Z}_+.$$

Each of these p-adic intervals is both closed and open under the p-adic topology which is generated by the collection of p-adic intervals [31]. The collection $\left\{[0,p^{-j}) : j \in \mathbb{Z}\right\}$ forms a fundamental system of the p-adic topology on $\mathbb{R}^+$. Therefore, the generalized Walsh functions $w_j(\xi), 0 \le j \le p^n - 1$, assume constant values on each of p-adic interval I_n^ℓ and hence continuous on these intervals. Thus, $w_j(\xi) = 1$ for $\xi \in I_n^0$.

Let $\mathcal{E}_n(\mathbb{R}^+)$ denotes the space of p-adic entire functions of order n. Thus, for every $\phi \in \mathcal{E}_n(\mathbb{R}^+)$, we have

$$\phi(\xi) = \sum_{k\in\mathbb{Z}^+} \phi(p^{-n}k)\chi_{I_n^k}(\xi), \quad \xi \in \mathbb{R}_+. \tag{2.7}$$

It is clear that $\mathcal{E}_n(\mathbb{R}^+)$ contains each Walsh function of order up to p^{n-1}. The set $\mathcal{E}(\mathbb{R}^+)$ of p-adic entire functions on $\mathbb{R}^+$ is the union of all the spaces $\mathcal{E}_n(\mathbb{R}^+)$ and is dense in $L^p(\mathbb{R}^+), 1 \le p < \infty$ and each function in $\mathcal{E}(\mathbb{R}^+)$ is of compact support. Thus, we consider the following set of functions

$$\mathcal{E}^0(\mathbb{R}^+) = \left\{\phi \in \mathcal{E}(\mathbb{R}^+) : \ \widehat{\phi} \in L^\infty(\mathbb{R}^+) \ \textit{and supp} \ \phi \subset \mathbb{R}^+ \setminus \{0\}\right\}. \tag{2.8}$$

Let $N \ge 1$ be a given integer and r be an odd integer which are relatively prime such that $1 \le r \le N-1$, we consider the translation set Λ^+ as

$$\Lambda^+ = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+. \tag{2.9}$$

It can be easily seen that the translation set Λ^+ is not necessarily a group nor a uniform discrete set. The set Λ^+n is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Furthermore, the translation set Λ^+ is the spectrum for the spectral set $\Gamma_N = \left[0, \frac{1}{2}\right) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$ and the pair (Λ^+, Γ_N) is called a *spectral pair* [5].

Definition 2.1 Let $N \ge 1$ be a given integer and r be an odd integer which are relatively prime such that $1 \le r \le N-1$, an associated nonuniform MRA is a sequence of closed subspaces $\left\{\mathcal{V}_j : j \in \mathbb{Z}\right\}$ of $L^2(\mathbb{R}^+)$ satisfying the following properties:

(a) $\mathcal{V}_j \subset \mathcal{V}_{j+1} \ \forall \ j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j$ is dense in $L^2(\mathbb{R}^+)$;
(c) $\bigcap_{j\in\mathbb{Z}} \mathcal{V}_j = \{0\}$;
(d) $g(x) \in \mathcal{V}_j \iff g(Nx) \in \mathcal{V}_{j+1}\ \forall\, j \in \mathbb{Z}$;
(e) There exists a function $\phi \in \mathcal{V}_0$ such that $\{\phi(x \ominus \sigma) : \sigma \in \Lambda^+\}$, is a complete orthonormal basis for $\mathcal{V}_0$.

It should be noted that the definition of dyadic dilation multiresolution analysis in one dimension can be deduced from the above definitio when $N = 1$. For $N > 1$, the dilation factor of N corroborates that $N\Lambda^+ \subset \mathbb{Z}^+ \subset \Lambda^+$.

For every $j \in \mathbb{Z}$, define $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$. Thus we can write

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j \quad \text{and} \quad \mathcal{W}_m \perp \mathcal{W}_{m'} \quad \text{if } m \neq m'. \tag{2.12}$$

Therefore, it implies that for $j > M$,

$$\mathcal{V}_j = \mathcal{V}_M \oplus \bigoplus_{m=0}^{j-M-1} \mathcal{W}_{j-m}\,. \tag{2.13}$$

By invoking Definition 2.2. (b), this follows that

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} \mathcal{W}_j, \tag{2.14}$$

a decomposition of $L^2(\mathbb{R}^+)$ into mutually orthogonal subspaces.

There exists $N - 1$ functions whose translated and dilated family form an orthonormal basis for $L^2(\mathbb{R}^+)$.

Definition 2.3 A set $\{\psi_\ell : 1 \le \ell \le N-1\} \subset L^2(\mathbb{R})$ is said to be a *set of basic wavelets* associated with the nonuniform multiresolution analysis $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ if the family of functions $\{\psi_\ell(x \ominus \sigma) : 1 \le \ell \le N-1, \sigma \in \Lambda^+\}$ forms an orthonormal basis for $\mathcal{W}_0$.

Results and Discussion

Lemma 3.1 *Let $\phi, \widetilde{\phi} \in L^2(\mathbb{R}^+)$ be given. Then the collection $\{\phi(x \ominus \sigma) : \sigma \in \Lambda^+\}$ is biorthogonal to $\{\widetilde{\phi}(x \ominus \sigma) : \sigma \in \Lambda^+\}$ if and only if*

$$\sum_{\sigma\in\Lambda^+_+} \widehat{\phi}(\zeta \oplus \sigma)\overline{\widehat{\widetilde{\phi}}(\zeta \oplus \sigma)} = 1 \quad a.e\ \zeta \in \mathbb{R}^+.$$

Proof For $\gamma \in \Lambda^+$, it follows that $\left\langle \phi(x \ominus \sigma), \widetilde{\phi}(x \ominus \gamma) \right\rangle = \delta_{\sigma,\gamma} \Leftrightarrow \left\langle \phi, \widetilde{\phi}(x \ominus \gamma) \right\rangle = \delta_{0,\gamma}$. Moreover, we have

$$\begin{aligned}\left\langle \phi, \widetilde{\phi}(x \ominus \gamma)\right\rangle &= \left\langle \widehat{\phi}, \widehat{\widetilde{\phi}}(x \ominus \gamma)\right\rangle \\ &= \int_{\mathbb{R}^+} \widehat{\phi}(\zeta)\overline{\widehat{\widetilde{\phi}}(\zeta)}\,\overline{\chi(\gamma,\zeta)}\mathrm{d}\zeta \\ &= \int_0^{1/2}\left\{\sum_{m\in\mathbb{Z}} \widehat{\phi}\Big(\zeta \oplus \frac{m}{2}\Big)\overline{\widehat{\widetilde{\phi}}\Big(\zeta \oplus \frac{m}{2}\Big)}\chi(\gamma, m)\right\}\overline{\chi(\gamma,\zeta)}\mathrm{d}\zeta.\end{aligned}$$

Using the fact that $\left\{\overline{\chi(\gamma,\zeta)} : \gamma \in \Lambda^+\right\}$ is an orthonormal basis of $L^2\left[0, \frac{1}{2}\right)$, we obtain the desired result. □

Now we proceed to establish a sufficient condition for the translates of a function to be linearly independent.

Lemma 3.2 *Let $\phi \in L^2(\mathbb{R}^+)$. Suppose there exists two constants $C, D > 0$ such that*

$$C \le \sum_{\sigma\in\Lambda^+}\left|\widehat{\phi}(\zeta \oplus \sigma)\right|^2 \le D \quad \text{for a.e } \zeta \in \mathbb{R}^+. \tag{3.1}$$

Then, the set $\left\{\phi(x \ominus \sigma) : \sigma \in \Lambda^+\right\}$ is linearly independent.

Proof For the proof of the lemma, it is sufficient to find another function say $\widetilde{\phi}$ whose translates are biorthogonal to ϕ. To do this, we define the function $\widetilde{\phi}$ by

$$\widehat{\widetilde{\phi}}(\zeta) = \frac{\widehat{\phi}(\zeta)}{\sum_{\sigma\in\Lambda^+}\left|\widehat{\phi}(\zeta \oplus \sigma)\right|^2}.$$

Equation (3.1) implies that $\widetilde{\phi}$ is well defined and

$$\begin{aligned}\sum_{\gamma\in\Lambda^+}\widehat{\phi}(\zeta\oplus\gamma)\overline{\widehat{\widetilde{\phi}}(\zeta\oplus\gamma)} &= \sum_{\gamma\in\Lambda^+}\widehat{\phi}(\zeta\oplus\gamma)\frac{\overline{\widehat{\phi}(\zeta\oplus\gamma)}}{\sum_{\sigma\in\Lambda^+}\left|\widehat{\phi}(\zeta\oplus\sigma\oplus\gamma)\right|^2} \\ &= \frac{\sum_{\gamma\in\Lambda^+}\left|\widehat{\phi}(\zeta\oplus\gamma)\right|^2}{\sum_{\nu\in\sigma}\left|\widehat{\phi}(\zeta+\nu)\right|^2} \\ &= 1.\end{aligned}$$

Applying Lemma 3.1, it follows that the set $\left\{\phi(x \ominus \sigma) : \sigma \in \Lambda^+\right\}$ is linearly independent. Thus the proof is completed . □

Lemma 3.3 *Assume that the scaling function ϕ satisfies inequality (3.1). Let $g = \sum_{\sigma\in\Lambda^+} h_\sigma \phi(x \ominus \sigma)$, where $g \in \operatorname{span}\left\{\phi(x \ominus \sigma) : \sigma \in \Lambda^+\right\}$ and $\left\{h_\sigma\right\}$ is a finite sequence. Define the Fourier transform of h by $\widehat{h}(\zeta) = \sum_{\sigma\in\Lambda^+} h_\sigma \overline{\chi(\sigma,\zeta)}$. Then*

$$C\int_0^{1/2}\left|\widehat{h}(\zeta)\right|^2 d\zeta \le \|g\|_2^2 \le D\int_0^{1/2}\left|\widehat{h}(\zeta)\right|^2 d\zeta.$$

Proof By using Placherel's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^+}\left|g(x)\right|^2 dx &= \int_{\mathbb{R}^+}\left|\sum_{\sigma\in\Lambda^+} h_\sigma\phi(x\ominus\sigma)\right|^2 dx\\
&= \int_{\mathbb{R}^+}\left|\sum_{\sigma\in\Lambda^+} h_\sigma\widehat{\phi}(\zeta)\overline{\chi(\sigma,\zeta)}\right|^2 d\zeta\\
&= \int_{\mathbb{R}^+}\left|\widehat{\phi}(\zeta)\right|^2\left|\sum_{\sigma\in\Lambda^+} h_\sigma\overline{\chi(\sigma,\zeta)}\right|^2 d\zeta\\
&= \int_{\mathbb{R}^+}\left|\widehat{\phi}(\zeta)\right|^2\left|\widehat{h}(\zeta)\right|^2 d\zeta\\
&= \int_0^{1/2}\sum_{m\in\mathbb{Z}}\left|\widehat{\phi}\left(\zeta\oplus\frac{m}{2}\right)\right|^2\left|\widehat{h}(\zeta)\right|^2 d\zeta.
\end{aligned}$$

Using identity (3.1), the result follows. □

Theorem 3.4 *Let $\{\phi(x\ominus\sigma):\sigma\in\Lambda^+\}$ be a Riesz basis for its closed linear span. Suppose that there exists a function $\{\widetilde{\phi}(x\ominus\sigma):\sigma\in\Lambda^+\}$ which is biorthogonal to $\{\phi(x\ominus\sigma):\sigma\in\Lambda^+\}$. Then, for every $f\in\overline{\text{span}}\{\phi(x\ominus\sigma):\sigma\in\Lambda^+\}$, we have*

$$f=\sum_{\sigma\in\Lambda^+}\left\langle f,\widetilde{\phi}(x\ominus\sigma)\right\rangle\phi(x\ominus\sigma); \tag{3.2}$$

and there exists constants $C, D > 0$ such that

$$C\|f\|_2^2\le\sum_{\sigma\in\Lambda^+}\left|\left\langle f,\widehat{\widetilde{\phi}}(\zeta\ominus\sigma)\right\rangle\right|^2\le D\|f\|_2^2. \tag{3.3}$$

Proof We first prove (3.2) and (3.3) for any $f\in\text{span}\{\phi(x\ominus\sigma):\sigma\in\Lambda^+\}$ and then generalize it to $\overline{\text{span}}\{\phi(x\ominus\sigma):\sigma\in\Lambda^+\}$. Let $f\in\text{span}\{\phi(x\ominus\sigma):\sigma\in\Lambda^+\}$, then there exists a finite sequence $\{h_\sigma:\sigma\in\Lambda^+\}$ such that $f=\sum_{\sigma\in\Lambda^+}h_\sigma\phi(x\ominus\sigma)$. Also, the biorthogonality condition implies that

$$\begin{aligned}
\left\langle f,\widetilde{\phi}(x\ominus\gamma)\right\rangle &= \left\langle\sum_{\sigma\in\Lambda^+}h_\sigma\phi(x\ominus\sigma),\widetilde{\phi}(x\ominus\gamma)\right\rangle\\
&= \sum_{\sigma\in\Lambda^+}h_\sigma\left\langle\phi(x\ominus\sigma),\widetilde{\phi}(x\ominus\gamma)\right\rangle\\
&= h_\sigma,
\end{aligned}$$

which proves (3.2). In order to prove (3.3), we make the use of Lemma 3.3 to get

$$D^{-1}\|f\|_2^2 \leq \int_0^{1/2} \left|\widehat{h}(\zeta)\right|^2 d\zeta \leq C^{-1}\|f\|_2^2.$$

Using the Placherel formula for Fourier series and the fact that $h_\sigma = \left\langle f, \widetilde{\phi}(x \ominus \sigma)\right\rangle$, we obtain

$$\int_0^{1/2} \left|\widehat{h}(\zeta)\right|^2 d\zeta = \sum_{\sigma\in\Lambda^+} \left|h_\sigma\right|^2 = \sum_{\sigma\in\Lambda^+} \left|\left\langle f, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2.$$

This proves (3.3). We now generalize the results to $\overline{\text{span}}\left\{\phi(x\ominus\sigma) : \sigma \in \Lambda^+\right\}$. For $f \in \overline{\text{span}}\left\{\widetilde{\phi}(x\ominus\sigma) : \sigma \in \Lambda^+\right\}$, there exists a sequence $\{f_m : m \in \mathbb{Z}\}$ in $\text{span}\left\{\widetilde{\phi}(x\ominus\sigma) : \sigma \in \Lambda^+\right\}$ such that $\|f_m - f\|_2 \to 0$ as $m \to \infty$. Thus, for each $\sigma \in \Lambda^+$, we have

$$\left\langle f_m, \widetilde{\phi}(x\ominus\sigma)\right\rangle \to \left\langle f, \widetilde{\phi}(x\ominus\sigma)\right\rangle \quad \text{as } m \to \infty.$$

Hence, the result holds for each f_m. Thus, we have

$$\begin{aligned}\sum_{\sigma\in\Lambda^+} \left|\left\langle f, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2 &= \sum_{\sigma\in\Lambda^+} \lim_{m\to\infty} \left|\left\langle f_m, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2 \\ &= \lim_{m\to\infty} \sum_{\sigma\in\Lambda^+} \left|\left\langle f_m, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2 \\ &\leq D \lim_{m\to\infty} \|f_m\|_2^2 \\ &= D\|f\|_2^2.\end{aligned} \tag{3.4}$$

Moreover, we have

$$\left\{\sum_{\sigma\in\Lambda^+} \left|\left\langle f_m, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2\right\}^{1/2} \leq \left\{\sum_{\sigma\in\Lambda^+} \left|\left\langle f_m - f, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2\right\}^{1/2} + \left\{\sum_{\sigma\in\Lambda^+} \left|\left\langle f, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2\right\}^{1/2}.$$

As the upper bound in (3.3) holds for $f_m - f$ and lower bound for each f_m, we infer that

$$C^{1/2}\|f\|_2 \leq D^{1/2}\|f_m - f\|_2 + \left(\sum_{\sigma\in\Lambda^+} \left|\left\langle f_m, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2\right)^{1/2},$$

from which we conclude that

$$C\|f\|_2^2 \leq \sum_{\sigma\in\Lambda^+} \left|\left\langle f, \widetilde{\phi}(x\ominus\sigma)\right\rangle\right|^2. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain (3.3). Similarly, we can prove (3.2) for

$$f \in \overline{\mathrm{span}}\{\phi(x \ominus \sigma) : \sigma \in \Lambda^{+}\}$$

and the proof is completed. □

Now we proceed to establish the properties of Nonuniform Biorthogonal wavelets on positive half line.

Let $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ and $\{\widetilde{\mathcal{V}}_j : j \in \mathbb{Z}\}$ be biorthogonal NUMRA's with scaling functions ϕ and $\widetilde{\phi}$. Then there exists integral periodic functions m_0 and $\widetilde{m}_0$ with the property $\widehat{\phi}(\zeta) = m_0(\zeta/N)\widehat{\phi}(\zeta/N)$ and $\widehat{\widetilde{\phi}}(\zeta) = \widetilde{m}_0(\zeta/N)\widehat{\widetilde{\phi}}(\zeta/N)$. Suppose there exists integral periodic functions m_ℓ and $\widetilde{m}_\ell, 1 \le \ell \le N-1$ such that

$$\mathcal{M}(\zeta)\overline{\widetilde{\mathcal{M}}(\zeta)} = I, \tag{3.6}$$

where

$$\mathcal{M}(\zeta) = \begin{pmatrix} m_0\left(\frac{\zeta}{N}\right) & m_0\left(\frac{\zeta}{N} \oplus \frac{1}{2N}\right) & \dots & m_0\left(\frac{\zeta}{N} \oplus \frac{N-1}{2N}\right) \\ m_1\left(\frac{\zeta}{N}\right) & m_2\left(\frac{\zeta}{N} \oplus \frac{1}{2N}\right) & \dots & m_2\left(\frac{\zeta}{N} \oplus \frac{N-1}{2N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1}\left(\frac{\zeta}{N}\right) & m_{N-1}\left(\frac{\zeta}{N} \oplus \frac{1}{2N}\right) & \dots & m_{N-1}\left(\frac{\zeta}{N} \oplus \frac{N-1}{2N}\right) \end{pmatrix}$$

and

$$\widetilde{\mathcal{M}}(\zeta) = \begin{pmatrix} \widetilde{m}_0\left(\frac{\zeta}{N}\right) & \widetilde{m}_0\left(\frac{\zeta}{N} \oplus \frac{1}{2N}\right) & \dots & \widetilde{m}_0\left(\frac{\zeta}{N} \oplus \frac{N-1}{2N}\right) \\ \widetilde{m}_1\left(\frac{\zeta}{N}\right) & \widetilde{m}_2\left(\frac{\zeta}{N} \oplus \frac{1}{2N}\right) & \dots & \widetilde{m}_2\left(\frac{\zeta}{N} \oplus \frac{N-1}{2N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{m}_{N-1}\left(\frac{\zeta}{N}\right) & \widetilde{m}_{N-1}\left(\frac{\zeta}{N} \oplus \frac{1}{2N}\right) & \dots & \widetilde{m}_{N-1}\left(\frac{\zeta}{N} \oplus \frac{N-1}{2N}\right) \end{pmatrix}.$$

For $1 \le \ell \le N-1$, define the associated biorthgonal wavelets as ψ_ℓ and $\widetilde{\psi}_\ell$ by

$$\widehat{\psi}_\ell(\zeta) = m_\ell(\zeta/N)\widehat{\phi}(\zeta/N) \quad \text{and} \quad \widehat{\widetilde{\psi}}_\ell(\zeta) = \widetilde{m}_\ell(\zeta/N)\widehat{\widetilde{\phi}}(\zeta/N).$$

Definition 3.5 A pair of NUMRA's $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ and $\{\widetilde{\mathcal{V}}_j : j \in \mathbb{Z}\}$ with scaling functions ϕ and $\widetilde{\phi}$, respectively are said to be biorthogonal to each other if $\{\phi(\cdot \ominus \sigma) : \sigma \in \Lambda^{+}\}$ and $\{\widetilde{\phi}(\cdot \ominus \sigma) : \sigma \in \Lambda^{+}\}$ are biorthogonal.

Definition 3.6 Let ϕ and $\widetilde{\phi}$ be scaling functions for biorthogonal NUMRA's. For each $j \in \mathbb{Z}$, define the operators P_j and $\widetilde{P}_j$ on $L^2(\mathbb{R}^{+})$ by

$$P_jf = \sum_{\sigma\in\Lambda^+} \langle f, \widetilde{\phi}_{j,\sigma}\rangle \phi_{j,\sigma} \quad \text{and} \quad \widetilde{P}_jf = \sum_{\sigma\in\Lambda^+} \langle f, \phi_{j,\sigma}\rangle \widetilde{\phi}_{j,\sigma},$$

respectively. It is easy to verify that these operators are uniformly bounded on $L^2(\mathbb{R}^+)$ and both the series are convergent in $L^2(\mathbb{R}^+)$.

Remark 3.7

The operators P_j and $\widetilde{P}_j$ satisfy the following properties.

(a) $P_jf = f \iff f \in V_j$ and $\widetilde{P}_jf = f \iff f \in \widetilde{V}_j$.
(b) $\lim_{j\to\infty} \|P_jf - f\|_2 = 0$ and $\lim_{j\to-\infty} \|P_jf\|_2 = 0$ for every $f \in L^2(\mathbb{R}^+)$.

Lemma 3.8 *Let ϕ and $\widetilde{\phi}$ be the scaling functions for biorthogonal NUMRA's and ψ_ℓ and $\widetilde{\psi}_\ell, 1 \le \ell \le N-1$ be the associated wavelets satisfying (3.6). Then, we have the following*

(*a*) $\{\psi_{\ell,0,\sigma} : \sigma \in \Lambda^+\}$ *is biorthogonal to* $\{\widetilde{\psi}_{\ell,0,\gamma} : \gamma \in \Lambda^+\}$,
(*b*) $\langle \psi_{\ell,0,\sigma}, \widetilde{\phi}_{0,\gamma}\rangle = \langle \widetilde{\psi}_{\ell,0,\sigma}, \phi_{0,\gamma}\rangle$, for all $\sigma, \gamma \in \Lambda^+$.

Proof we have

$$\begin{aligned}
&\sum_{t\in\mathbb{Z}} \widehat{\psi}_\ell\left(\zeta \oplus \frac{t}{2}\right)\overline{\widehat{\widetilde{\psi}}_\ell\left(\zeta \oplus \frac{t}{2}\right)}\\
&= \sum_{t\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{N} \oplus \frac{t}{2N}\right)\widehat{\phi}\left(\frac{\zeta}{N} \oplus \frac{t}{2N}\right)\overline{\widetilde{m}_\ell\left(\frac{\zeta}{N} \oplus \frac{t}{2N}\right)}\,\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N} \oplus \frac{t}{2N}\right)}\right\}\\
&= \sum_{s=0}^{N-1}\sum_{t\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{N} \oplus \frac{t}{2} \oplus \frac{s}{2N}\right)\widehat{\phi}\left(\frac{\zeta}{N} \oplus \frac{t}{2} \oplus \frac{s}{2N}\right)\overline{\widetilde{m}_\ell\left(\frac{\zeta}{N} \oplus \frac{t}{2} \oplus \frac{s}{2N}\right)}\,\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N} \oplus \frac{t}{2} \oplus \frac{s}{2N}\right)}\right\}\\
&= \sum_{s=0}^{N-1} \left\{ m_\ell\left(\frac{\zeta}{N} \oplus \frac{s}{2N}\right)\overline{\widetilde{m}_\ell\left(\frac{\zeta}{N} \oplus \frac{s}{2N}\right)}\right\}\\
&= 1.
\end{aligned}$$

Hence, by Lemma 3.1, $\{\psi_{\ell,0,\sigma} : \sigma \in \Lambda^+\}$ is biorthogonal to $\{\widetilde{\psi}_{\ell,0,\sigma} : \sigma \in \Lambda^+\}$. This proves part (a). To prove part (b), we have for, $\sigma, \gamma \in \Lambda^+$

$$
\begin{aligned}
\langle \psi_{\ell,0,\sigma}, \widetilde{\phi}_{0,\gamma} \rangle &= \langle \psi_\ell(x \ominus \sigma), \widetilde{\phi}(x \ominus \gamma) \rangle \\
&= \left\langle \widehat{\psi}_\ell\, \overline{\chi(\sigma.\zeta)}, \widehat{\widetilde{\phi}}\, \overline{\chi(\gamma,\zeta)} \right\rangle \\
&= \int_{\mathbb{R}^+} m_\ell\left(\frac{\zeta}{N}\right) \widehat{\phi}\left(\frac{\zeta}{N}\right) \overline{\chi(\sigma,\zeta)}\, \overline{\widetilde{m}_0\left(\frac{\zeta}{N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\right)} \overline{\chi(\gamma,\zeta)} \mathrm{d}\zeta \\
&= \int_0^{1/2} \sum_{p\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{N} \oplus \frac{p}{2N}\right) \widehat{\phi}\left(\frac{\zeta}{N} \oplus \frac{p}{2N}\right) \right. \\
&\qquad \left. \times \overline{\widetilde{m}_0\left(\frac{\zeta}{N} \oplus \frac{p}{2N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N} \oplus \frac{p}{2N}\right)} \right\} \chi(\gamma \ominus \sigma, \zeta) \mathrm{d}\zeta \\
&= \int_0^{1/2} \sum_{s=0}^{N-1} \sum_{p\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{N} \oplus \frac{p}{2} \oplus \frac{s}{2N}\right) \widehat{\phi}\left(\frac{\zeta}{N} \oplus \frac{p}{2} \oplus \frac{s}{2N}\right) \right. \\
&\qquad \left. \times \overline{\widetilde{m}_0\left(\frac{\zeta}{N} \oplus \frac{p}{2} \oplus \frac{s}{2N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N} \oplus \frac{p}{2} \oplus \frac{s}{2N}\right)} \right\} \chi(\gamma \ominus \sigma, \zeta) \mathrm{d}\zeta \\
&= \int_0^{1/2} \sum_{s=0}^{N-1} \left\{ m_\ell\left(\frac{\zeta}{N} \oplus \frac{s}{2N}\right) \overline{\widetilde{m}_0\left(\frac{\zeta}{N} \oplus \frac{s}{2N}\right)} \right\} \chi(\gamma \ominus \sigma, \zeta) \mathrm{d}\zeta \\
&= 0.
\end{aligned}
$$

The dual one can also be shown equal to zero in a similar manner. This proves part (b) and hence the proof is completed. □

Theorem 3.9 *Let $\phi, \widetilde{\phi}, \psi_\ell$ and $\widetilde{\psi}_\ell, 1 \le \ell \le N-1$ be as in Theorem 3.8. Let $\psi_0 = \phi$ and $\widetilde{\psi}_0 = \widetilde{\phi}$. Then, for every $f \in L^2(\mathbb{R})$, we have*

$$
P_1 f = P_0 f + \sum_{\ell=1}^{N-1} \sum_{\sigma\in\Lambda^+} \langle f, \widetilde{\psi}_{\ell,0,\sigma} \rangle \psi_{\ell,0,\sigma} \tag{3.7}
$$

and

$$
\widetilde{P}_1 f = \widetilde{P}_0 f + \sum_{\ell=1}^{N-1} \sum_{\sigma\in\Lambda^+} \langle f, \psi_{\ell,0,\sigma} \rangle \widetilde{\psi}_{\ell,0,\sigma}. \tag{3.8}
$$

where the series in (3.7) and (3.8) converges in $L^2(\mathbb{R}^+)$.

Proof For the sake of convenience, we will only prove (3.7), as (3.8) is an easy consequence. In particular, we will prove it in the weak sense only. For this, let $f, g \in L^2(\mathbb{R})$. Then, we have

$$
\begin{aligned}
&\sum_{\ell=0}^{N-1}\sum_{\sigma\in\Lambda^+}\left\langle f,\widetilde{\psi}_{\ell,0,\sigma}\right\rangle\overline{\left\langle g,\psi_{\ell,0,\sigma}\right\rangle}\\
&=\sum_{\ell=0}^{N-1}\sum_{\sigma\in\Lambda^+}\left\{\int_{\mathbb{R}^+}\widehat{f}(\zeta)\overline{\widehat{\widetilde{\psi}}_\ell(\zeta)}\chi(\sigma,\zeta)\mathrm{d}\zeta\right\}\left\{\int_{\mathbb{R}^+}\overline{\widehat{g}(\zeta)}\widehat{\psi}_\ell(\zeta)\overline{\chi(\sigma,\zeta)}\mathrm{d}\zeta\right\}\\
&=\sum_{\ell=0}^{N-1}\sum_{\sigma\in\Lambda^+}\left\{\int_0^{1/2}\sum_{p\in\mathbb{Z}}\widehat{f}\left(\zeta\oplus\frac{p}{2}\right)\overline{\widehat{\widetilde{\psi}}_\ell\left(\zeta\oplus\frac{p}{2}\right)}\chi(\sigma,\zeta)\mathrm{d}\zeta\right\}\\
&\quad\times\left\{\int_0^{1/2}\sum_{q\in\mathbb{Z}}\overline{\widehat{g}\left(\zeta\oplus\frac{q}{2}\right)}\widehat{\psi}_\ell\left(\zeta\oplus\frac{q}{2}\right)\overline{\chi(\sigma,\zeta)}\mathrm{d}\zeta\right\}\\
&=\sum_{\ell=0}^{N-1}\int_0^{1/2}\left\{\sum_{p\in\mathbb{Z}}\widehat{f}\left(\zeta\oplus\frac{p}{2}\right)\overline{\widehat{\widetilde{\psi}}_\ell\left(\zeta\oplus\frac{p}{2}\right)}\right\}\left\{\sum_{q\in\mathbb{Z}}\overline{\widehat{g}\left(\zeta\oplus\frac{q}{2}\right)}\widehat{\psi}_\ell\left(\zeta\oplus\frac{q}{2}\right)\right\}\mathrm{d}\zeta
\end{aligned}
$$

$$
\begin{aligned}
&=\int_0^{1/2}\sum_{\ell=0}^{N-1}\left\{\sum_{p\in\mathbb{Z}}\widehat{f}\left(\zeta\oplus\frac{p}{2}\right)\overline{\widetilde{m}_\ell\left(\frac{\zeta}{N}\oplus\frac{p}{2N}\right)}\,\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{p}{2N}\right)}\right.\\
&\quad\left.\times\sum_{q\in\mathbb{Z}}\overline{\widehat{g}\left(\zeta\oplus\frac{q}{2}\right)}m_\ell\left(\frac{\zeta}{N}\oplus\frac{q}{2N}\right)\widehat{\phi}\left(\frac{\zeta}{N}\oplus\frac{q}{2N}\right)\right\}\mathrm{d}\zeta\\
&=\int_0^{1/2}\sum_{\ell=0}^{N-1}\left\{\sum_{r=0}^{N-1}\sum_{p'\in\mathbb{Z}}\widehat{f}\left(\zeta\oplus\frac{p'}{2}N\oplus\frac{r}{2}\right)\overline{\widetilde{m}_\ell\left(\frac{\zeta}{N}\oplus\frac{r}{2N}\oplus\frac{p'}{2}\right)}\,\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{r}{2N}\oplus\frac{p'}{2}\right)}\right.\\
&\quad\left.\times\sum_{s=0}^{N-1}\sum_{q'\in\mathbb{N}_0}\overline{\widehat{g}\left(\zeta\oplus\frac{q'}{2}N\oplus\frac{s}{2}\right)}m_\ell\left(\frac{\zeta}{N}\oplus\frac{s}{2N}\oplus\frac{q'}{2}\right)\widehat{\phi}\left(\frac{\zeta}{N}\oplus\frac{s}{2N}\oplus\frac{q'}{2}\right)\right\}\mathrm{d}\zeta\\
&=\int_0^{1/2}\sum_{r=0}^{N-1}\sum_{p'\in\mathbb{N}_0}\sum_{s=0}^{N-1}\sum_{q'\in\mathbb{N}_0}\left\{\sum_{\ell=0}^{N-1}\overline{\widetilde{m}_\ell\left(\frac{\zeta}{N}\oplus\frac{r}{2N}\right)}m_\ell\left(\frac{\zeta}{N}\oplus\frac{s}{2N}\right)\right\}\\
&\quad\times\widehat{f}\left(\zeta\oplus\frac{p'}{2}N\oplus\frac{r}{2}\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{r}{2N}+\frac{p'}{2}\right)}\,\overline{\widehat{g}\left(\zeta\oplus\frac{q'}{2}N\oplus\frac{s}{2}\right)}\widehat{\phi}\left(\frac{\zeta}{N}\oplus\frac{s}{2N}\oplus\frac{q'}{2}\right)\mathrm{d}\zeta\\
&=\int_0^{1/2}\sum_{p'\in\mathbb{N}_0}\sum_{q'\in\mathbb{N}_0}\sum_{s=0}^{N-1}\widehat{f}\left(\zeta\oplus\frac{p'}{2}N\oplus\frac{s}{2}\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{s}{2N}\oplus\frac{p'}{2}\right)}\\
&\quad\times\overline{\widehat{g}\left(\zeta\oplus\frac{q'}{2}N\oplus\frac{s}{2}\right)}\widehat{\phi}\left(\frac{\zeta}{N}\oplus\frac{s}{2N}\oplus\frac{p'}{2}\right)\mathrm{d}\zeta\\
&=\sum_{s=0}^{N-1}\int_0^{s+1/2}\sum_{p'\in\mathbb{N}_0}\sum_{q'\in\mathbb{N}_0}\widehat{f}\left(\zeta\oplus\frac{p'}{2}N\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{p'}{2}\right)}\,\overline{\widehat{g}\left(\zeta\oplus\frac{q'}{2}N\right)}\widehat{\phi}\left(\frac{\zeta}{N}\oplus\frac{p'}{2}\right)\mathrm{d}\zeta.
\end{aligned}
\tag{3.9}
$$

Furthermore, we have

$$\begin{aligned}
&\sum_{\sigma\in\Lambda^+}\left\langle f,\widetilde{\phi}_{1,\sigma}\right\rangle\overline{\langle g,\phi_{1,\sigma}\rangle}\\
&=\sum_{\sigma\in\Lambda^+}\left\{\int_{\mathbb{R}}\widehat{f}(\zeta)\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\right)}e^{2\pi i\zeta/N}\mathrm{d}\zeta\right\}\left\{\int_{\mathbb{R}}\overline{\widehat{g}(\zeta)}\widehat{\phi}\left(\frac{\zeta}{N}\right)e^{-2\pi i\zeta/N}\mathrm{d}\zeta\right\}\\
&=\int_0^{1/2}\sum_{p\in\mathbb{Z}}\widehat{f}\left(\zeta\oplus\frac{p}{2}N\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{p}{2}\right)}\mathrm{d}\zeta\int_0^{1/2}\sum_{\mathrm{q}\in\mathbb{Z}}\overline{\widehat{\mathrm{g}}\left(\zeta\oplus\frac{\mathrm{q}}{2}\mathrm{N}\right)}\widehat{\phi}\left(\frac{\zeta}{\mathrm{N}}\oplus\frac{\mathrm{q}}{2}\right)\mathrm{d}\zeta\\
&=\int_0^{1/2}\sum_{p\in\mathbb{Z}}\widehat{f}\left(\zeta\oplus\frac{p}{2}N\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{p}{2}\right)}\mathrm{d}\zeta\int_0^{1/2}\sum_{\mathrm{q}\in\mathbb{Z}}\overline{\widehat{\mathrm{g}}\left(\zeta\oplus\frac{\mathrm{q}}{2}\mathrm{N}\right)}\widehat{\phi}\left(\frac{\zeta}{\mathrm{N}}\oplus\frac{\mathrm{q}}{2}\right)\mathrm{d}\zeta\\
&=\int_0^{1/2}\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\widehat{f}\left(\zeta\oplus\frac{p}{2}N\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{N}\oplus\frac{p}{2}\right)}\overline{\widehat{g}\left(\zeta\oplus\frac{q}{2}N\right)}\widehat{\phi}\left(\frac{\zeta}{N}\oplus\frac{q}{2}\right)\mathrm{d}\zeta.
\end{aligned}\tag{3.10}$$

Combing (3.9) and (3.10), we get the desired result. □

Theorem 3.10 *Let ϕ and $\widetilde{\phi}$ be the scaling functions for biorthogonal NUMRA's and ψ_ℓ and $\widetilde{\psi}_\ell, 1\le\ell\le N-1$ be the associated wavelets satisfying the matrix condition* (3.6)*. Then, the collection $\left\{\psi_{\ell,j,\sigma}:1\le\ell\le N-1, j\in\mathbb{Z},\sigma\in\Lambda^+\right\}$ and $\left\{\widetilde{\psi}_{\ell,j,\sigma}:1\le\ell\le N-1, j\in\mathbb{Z},\sigma\in\Lambda^+\right\}$ are biorthogonal. Moreover, if*

$$\left|\widehat{\phi}(\zeta)\right|\le K(1+|\zeta|)^{-\frac{1}{2}-\epsilon},\ \left|\widehat{\widetilde{\phi}}(\zeta)\right|\le K(1+|\zeta|)^{-\frac{1}{2}-\epsilon},\ \left|\widehat{\psi}_\ell(\zeta)\right|\le K|\zeta|\ \textit{and}\ \left|\widehat{\widetilde{\psi}}(\zeta)\right|\le K|\zeta|,\tag{3.11}$$

for some constant $K>0,\epsilon>0$ and for a.e. $\zeta\in\mathbb{R}$, then $\left\{\psi_{\ell,j,\sigma}:1\le\ell\le N-1, j\in\mathbb{Z},\sigma\in\Lambda^+\right\}$ and $\left\{\widetilde{\psi}_{\ell,j,\sigma}:1\le\ell\le N-1, j\in\mathbb{Z},\sigma\in\Lambda^+\right\}$ form Riesz bases for $L^2(\mathbb{R})$.

Proof First we show that $\left\{\psi_{\ell,j,\sigma}:1\le\ell\le N-1, j\in\mathbb{Z},\sigma\in\Lambda^+\right\}$ and $\left\{\widetilde{\psi}_{\ell,j,\sigma}:1\le\ell\le N-1, j\in\mathbb{Z},\sigma\in\Lambda^+\right\}$ are biorthogonal to each other. For this, we will show that for each ℓ, $1\le\ell\le N-1$ and $j\in\mathbb{Z}$,

$$\left\langle\psi_{\ell,j,\sigma},\widetilde{\psi}_{\ell,j,\gamma}\right\rangle=\delta_{\sigma,\gamma}.\tag{3.12}$$

In fact, we have already proved (3.12) for $j=0$. For $j\ne0$, we have

$$\left\langle\psi_{\ell,j,\sigma},\widetilde{\psi}_{\ell,j,\gamma}\right\rangle=\left\langle D_{-j}\psi_{\ell,0,\sigma},D_{-j}\widetilde{\psi}_{\ell,0,\gamma}\right\rangle=\left\langle\psi_{\ell,0,\sigma},\widetilde{\psi}_{\ell,0,\gamma}\right\rangle=\delta_{\sigma,\gamma}.$$

Also, for fixed $\sigma,\gamma\in\Lambda^+$ and $j,j'\in\mathbb{Z}$ with $j<j'$, we claim that

$$\left\langle\psi_{\ell,j,\sigma},\widetilde{\psi}_{\ell',j',\gamma}\right\rangle=0.\tag{3.13}$$

As $\psi_{\ell,0,\sigma}\in\mathcal{V}_1$, hence $\psi_{\ell,j,\sigma}=D_{-j}\psi_{\ell,0,\sigma}\in\mathcal{V}_{j+1}\subseteq\mathcal{V}_{j'}$. Therefore, it is enough to show that $\widetilde{\psi}_{\ell',j',\gamma}$ is orthogonal to every element of $\mathcal{V}_{j'}$. Let $g\in\mathcal{V}_{j'}$. Since $\left\{\phi_{j',\sigma}:\sigma\in\Lambda^+\right\}$ is a Riesz basis for $\mathcal{V}_{j'}$, hence there exists an l^2-sequence $\left\{d_\sigma:\sigma\in\Lambda^+\right\}$ such that $g=\sum_{\sigma\in\Lambda^+}d_\sigma\phi_{j',\sigma}$ in $L^2(\mathbb{R}^+)$. Using part (b) of Lemma 3.6, we have

$$\left\langle \widetilde{\psi}_{\ell',j',\gamma}, \phi_{j',\sigma} \right\rangle = \left\langle D_{-j'} \widetilde{\psi}_{\ell',0,\gamma}, D_{-j'} \phi_{0,\sigma} \right\rangle = 0.$$

Therefore,

$$\left\langle \widetilde{\psi}_{\ell',j',\gamma}, g \right\rangle = \left\langle \widetilde{\psi}_{\ell',j',\gamma}, \sum_{\sigma\in\Lambda^+} d_\sigma \phi_{j',\sigma} \right\rangle = \sum_{\sigma\in\Lambda^+} d_\sigma \left\langle \widetilde{\psi}_{\ell',j',\gamma}, \phi_{j',\sigma} \right\rangle = 0.$$

We now show that these two collections form Riesz bases for $L^2(\mathbb{R}^+)$. The linear independence is clear from the fact that these collections are biorthogonal to each other. So, we have to check the frame inequalities only i.e., there exists constants $C, \widetilde{C}, D, \widetilde{D} > 0$ such that

$$C\|g\|_2^2 \leq \sum_{\ell=1}^{N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda^+} \left|\left\langle g, \psi_{\ell,j,\sigma} \right\rangle\right|^2 \leq D\|g\|_2^2, \quad \forall f \in L^2(\mathbb{R}^+), \tag{3.14}$$

and

$$\widetilde{C}\|g\|_2^2 \leq \sum_{\ell=1}^{N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda^+} \left|\left\langle g, \widetilde{\psi}_{\ell,j,\sigma} \right\rangle\right|^2 \leq \widetilde{D}\|g\|_2^2, \quad \forall f \in L^2(\mathbb{R}^+). \tag{3.15}$$

Let us first check the existence of the upper bounds in (3.14) and (3.15). For this, we have

$$\begin{aligned}
\sum_{\sigma\in\Lambda^+} \left|\left\langle g, \widetilde{\psi}_{\ell,j,\sigma} \right\rangle\right|^2 &= \sum_{\sigma\in\Lambda^+} \left| \int_{\mathbb{R}^+} \widehat{g}(\zeta)(N)^{-j/2}\, \overline{\widehat{\psi}_\ell\left(N^{-j}\zeta\right)}\, \chi(\sigma, N^{-j}\zeta) \mathrm{d}\zeta \right|^2 \\
&= N^{-j} \sum_{\sigma\in\Lambda^+} \left| \int_0^{1/2} \sum_{p\in\mathbb{Z}} \widehat{g}\left(\zeta \oplus (N)^j \frac{p}{2}\right) \overline{\widehat{\psi}_\ell\left((N)^{-j}\zeta \oplus \frac{p}{2}\right)}\, \chi(\sigma, N^{-j}\zeta) \mathrm{d}\zeta \right|^2 \\
&= \int_0^{1/2} \left| \sum_{p\in\mathbb{Z}} \widehat{g}\left(\zeta \oplus (N)^j \frac{p}{2}\right) \overline{\widehat{\psi}_\ell\left((N)^{-j}\zeta \oplus \frac{p}{2}\right)} \right|^2 \mathrm{d}\zeta \\
&= \int_0^{1/2} \left\{ \sum_{p\in\mathbb{Z}} \left|\widehat{g}\left(\zeta \oplus (N)^j \frac{p}{2}\right)\right|^2 \left|\widehat{\psi}_\ell\left((N)^{-j}\zeta \oplus \frac{p}{2}\right)\right|^{2\delta} \right\} \\
&\qquad \times \left\{ \sum_{q\in\mathbb{Z}} \left|\widehat{\psi}_\ell\left((N)^{-j}\zeta \oplus \frac{q}{2}\right)\right|^{2(1-\delta)} \right\} \mathrm{d}\zeta \\
&= \int_{\mathbb{R}^+} \left|\widehat{g}(\zeta)\right|^2 \left|\widehat{\psi}_\ell\left((N)^{-j}\zeta\right)\right|^{2\delta} \sum_{q\in\mathbb{Z}} \left|\widehat{\psi}_\ell\left((N)^{-j}\zeta \oplus \frac{q}{2}\right)\right|^{2(1-\delta)} \mathrm{d}\zeta.
\end{aligned}$$

By our assumption (3.11), we have $|\widehat{\psi}_\ell(\zeta)| \leq K\left(1 + |(N)^{-1}\zeta|\right)^{-1/2-\epsilon}$ and therefore, it follows that $\sum_{q\in\mathbb{Z}} \left|\widehat{\psi}_\ell\left((N)^{-j}\zeta + q/2\right)\right|^{2(1-\delta)}$ is uniformly bounded if $\delta < 2\epsilon(1+2\epsilon)^{-1}$. Thus, there exists a constant $K > 0$ such that

$$\sum_{\sigma\in\Lambda^+}\left|\left\langle g,\widetilde{\psi}_{\ell,j,\sigma}\right\rangle\right|^2 \le K\int_{\mathbb{R}^+}\left|\widehat{g}(\zeta)\right|^2\sum_{j\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((N)^{-j}\zeta\right)\right|^{2\delta}d\zeta$$
$$\le K\sup\left\{\sum_{j\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((N)^{-j}\zeta\right)\right|^{2\delta} : 1\le\zeta\le N\right\}\|g\|_2^2.$$

Also for $\zeta\in[1,N]$, we have

$$\sum_{j=-\infty}^{0}\left|\widehat{\psi}_\ell\left((N)^{-j}\zeta\right)\right|^{2\delta} \le \sum_{j=-\infty}^{0}\frac{K^{2\delta}}{\left(1+|(N)^{j-1}\zeta|\right)^{\delta(1+2\epsilon)}}$$
$$\le \sum_{j=-\infty}^{0}\frac{K^{2\delta}}{(N)^{(j-1)\delta(1+2\epsilon)}}$$
$$\le K^{2\delta}\frac{q^{\delta(1+2\epsilon)}}{1-(N)^{-\delta(1+2\epsilon)}}.$$

Furthermore, we have

$$\sum_{j=1}^{\infty}\left|\widehat{\psi}_\ell\left((N)^{-j}\zeta\right)\right|^{2\delta} \le \sum_{j=1}^{\infty}\left(K(N)^{-j}|\zeta|\right)^{2\delta} \le K^{2\delta}\sum_{j=1}^{\infty}(N)^{(-j+1)2\delta} = K^{2\delta}\frac{1}{1-(N)^{-2\delta}},$$

and hence, it follows that $\sup\left\{\sum_{j\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((N)^{-j}\zeta\right)\right|^{2\delta} : 1\le\zeta\le N\right\}$ is finite. Therefore, there exist $D>0$ such that of (3.15) holds. Similarly, we can show for dual one also. The existence of lower bounds for both the cases can be shown in similar fashion. Thus, we have

$$\|g\|_2^2 = \langle g,g\rangle$$
$$= \left\langle \sum_{\ell=1}^{N-1}\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda^+}\langle g,\widetilde{\psi}_{\ell,j,\sigma}\rangle\psi_{\ell,j,\sigma}, g\right\rangle$$
$$= \sum_{\ell=1}^{N-1}\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda^+}\langle g,\widetilde{\psi}_{\ell,j,\sigma}\rangle\langle\psi_{\ell,j,\sigma},g\rangle$$
$$\le \left(\sum_{\ell=1}^{N-1}\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda^+}\left|\langle g,\widetilde{\psi}_{\ell,j,\sigma}\rangle\right|^2\right)^{1/2}\left(\sum_{\ell=1}^{N-1}\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda^+}\left|\langle g,\psi_{\ell,j,\sigma}\rangle\right|^2\right)^{1/2}$$
$$\le (\widetilde{D})^{1/2}\|g\|_2\left(\sum_{\ell=1}^{N-1}\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda^+}\left|\langle g,\psi_{\ell,j,\sigma}\rangle\right|^2\right)^{1/2}.$$

Hence,

$$\frac{1}{\widetilde{D}}\|g\|_2^2 \le \sum_{\ell=1}^{N-1}\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda^+}\left|\langle g,\psi_{\ell,j,\sigma}\rangle\right|^2.$$

The dual case can be proved in similar lines. This completes the proof. $\square$

Conclusions

In this paper, we develop the comprehensive theory of nonuniform biorthogonal wavelets on the positive half line . We provide the complete characterization for the translates of a single function to form Reisz basis and the associated biorthogonal families with respect to NUMRA in $L^2(\mathbb{R}^+)$. Under some mild assuptions on wavelets associated with NUMRA and the scaling function, we show the wavelets can generate Reisz bases.The results established in this paper are theoretical in nature and will definitely provide new directions to the development of wavelet analysis and widen its field of applications.

Abbreviations

OA: Owais Ahmad; NAS: Neyaz Ahmad Sheikh; MA: Mobin Ahmad.

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Author's contributions

OA wrote abstract and mathematical analysis of the results. NAS wrote the introduction and fixed many grammatical errors. MA wrote conclusion and typed the paper in Latex. All authors read and approved the final manuscript.

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Availability of data and materials

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests

The author declares that there are no competing interests.

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