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Original Article

New types of continuity and openness in fuzzifying bitopological spaces

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Abstract In this paper, we introduce and study the concepts of semicontinuous mappings, α–continuous mappings, semiopen mappings and α–open mappings in fuzzifying bitopological spaces. The characterizations of these mappings along with their relationship with certain other mappings are investigated.

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1. Introduction

In 1965 [1], Zadeh introduced the fundamental concept of fuzzy sets. Since Chang introduced fuzzy sets theory into topology in 1968 [2], Wong, Lowen, Hutton, Pu and Liu, etc., discussed respectively various aspects of fuzzy topology [3–6].

In 1991–1993 [7–9], Ying introduced the concept of the fuzzifying topology with the sematic method of continuous valued logic. In 1994 [10], Park and Lee introduced and discussed the concepts of fuzzy semi-preopen sets and fuzzy semi-precontinuous mappings. Also, in 1994 Kumar [11,12] studied the concepts of fuzzy pairwise α-continuity and pairwise pre-continuity and studied the concepts of semiopen sets, semi continuity and semiopen mappings in fuzzy bitopological spaces. In 1999 Khedr et al. [13], introduced the concepts of semi-open sets and semi-continuity in fuzzifying topology.

The study of bitopological spaces was first initiated by Kelley [14] in 1963. In 2003 Zhang and Liu [15], studied the concept of fuzzy $\theta_{(i,j)}$-closed, $\theta_{(i,j)}$-open sets in fuzzifying bitopological spaces. Also in [16], Gowrisankar et al. studied the concepts of (i,j)-pre open sets in fuzzifying bitopological spaces.

The structure of this paper is organized as follows: In Section (3) we study fuzzy continuity, open mapping in fuzzifying bitopological spaces and we introduce some results. In Section (4) we study α-open set in fuzzifying bitopological spaces and we introduce the relationship between this set and preopen (resp. semiopen) sets. In Section (5) we define the concepts of semicontinuity, α-semicontinuity in fuzzifying bitopological spaces and study the relationship between them. In Section (6) we define the concepts of fuzzy semiopen mapping,

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α-open mapping, preopen mapping in fuzzifying bitopological spaces and study the relationship between them.

2. Preliminaries

Firstly, we display the fuzzy logical and corresponding set-theoretical notations used in this paper:

(1) A formula φ is valid, we write $\models \varphi$ if and only if $[\varphi] = 1$ for every interpretation.
(2) $[\neg\alpha] = 1 - [\alpha]$,$[\alpha \wedge \beta] = \min([\alpha],[\beta])$,$[\alpha \to \beta] = \min(1,1 - [\alpha] + [\beta])$,
$[\forall x\alpha(x)] = \inf_{x\in X}[\alpha(x)]$, where X is the universe of discourse.
(3) $[\alpha \vee \beta] := [\neg(\neg\alpha \wedge \neg\beta)]$; $[\alpha \leftrightarrow \beta] := [(\alpha \to \beta) \wedge (\beta \to \alpha)]$;$[\exists x\alpha(x)] := [\neg(\forall x\neg\alpha(x))]$;-
$[\tilde{A}\tilde{\subseteq}\tilde{B}] := [\forall x(x \tilde{\in}\tilde{A} \to x \tilde{\in}\tilde{B})] = \inf_{x\in X}\min(1,1- \tilde{A}(x)+ \tilde{B}(x))$;$[A \equiv B] := [(\tilde{A}\tilde{\subseteq}\tilde{B}) \wedge (\tilde{B}\tilde{\subseteq}\tilde{A})]$, where $\tilde{A}$, $\tilde{B} \in \Im(X)$ and $\Im(X)$ is the family of all fuzzy sets in X.
(4) $[\alpha \dot{\wedge} \beta] := [\neg(\alpha \to \neg\beta)] = \max(0,[\alpha] + [\beta] - 1)$;
$[\alpha \dot{\vee} \beta] := [\neg\alpha \to \beta] = \min(1,[\alpha] + [\beta])$.

Secondly, we give the following definitions which are used in the sequel.

Definition 2.1 [7] **.** Let X be a universe of discourse, $P(X)$ is the family of subsets of X and $\tau \in \Im(P(X))$ satisfy the following conditions:

(1) $\tau(X) = 1$ and $\tau(\phi) = 1$;
(2) for any $A,B,\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\},\tau(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \tau(A_\lambda)$.

Then τ is a fuzzifying topological space.

Definition 2.2 [7] **.** Let (X,τ) be a fuzzifying topological space.

(1) The family of all fuzzifying closed sets is denoted by $F \in \Im(P(X))$, and defined as follows: $A \in F := X \sim A \in \tau$, where $X \sim A$ is the complement of A.
(2) The neighborhood system of $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as follows:
$N_x(A) = \sup_{x\in B\subseteq A} \tau(B)$.
(3) The closure $cl(A)$ of $A \subseteq X$ is defined as follows: $cl(A)(x) = 1 - N_x(X \sim A)$.
(4) The interior of $A \subseteq X$ is denoted by $int(A) \in \Im(P(X))$ and defined as follows: $int(A) = N_x(A)$.

Definition 2.3 [9] **.** Let (X,τ) and (Y,σ) be two fuzzifying topological spaces.

(1) A unary fuzzy predicate $C \in \Im(Y^X)$, called fuzzy continuity, is given as follows:
$f \in C := \forall u(u \in \sigma \to f^{-1}(u) \in \tau)$. i.e.,

$$C(f) = \inf_{u\in P(Y)} \min(1,1 - \sigma(u) + \tau(f^{-1}(u))).$$

(2) A unary fuzzy predicate $O \in \Im(Y^X)$, called fuzzy openness, is given as follows:
$f \in O := \forall u(u \in \tau \to f(u) \in \sigma)$. i.e.,

$$O(f) = \inf_{u\in P(X)} \min(1,1 - \tau(u) + \sigma(f(u))).$$

Definition 2.4 [15] **.** Let (X,τ_1) and (X,τ_2) be two fuzzifying topological spaces. Then a system (X,τ_1,τ_2) consisting of a universe of discourse X with two fuzzifying topologies τ_1 and τ_2 on X is called a fuzzifying bitopological space.

Definition 2.5 [17] **.** Let (X,τ_1,τ_2) be a fuzzifying bitopological space.

(1) The family of all fuzzifying (i,j)-semiopen sets, denoted by $s\tau_{(i,j)} \in \Im(P(X))$, is defined as follows:
$A \in s\tau_{(i,j)} := \forall x(x \in A \to x \in cl_j(int_i(A)))$, i.e.,
$s\tau_{(i,j)}(A) = \inf_{x\in A} cl_j(int_i(A))(x)$.
(2) The family of all fuzzifying (i,j)-semiclosed sets, denoted by $sF_{(i,j)} \in \Im(P(X))$, is defined as follows:
$A \in sF_{(i,j)} := X \sim A \in s\tau_{(i,j)}$.

Definition 2.6 [17] **.** Let (X,τ_1,τ_2) be a fuzzifying bitopological space and $x \in X$.

(1) The (i,j)-semi neighborhood system of x is denoted by $sN_x^{(i,j)} \in \Im(P(X))$ and defined as
$A \in sN_x^{(i,j)} := \exists B(B \in s\tau_{(i,j)} \wedge x \in B \subseteq A)$, i.e.,
$sN_x^{(i,j)}(A) = \sup_{x\in B\subseteq A} s\tau_{(i,j)}(B)$.
(2) The (i,j)-semi derived set $sd_{(i,j)}(A)$ of A is defined as follows:
$x \in sd_{(i,j)}(A) := \forall B(B \in sN_x^{(i,j)} \to B \cap (A \sim \{x\}) \neq \phi)$,
i.e., $sd_{(i,j)}(A)(x) = \inf_{B\cap(A\sim\{x\})=\phi}(1 - sN_x^{(i,j)}(B))$.
(3) The Fuzzifying (i,j)-semi closure of $A \subseteq X$, is denoted by $scl_{(i,j)}(A)$ and defined as follows:
$x \in scl_{(i,j)}(A) := \forall B((B \supseteq A) \wedge (B \in sF_{(i,j)}) \to x \in B)$,
i.e., $scl_{(i,j)}(A)(x) = \inf_{x\notin B\supseteq A}(1 - sF_{(i,j)}(B))$.
(4) The (i,j)-semi interior of $A \subseteq X$ is defined as follows:
$sint_{(i,j)}(A)(x) = sN_x^{(i,j)}(A)$.
(5) The (i,j)-semi exterior of $A \subseteq X$ is defined as follows:
$x \in sext_{(i,j)}(A) := x \in sint_{(i,j)}(X \sim A)$, i.e.,
$sext_{(i,j)}(A)(x) = sint_{(i,j)}(X \sim A)(x)$.
(6) The (i,j)-semi boundary of $A \subseteq X$ is defined as follows:
$x \in sb_{(i,j)}(A) := (x \notin sint_{(i,j)}(A)) \wedge (x \notin sint_{(i,j)}(X \sim A))$,
i.e., $sb_{(i,j)}(A)(x) = \min(1 - sint_{(i,j)}(A)(x),1 - sint_{(i,j)}(X \sim A)(x))$.

Definition 2.7 [16]**.** Let (X,τ_1,τ_2) be a fuzzifying bitopological space. The family of fuzzifying (i,j)-preopen sets, denoted by $p\tau_{(i,j)} \in \Im(P(X))$, is defined as follows:
$A \in p\tau_{(i,j)} := \forall x(x \in A \to x \in int_i(cl_j(A)))$. i.e.,
$p\tau_{(i,j)}(A) = \inf_{x\in A} int_i(cl_j(A))(x)$.

Definition 2.8 [16] **.** Let (X,τ_1,τ_2) and (X,σ_1,σ_2) be two fuzzifying bitopological spaces. A unary fuzzy predicate $PC_{(i,j)} \in \Im(Y^X)$, called fuzzy precontinuity, is given as follows:
$PC_{(i,j)}(f) := \forall v(v \in \sigma_i \to f^{-1}(v) \in p\tau_{(i,j)})$. i.e.,

$$PC_{(i,j)}(f) = \inf_{v\in P(Y)} \min(1,1 - \sigma_i(v) + p\tau_{(i,j)}(f^{-1}(v))).$$

3. Continuous mapping and open mapping in fuzzifying bitopological spaces

Theorem 3.1. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces.*

If $f:(X,\tau)\to(Y,\sigma)$, then $\models f\in C\leftrightarrow\forall v(f^{-1}(int(v))\subseteq int(f^{-1}(v)))$.

Proof. From Theorem (2.1) in [9], we have

$$\begin{aligned}
&[\forall v(f^{-1}(int(v))\subseteq int(f^{-1}(v)))]\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-f^{-1}(int(v))(x)+int(f^{-1}(v))(x))\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-int(v)(f(x))+int(f^{-1}(v))(x))\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-N_{f(x)}(v)+N_x(f^{-1}(v)))\\
&=[\forall v\forall x(v\in N_{f(x)}\to f^{-1}(v)\in N_x)]\\
&=[f\in C]. \qquad \square
\end{aligned}$$

Theorem 3.2. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces.*

If $f:(X,\tau)\to(Y,\sigma)$ is a fuzzy open mapping, we set

(1) $f\in O^1 := \forall\upsilon\forall x(f^{-1}(v)\in N_x\to\upsilon\in N_{f(x)})$;
(2) $f\in O^2 := \forall\upsilon(f^{-1}(cl(\upsilon))\subseteq cl(f^{-1}(\upsilon)))$;
(3) $f\in O^3 := \forall\upsilon(int(f^{-1}(\upsilon))\subseteq f^{-1}(int(\upsilon)))$;
(4) $f\in O^4 := \forall u(f(int(u))\subseteq int(f(u)))$.

Then $\models O(f)\leftrightarrow O^n(f), n=1,2,3,4$.

Proof.

(1) Firstly, we prove that $[f\in O]\geq[f\in O^1]$. It is clear that for each $u\subseteq X$, there exists $v\subseteq Y$, such that $v=f(u)$, then $u\subseteq f^{-1}(v)$. Therefore

$$\begin{aligned}
[f\in O]&=\inf_{u\in P(X)}\min(1,1-\tau(u)+\sigma(f(u)))\\
&=\inf_{u\in P(X)}\min(1,1-\inf_{x\in u}N_x(u)+\inf_{f(x)\in f(u)}N_{f(x)}(f(u)))\\
&\geq\inf_{u\in P(X),v=f(u)}\min(1,1-\inf_{x\in u}N_x(f^{-1}(v))\\
&\quad+\inf_{x\in u}N_{f(x)}(v))\\
&\geq\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-N_x(f^{-1}(v))\\
&\quad+N_{f(x)}(v))=[f\in O^1].
\end{aligned}$$

Secondly, we prove that $[f\in O]\leq[f\in O^1]$. It is easy to show that if $N_x(f^{-1}(v))\leq N_{f(x)}(v)$, then the result holds. For the case $N_x(f^{-1}(v))>N_{f(x)}(v)$; from Lemma (1.2) in [9], we have that $x\in A\subseteq f^{-1}(v)$, then $f(x)\in f(A)\subseteq v$. So

$$\begin{aligned}
N_x(f^{-1}(v))-N_{f(x)}(v)&=\sup_{x\in A\subseteq f^{-1}(v)}\tau(A)-\sup_{f(x)\in B\subseteq v}\sigma(B)\\
&\leq\sup_{x\in A\subseteq f^{-1}(v)}(\tau(A)-\sigma(f(A))).
\end{aligned}$$

So,

$$\begin{aligned}
&\min(1,1-N_x(f^{-1}(v))+N_{f(x)}(v))\\
&\geq\inf_{x\in A\subseteq f^{-1}(v)}\min(1,1-\tau(A)+\sigma(f(A)))\\
&\geq\inf_{u\in P(X)}\min(1,1-\tau(u)+\sigma(f(u)))=[f\in O].
\end{aligned}$$

Hence
$[f\in O^1]=\inf_{x\in X}\inf_{v\in P(Y)}\min(1,1-N_x(f^{-1}(v))+N_{f(x)}(v))\geq[f\in O]$.

(2) Now to prove that $[f\in O^2]=[f\in O^1]$, we show that

$$\begin{aligned}
&f^{-1}(cl(v))(x)-cl(f^{-1}(v))(x)\\
&=cl(v)(f(x))-cl(f^{-1}(v))(x)\\
&=1-N_{f(x)}(Y\sim v)-(1-N_x(X\sim f^{-1}(v)))\\
&=N_x(f^{-1}(Y\sim v))-N_{f(x)}(Y\sim v).
\end{aligned}$$

Therefore

$$\begin{aligned}
&[f\in O^2]=[\forall v(f^{-1}(cl(v))\subseteq cl(f^{-1}(v)))]\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-f^{-1}(cl(v))(x)+cl(f^{-1}(v))(x))\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-N_x(f^{-1}(Y\sim v))\\
&\quad+N_{f(x)}(Y\sim v))\\
&=\inf_{w\in P(Y)}\inf_{x\in X}\min\big(1,1-N_x(f^{-1}(w))+N_{f(x)}(w)\big)\\
&=[f\in O^1].
\end{aligned}$$

(3) We prove that $[f\in O^3]=[f\in O^1]$, as follows

$$\begin{aligned}
[f\in O^3]&=[\forall v(int(f^{-1}(v))\subseteq f^{-1}(int(v)))]\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-int(f^{-1}(v))(x)\\
&\quad+f^{-1}(int(v))(x))\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-int(f^{-1}(v))(x)\\
&\quad+int(v)(f(x)))\\
&=\inf_{v\in P(Y)}\inf_{x\in X}\min(1,1-N_x(f^{-1}(v))+N_{f(x)}(v))\\
&=[f\in O^1].
\end{aligned}$$

(4) We now prove that $[f\in O^4]=[f\in O^3]$. Note that, for every $v\subseteq Y$, we have $[f(f^{-1}(v))\subseteq v]=1$. So $[int(f(f^{-1}(v)))\subseteq int(v)]=1$. Therefore from Lemma (1.2) in [9], we have $[f^{-1}(int(f(f^{-1}(v))))\subseteq f^{-1}(int(v))]=1$. Furthermore $[int(f^{-1}(v))\subseteq f^{-1}(f(int(f^{-1}(v))))]=1$. So

$$\begin{aligned}
&[int(f^{-1}(v))\subseteq f^{-1}(int(v))]\\
&\geq[f^{-1}(f(int(f^{-1}(v))))\subseteq f^{-1}(int(v))]\\
&\geq[f^{-1}(f(int(f^{-1}(v))))\subseteq f^{-1}(int(f(f^{-1}(v))))]\\
&\geq[f(int(f^{-1}(v)))\subseteq int(f(f^{-1}(v)))].
\end{aligned}$$

Therefore

$$\begin{aligned}
[f\in O^3]&=\inf_{v\in P(Y)}[int(f^{-1}(v))\subseteq f^{-1}(int(v))]\\
&\geq\inf_{v\in P(Y)}[f(int(f^{-1}(v)))\subseteq int\big(f(f^{-1}(v))\big)]\\
&\geq\inf_{u\in P(X)}[f(int(u))\subseteq int(f(u))]=[f\in O^4].
\end{aligned}$$

Now, for each $u\subseteq X$, there exists $v\subseteq Y$, such that $f(u)=v$, then $u\subseteq f^{-1}(v)$. Hence

$$\begin{aligned}
&[int(f^{-1}(v))\subseteq f^{-1}(int(v))]\\
&\leq[int(u)\subseteq f^{-1}(int(f(u)))]\\
&\leq[f(int(u))\subseteq f(f^{-1}(int(f(u))))]\\
&\leq[f(int(u))\subseteq int(f(u))].
\end{aligned}$$

Therefore

$$\begin{aligned}
&[f\in O^4]=\inf_{u\in P(X)}[f(int(u))\subseteq int(f(u))]\\
&\geq\inf_{u\in P(X),v=f(u)}[int(f^{-1}(v))\subseteq f^{-1}(int(v))]\\
&\geq\inf_{v\in P(Y)}[int(f^{-1}(v))\subseteq f^{-1}(int(v))]=[f\in O^3]. \qquad \square
\end{aligned}$$

Definition 3.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces. A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise fuzzy continuous (resp. pairwise fuzzy open) if $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ are fuzzy continuous (resp. fuzzy open).

Theorem 3.3. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise fuzzy continuous and pairwise fuzzy open mapping. Then for each $A \in P(X)$ and $B \in P(Y)$, we have*

(1) $\models A \in s\tau_{(i,j)} \to f(A) \in s\sigma_{(i,j)}$;
(2) $\models B \in s\sigma_{(i,j)} \to f^{-1}(B) \in s\tau_{(i,j)}$.

Proof.

(1) From Theorem (2.1) in [9], Lemma (1.2) in [9] and part (4) of the above Theorem 3.2, we have

$$\begin{aligned}[A \in s\tau_{(i,j)}] &= [A \subseteq cl_j(int_i(A))] \\ &\le [f(A) \subseteq f(cl_j(int_i(A)))] \\ &\le [f(A) \subseteq cl_j(f(int_i(A)))] \\ &\le [f(A) \subseteq cl_j(int_i(f(A)))] \\ &= [f(A) \in s\sigma_{(i,j)}].\end{aligned}$$

(2) From Lemma (1.2) in [9], Theorem 3.1 and part (2) of the Theorem (3.2), we have

$$\begin{aligned}[B \in s\sigma_{(i,j)}] &= [B \subseteq cl_j(int_i(B))] \\ &\le [f^{-1}(B) \subseteq f^{-1}(cl_j(int_i(B)))] \\ &\le [f^{-1}(B) \subseteq cl_j(f^{-1}(int_i(B)))] \\ &\le [f^{-1}(B) \subseteq cl_j(int_i(f^{-1}(B)))] \\ &= [f^{-1}(B) \in s\tau_{(i,j)}]. \qquad \square\end{aligned}$$

4. α-open set in fuzzifying bitopological spaces

Definition 4.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.

(1) The family of all fuzzifying $\alpha_{(i,j)}$-open sets, denoted by $\alpha\tau_{(i,j)} \in \Im(P(X))$, is defined as follows:
$A \in \alpha\tau_{(i,j)} := \forall x(x \in A \to x \in int_i(cl_j(int_i(A))))$,
i.e., $\alpha\tau_{(i,j)}(A) = \inf_{x \in A} int_i(cl_j(int_i(A)))(x)$.
(2) The family of all fuzzifying $\alpha_{(i,j)}$-closed sets, denoted by $\alpha F_{(i,j)} \in \Im(P(X))$, is defined as follows:
$A \in \alpha F_{(i,j)} := X \sim A \in \alpha\tau_{(i,j)}$.

Lemma 4.1. *Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$. Then*

(1) $\models X \sim (int_i(cl_j(int_i(A)))) \equiv cl_i(int_j(cl_i(X \sim A)))$;
(2) $\models X \sim (cl_i(int_j(cl_i(A)))) \equiv int_i(cl_j(int_i(X \sim A)))$.

Proof. It is obvious. $\square$

Theorem 4.1. *Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then*

(1) $\models A \in \alpha\tau_{(i,j)} \leftrightarrow \forall x(x \in A \to \exists B(B \in \alpha\tau_{(i,j)} \wedge x \in B \subseteq A))$;
(2) $\models A \in \alpha F_{(i,j)} \leftrightarrow \forall x(x \in cl_i(int_j(cl_i(A))) \to x \in A)$.

Proof. It is similar to the proof of Theorem (3.4) [17]. $\square$

Theorem 4.2. *Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$, then*

(1) $\models A \in \tau_i \to A \in \alpha\tau_{(i,j)}$;
(2) $\models A \in \alpha\tau_{(i,j)} \to A \in p\tau_{(i,j)}$;
(3) $\models A \in \alpha\tau_{(i,j)} \to A \in s\tau_{(i,j)}$;
(4) $\models A \in \alpha\tau_{(i,j)} \leftrightarrow A \in p\tau_{(i,j)} \wedge A \in s\tau_{(i,j)}$.

Proof.

(1) $[A \in \tau_i] = [A \subseteq int_i(A)] \le [A \subseteq cl_j(int_i(A))] \le [int_i(A) \subseteq int_i(cl_j(int_i(A)))]$.
Therefore $[A \in \tau_i] = [A \subseteq int_i(A)] = [A \subseteq int_i(A)] \wedge [int_i(A) \subseteq int_i(cl_j(int_i(A)))] \le [A \subseteq int_i(cl_j(int_i(A)))] = [A \in \alpha\tau_{(i,j)}]$.
(2) Since $[int_i(A) \subseteq A] = 1$, then $[A \in \alpha\tau_{(i,j)}] = [A \subseteq int_i(cl_j(int_i(A)))] \le [A \subseteq int_i(cl_j(A))] = [A \in p\tau_{(i,j)}]$.
(3) It is similar to (2).
(4) From Definition 5.3 [7], we have

$$\begin{aligned}[A \in s\tau_{(i,j)}] &= [A \subseteq cl_j(int_i(A))] \\ &\le [cl_j(A) \subseteq cl_j(cl_j(int_i(A)))] \\ &\le [cl_j(A) \subseteq cl_j(int_i(A))] \\ &\le [int_i(cl_j(A)) \subseteq int_i(cl_j(int_i(A)))].\end{aligned}$$

Now

$$\begin{aligned}&[A \in p\tau_{(i,j)} \wedge A \in s\tau_{(i,j)}] \\ &= [A \subseteq int_i(cl_j(A)) \wedge A \subseteq cl_j(int_i(A))] \\ &\le [A \subseteq int_i(cl_j(A)) \wedge int_i(cl_j(A)) \subseteq int_i(cl_j(int_i(A)))] \\ &\le [A \subseteq int_i(cl_j(int_i(A)))] \\ &= [A \in \alpha\tau_{(i,j)}].\end{aligned}$$

Conversely from (2) and (3) above, we have $[A \in \alpha\tau_{(i,j)}] \le [A \in p\tau_{(i,j)} \wedge A \in s\tau_{(i,j)}]$. $\square$

Remark 4.1. It is clear from the above Theorem that the following implications are true:

$$\tau_i \to \alpha\tau_{(i,j)} \begin{array}{l} \nearrow s\tau_{(i,j)} \\ \searrow p\tau_{(i,j)} \end{array}$$

The following examples show that generally the reverse of these implications need not be true.

Example 4.1. Let $X = \{a, b, c\}$, $B = \{a, b\}$ and τ_1, τ_2 be two fuzzifying topologies on X defined as follows:

$$\tau_1(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a\}\}, \\ 3/4 & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ 0 & \text{if } o.w. \end{cases},$$

$$\tau_2(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X\}, \\ 1/2 & \text{if } A = \{c\}, \\ 0 & \text{if } o.w. \end{cases}$$

We have, $cl_2(int_1(B))(a) = cl_2(int_1(B))(b) = 1$, $cl_2(int_1(B))(c) = 1/2$, so $s\tau_{(1,2)}(B) = 1$ and $int_1(cl_2(int_1(B)))(a) = 1$, $int_1(cl_2(int_1(B)))(b) = 1/2$, $int_1(cl_2(int_1(B)))(c) = 1/2$, so $\alpha\tau_{(1,2)}(B) = 1/2$.

Therefore $s\tau_{(1,2)}(B) \nsubseteq \alpha\tau_{(1,2)}(B), \alpha\tau_{(1,2)}(B) \nsubseteq \tau_1(B)$ and $\alpha\tau_{(1,2)}(B) \nsubseteq \tau_2(B)$.

Example 4.2. Let $X = \{a, b, c\}$, $B = \{a, b\}$ and τ_1, τ_2 be two fuzzifying topologies on X defined as follows:

$$\tau_1(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X\}, \\ 1/4 & \text{if } A = \{c\}, \\ 0 & \text{if } o.w. \end{cases},$$

$$\tau_2(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a\}\}, \\ 3/4 & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ 0 & \text{if } o.w. \end{cases}$$

We have, $int_1(cl_2(B))(a) = int_1(cl_2(B))(b) = int_1(cl_2(B))(c) = 1/4$, so $p\tau_{(1,2)}(B) = 1/4$ and $cl_2(int_1(B)) \equiv \phi$, $int_1(cl_2(int_1(B))) \equiv \phi$, so $\alpha\tau_{(1,2)}(B) = 0$.

Therefore $p\tau_{(1,2)}(B) \nsubseteq \alpha\tau_{(1,2)}(B)$.

5. Semicontinuity and α-continuity in fuzzifying bitopological spaces

Definition 5.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces.

(1) A unary predicate $SC_{(i,j)} \in \Im(Y^X)$, called fuzzy semicontinuity, is given as follows:
$f \in SC_{(i,j)} := \forall v(v \in \sigma_i \to f^{-1}(v) \in s\tau_{(i,j)})$. i.e.,
$SC_{(i,j)}(f) = \inf_{v \in P(Y)} \min(1, 1 - \sigma_i(v) + s\tau_{(i,j)}(f^{-1}(v)))$.

(2) A unary predicate $\alpha C_{(i,j)} \in \Im(Y^X)$, called fuzzy α-continuity, is given as follows:
$f \in \alpha C_{(i,j)} := \forall v(v \in \sigma_i \to f^{-1}(v) \in \alpha\tau_{(i,j)})$. i.e.,

$$\alpha C_{(i,j)}(f) = \inf_{v \in P(Y)} \min(1, 1 - \sigma_i(v) + \alpha\tau_{(i,j)}(f^{-1}(v))).$$

Remark 5.1. It is clear from Theorem 4.2 and above Definition that the following implications are true:

$$C_i \rightarrow \alpha C_{(i,j)} \begin{array}{l} \nearrow SC_{(i,j)} \\ \searrow PC_{(i,j)} \end{array}$$

The following examples show that generally the reverse of these implications need not be true.

Example 5.1. For $X = \{a, b, c\}$, let τ_1, τ_2, γ_1 and γ_2 be four fuzzifying topologies on X defined as follows:

$$\tau_1(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a\}\}, \\ 3/4 & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ 0 & \text{if } o.w. \end{cases},$$

$$\tau_2(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X\}, \\ 1/2 & \text{if } A = \{c\}, \\ 0 & \text{if } o.w. \end{cases}$$

$$\gamma_1(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a, b\}\}, \\ 0 & \text{if } o.w. \end{cases},$$

$$\gamma_2(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a, b\}\}, \\ 1/2 & \text{if } A = \{a\}, \\ 0 & \text{if } o.w. \end{cases}$$

Consider the identity function f from (X, τ_1, τ_2) onto (X, γ_1, γ_2).

Then $C_1(f) = \inf_{v \in P(X)} \min(1, 1 - \gamma_1(v) + \tau_1(f^{-1}(v)))$.

Note that if $v = X$ or ϕ or $\gamma_1(v) = 0$, then $\min(1, 1 - \gamma_1(v) + \tau_1(f^{-1}(v))) = 1$, that is rejected, since we are looking for $\inf_{v \in P(X)} \min(1, 1 - \gamma_1(v) + \tau_1(f^{-1}(v)))$.

Therefore
$C_1(f) = \min(1, 1 - \gamma_1(\{a, b\}) + \tau_1(f^{-1}(\{a, b\}))) = 0$.

Similarly $C_2(f) = 0$, $SC_{(1,2)}(f) = 1$ and $\alpha C_{(1,2)}(f) = 1/2$.

Therefore $\alpha C_{(1,2)} \nsubseteq C_1, \alpha C_{(1,2)} \nsubseteq C_2$ and $SC_{(1,2)} \nsubseteq \alpha C_{(1,2)}$.

Example 5.2. For $X = \{a, b, c\}$, let τ_1, τ_2, γ_1 and γ_2 be four fuzzifying topologies on X defined as follows:

$$\tau_1(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X\}, \\ 1/4 & \text{if } A = \{c\}, \\ 0 & \text{if } o.w. \end{cases},$$

$$\tau_2(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a\}\}, \\ 3/4 & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ 0 & \text{if } o.w. \end{cases}$$

$$\gamma_1(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a, b\}\}, \\ 0 & \text{if } o.w. \end{cases},$$

$$\gamma_2(A) = \begin{cases} 1 & \text{if } A \in \{\phi, X, \{a, b\}\}, \\ 1/4 & \text{if } A = \{a\}, \\ 0 & \text{if } o.w. \end{cases}$$

Consider the identity function f from (X, τ_1, τ_2) onto (X, γ_1, γ_2). Then $\alpha C_{(1,2)}(f) = 0$, $PC_{(1,2)}(f) = 1/4$.

Therefore $PC_{(1,2)} \nsubseteq \alpha C_{(1,2)}$.

Definition 5.2. Let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$ be two fuzzifying bitopological spaces. For any $f \in Y^X$, we define the unary fuzzy predicate $SC^n_{(i,j)} \in \Im(Y^X)$, where $n = 1, 2, \ldots, 10$, as follows:

(1) $f \in SC^1_{(i,j)} := \forall v(v \in F_i \to f^{-1}(v) \in sF_{(i,j)})$;
(2) $f \in SC^2_{(i,j)} := \forall v \forall x(v \in N^i_{f(x)} \to f^{-1}(v) \in sN^{(i,j)}_x)$;
(3) $f \in SC^3_{(i,j)} := \forall x \forall v(v \in N^i_{f(x)} \to \exists u(f(u) \subseteq v \to u \in sN^{(i,j)}_x))$;
(4) $f \in SC^4_{(i,j)} := \forall u(f(scl_{(i,j)}(u)) \subseteq cl_i(f(u)))$;
(5) $f \in SC^5_{(i,j)} := \forall v(scl_{(i,j)}(f^{-1}(v)) \subseteq f^{-1}(cl_i(v)))$;
(6) $f \in SC^6_{(i,j)} := \forall u(f(sb_{(i,j)}(u)) \subseteq f(u) \cup b_i(f(u)))$;
(7) $f \in SC^7_{(i,j)} := \forall v(f^{-1}(int_i(v)) \subseteq sint_{(i,j)}(f^{-1}(v)))$;
(8) $f \in SC^8_{(i,j)} := \forall v(f^{-1}(ixt_i(v)) \subseteq sixt_{(i,j)}(f^{-1}(v)))$;
(9) $f \in SC^9_{(i,j)} := \forall u(f(sd_{(i,j)}(u)) \subseteq f(u) \cup d_i(f(u)))$;
(10) $f \in SC^{10}_{(i,j)} := \forall x \forall S(S \in N(X) \wedge S \rhd^s_{(i,j)} x \to f \circ S \rhd_i f(x))$.

Lemma 5.1. *Let $A \in P(X)$ and $\tilde{B}, \tilde{C} \in \Im(X)$, then*
$\models \tilde{B} \subseteq A \cup \tilde{C} \leftrightarrow \tilde{B} \cup A \subseteq A \cup \tilde{C}$.

Proof. It is clear. □

Theorem 5.1. *Let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$ be two fuzzifying bitopological spaces and $f \in Y^X$. Then*

(1) $\models f \in SC_{(i,j)} \leftrightarrow f \in SC^n_{(i,j)}, n = 1, 2, \ldots, 9$;
(2) $\models f \in SC_{(i,j)} \to f \in SC^{10}_{(i,j)}$.

Proof.

(1)

(a) We now prove that $[f \in SC_{(i,j)}] = [f \in SC^1_{(i,j)}]$.

$$[f \in SC^1_{(i,j)}] = \inf_{v \in P(Y)} \min(1, 1 - F_i(v) + sF_{(i,j)}(f^{-1}(v))) = \inf_{v \in P(Y)} \min(1, 1 - \sigma_i(Y \sim v) + s\tau_{(i,j)}(X \sim f^{-1}(v))) = \inf_{v \in P(Y)} \min(1, 1 - \sigma_i(Y \sim v) + s\tau_{(i,j)}(f^{-1}(Y \sim v))) = \inf_{w \in P(Y)} \min(1, 1 - \sigma_i(w) + s\tau_{(i,j)}(f^{-1}(w))) = [f \in SC_{(i,j)}].$$

(b) We prove that $[f \in SC_{(i,j)}] = [f \in SC^2_{(i,j)}]$.

$[f \in SC^2_{(i,j)}] = \inf_{x \in X} \inf_{v \in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v)))$.

Firstly, we show that $[f \in SC_{(i,j)}] \le [f \in SC^2_{(i,j)}]$. If $N^i_{f(x)}(v) \le sN^{(i,j)}_x(f^{-1}(v))$, then the result holds. Now suppose $N^i_{f(x)}(v) > sN^{(i,j)}_x(f^{-1}(v))$. From Lemma (1.2) in [9], we have that if $f(x) \in B \subseteq v$, then $x \in f^{-1}(B) \subseteq f^{-1}(v)$. So

$$N^i_{f(x)}(v) - sN^{(i,j)}_x(f^{-1}(v)) = \sup_{f(x) \in B \subseteq v} \sigma_i(B) - \sup_{x \in A \subseteq f^{-1}(v)} s\tau_{(i,j)}(A) \le \sup_{f(x) \in B \subseteq v} \sigma_i(B) - \sup_{f(x) \in B \subseteq v} s\tau_{(i,j)}(f^{-1}(B)) \le \sup_{f(x) \in B \subseteq v} (\sigma_i(B) - s\tau_{(i,j)}(f^{-1}(B))).$$

So

$$\min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v))) \ge \inf_{f(x) \in B \subseteq v} \min(1, 1 - \sigma_i(B) + s\tau_{(i,j)}(f^{-1}(B))) \ge \inf_{B \in P(Y)} \min(1, 1 - \sigma_i(B) + s\tau_{(i,j)}(f^{-1}(B))) = SC_{(i,j)}(f).$$

Hence $SC^2_{(i,j)}(f) = \inf_{x \in X} \inf_{v \in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v))) \ge SC_{(i,j)}(f)$. Secondly, we show that $[f \in SC_{(i,j)}] \ge [f \in SC^2_{(i,j)}]$, as follows

$$SC_{(i,j)}(f) = \inf_{v \in P(Y)} \min(1, 1 - \sigma_i(v) + s\tau_{(i,j)}(f^{-1}(v))) = \inf_{v \in P(Y)} \min(1, 1 - \inf_{f(x) \in v} N^i_{f(x)}(v) + \inf_{x \in f^{-1}(v)} sN^{(i,j)}_x(f^{-1}(v))) \ge \inf_{v \in P(Y)} \min(1, 1 - \inf_{x \in f^{-1}(v)} N^i_{f(x)}(v) + \inf_{x \in f^{-1}(v)} sN^{(i,j)}_x(f^{-1}(v))) \ge \inf_{v \in P(Y)} \inf_{x \in X} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v))) = SC^2_{(i,j)}(f).$$

Therefore $SC_{(i,j)}(f) = SC^2_{(i,j)}(f)$.

(c) We prove that $[f \in SC^2_{(i,j)}] = [f \in SC^3_{(i,j)}]$. With $f(f^{-1}(v)) \subseteq v$, we have

$$[f \in SC^3_{(i,j)}] = \inf_{x \in X} \inf_{v \in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + \sup_{u \in P(X), f(u) \subseteq v} sN^{(i,j)}_x(u)) \ge \inf_{x \in X} \inf_{v \in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v))) = [f \in SC^2_{(i,j)}] \quad (1)$$

Since $u \subseteq f^{-1}(v)$ when $f(u) \subseteq v$ then from Theorem (2.4) in [17], we have $sN^{(i,j)}_x(u) \le sN^{(i,j)}_x(f^{-1}(v))$. So

$$[f \in SC^3_{(i,j)}] = \inf_{x \in X} \inf_{v \in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + \sup_{u \in P(X), f(u) \subseteq v} sN^{(i,j)}_x(u)) \le \inf_{x \in X} \inf_{v \in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + \sup_{u \in P(X), f(u) \subseteq v} sN^{(i,j)}_x(f^{-1}(v))) = \inf_{x \in X} \inf_{v \in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v))) = [f \in SC^2_{(i,j)}] \quad (2)$$

(From equations (1) and (2)) we have $[f \in SC^2_{(i,j)}] = [f \in SC^3_{(i,j)}]$

(d) We prove that $[f \in SC^4_{(i,j)}] = [f \in SC^5_{(i,j)}]$.

Note that for every $v \in P(Y), [f(f^{-1}(v)) \subseteq v] = 1$, then $[cl_i(f(f^{-1}(v))) \subseteq cl_i(v)] = 1$. It is clear from Lemma (1.2) in [9], we have $[f^{-1}(cl_i(f(f^{-1}(v)))) \subseteq f^{-1}(cl_i(v))] = 1$. Furthermore $[scl_{(i,j)}(f^{-1}(v)) \subseteq f^{-1}(f(scl_{(i,j)}(f^{-1}(v))))] = 1$.

So

$$[scl_{(i,j)}(f^{-1}(v)) \subseteq f^{-1}(cl_i(v))] \ge [f^{-1}(f(scl_{(i,j)}(f^{-1}(v)))) \subseteq f^{-1}(cl_i(v))] \ge [f^{-1}(f(scl_{(i,j)}(f^{-1}(v)))) \subseteq f^{-1}(cl_i(f(f^{-1}(v))))] \ge [f(scl_{(i,j)}(f^{-1}(v))) \subseteq cl_i(f(f^{-1}(v)))].$$

Therefore

$$SC^5_{(i,j)}(f) = \inf_{v \in P(Y)} [scl_{(i,j)}(f^{-1}(v)) \subseteq f^{-1}(cl_i(v))] \ge \inf_{v \in P(Y)} [f(scl_{(i,j)}(f^{-1}(v))) \subseteq cl_i(f(f^{-1}(v)))] \ge \inf_{u \in P(X)} [f(scl_{(i,j)}(u)) \subseteq cl_i(f(u))] = SC^4_{(i,j)}(f) \quad (3)$$

Now for each $u \subseteq X$, there exists $v \subseteq Y$, such that $f(u) = v$, then $u \subseteq f^{-1}(v)$. Hence

$$[scl_{(i,j)}(f^{-1}(v)) \subseteq f^{-1}(cl_i(v))] \le [scl_{(i,j)}(u) \subseteq f^{-1}(cl_i(f(u)))] \le [f(scl_{(i,j)}(u)) \subseteq f(f^{-1}(cl_i(f(u))))] \le [f(scl_{(i,j)}(u)) \subseteq cl_i(f(u))].$$

So

$$SC^4_{(i,j)}(f) = \inf_{u \in P(X)} [f(scl_{(i,j)}(u)) \subseteq cl_i(f(u))] \ge \inf_{u \in P(X), v = f(u)} [scl_{(i,j)}(f^{-1}(v)) \subseteq f^{-1}(cl_i(v))] \ge \inf_{v \in P(Y)} [scl_{(i,j)}(f^{-1}(v)) \subseteq f^{-1}(cl_i(v))] = SC^5_{(i,j)}(f) \quad (4)$$

From equations (3) and (4) we have $[f \in SC^4_{(i,j)}] = [f \in SC^5_{(i,j)}]$

(e) We prove that $[f \in SC^2_{(i,j)}] = [f \in SC^5_{(i,j)}]$, as follows

$$SC^5_{(i,j)}(f) = \inf_{v \in P(Y)} \inf_{x \in X} \min(1, 1 - scl_{(i,j)}(f^{-1}(v))(x) + f^{-1}(cl_i(v))(x)) = \inf_{v \in P(Y)} \inf_{x \in X} \min(1, 1 - scl_{(i,j)}(f^{-1}(v))(x) + cl_i(v)(f(x))) = \inf_{v \in P(Y)} \inf_{x \in X} \min(1, 1 - (1 - sN^{(i,j)}_x(X \sim f^{-1}(v))) + (1 - N^i_{f(x)}(Y \sim v)))$$

$$= \inf_{v\in P(Y)} \inf_{x\in X} \min(1, 1 - N^i_{f(x)}(Y \sim v) + sN^{(i,j)}_x(f^{-1}(Y \sim v)))$$
$$= \inf_{w\in P(Y)} \inf_{x\in X} \min(1, 1 - N^i_{f(x)}(w) + sN^{(i,j)}_x(f^{-1}(w))) = SC^2_{(i,j)}(f).$$

(f) We prove that $[f \in SC^4_{(i,j)}] = [f \in SC^6_{(i,j)}]$, as follows. From Lemma 5.1, we have

$$[f \in SC^6_{(i,j)}]$$
$$= \inf_{u\in P(X)} [f(sb_{(i,j)}(u)) \subseteq f(u) \cup b_i(f(u))]$$
$$= \inf_{u\in P(X)} [f(sb_{(i,j)}(u)) \cup f(u) \subseteq f(u) \cup b_i(f(u))]$$
$$= \inf_{u\in P(X)} [f(sb_{(i,j)}(u) \cup u) \subseteq f(u) \cup b_i(f(u))]$$
$$= \inf_{u\in P(X)} [f(scl_{(i,j)}(u)) \subseteq cl_i(f(u))]$$
$$= [f \in SC^4_{(i,j)}].$$

(g) We prove that $[f \in SC^2_{(i,j)}] = [f \in SC^7_{(i,j)}]$. So

$$[f \in SC^7_{(i,j)}] = \inf_{v\in P(Y)} \inf_{x\in X} \min(1, 1 - f^{-1}(int_i(v))(x) + sint_{(i,j)}(f^{-1}(v))(x))$$
$$= \inf_{v\in P(Y)} \inf_{x\in X} \min(1, 1 - int_i(v)(f(x)) + sint_{(i,j)}(f^{-1}(v))(x))$$
$$= \inf_{v\in P(Y)} \inf_{x\in X} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v))) = [f \in SC^2_{(i,j)}].$$

(h) We prove that $[f \in SC^7_{(i,j)}] = [f \in SC^8_{(i,j)}]$, as follows

$$[f \in SC^8_{(i,j)}]$$
$$= \inf_{v\in P(Y)} [f^{-1}(ixt_i(v)) \subseteq sixt_{(i,j)}(f^{-1}(v))]$$
$$= \inf_{v\in P(Y)} [f^{-1}(int_i(Y \sim v)) \subseteq sint_{(i,j)}(X \sim f^{-1}(v))]$$
$$= \inf_{v\in P(Y)} [f^{-1}(int_i(Y \sim v)) \subseteq sint_{(i,j)}(f^{-1}(Y \sim v))]$$
$$= \inf_{w\in P(Y)} [f^{-1}(int_i(w)) \subseteq sint_{(i,j)}(f^{-1}(w))]$$
$$= [f \in SC^7_{(i,j)}].$$

(q) We prove that $[f \in SC^4_{(i,j)}] = [f \in SC^9_{(i,j)}]$. From Lemma 5.1, we have

$$[f \in SC^9_{(i,j)}] = \inf_{u\in P(X)} [f(sd_{(i,j)}(u)) \subseteq f(u) \cup d_i(f(u))]$$
$$= \inf_{u\in P(X)} [f(sd_{(i,j)}(u)) \cup f(u) \subseteq f(u) \cup d_i(f(u))]$$
$$= \inf_{u\in P(X)} [f(scl_{(i,j)}(u)) \subseteq cl_i(f(u))]$$
$$= [f \in SC^4_{(i,j)}].$$

(2) $[f \in SC^{10}_{(i,j)}] = \inf_{S\in N(X)} \inf_{x\in X} \min(1, 1 - [S \rhd^s_{(i,j)} x] + [f \circ S \rhd_i f(x)])$.

If $[S \rhd^s_{(i,j)} x] \le [f \circ S \rhd_i f(x)]$, then the result holds. Assume that $[S \rhd^s_{(i,j)} x] > [f \circ S \rhd_i f(x)]$. Since $f \circ S \overset{\not\subseteq}{\sim} v$ implies $S \overset{\not\subseteq}{\sim} f^{-1}(v)$, we have

$$[S \rhd^s_{(i,j)} x] - [f \circ S \rhd_i f(x)] = \inf_{u\in P(X), S \overset{\not\subseteq}{\sim} u} (1 - sN^{(i,j)}_x(u)) - \inf_{v\in P(Y), f\circ S \overset{\not\subseteq}{\sim} v} (1 - N^i_{f(x)}(v))$$
$$\le \inf_{v\in P(Y), f\circ S \overset{\not\subseteq}{\sim} v} (1 - sN^{(i,j)}_x(f^{-1}(v))) - \inf_{v\in P(Y), f\circ S \overset{\not\subseteq}{\sim} v} (1 - N^i_{f(x)}(v))$$
$$\le \sup_{v\in P(Y), f\circ S \overset{\not\subseteq}{\sim} v} (N^i_{f(x)}(v) - sN^{(i,j)}_x(f^{-1}(v))).$$

Hence

$$\min(1, 1 - [S \rhd^s_{(i,j)} x] + [f \circ S \rhd^s_{(i,j)} f(x)])$$
$$\ge \inf_{v\in P(Y), f\circ S \overset{\not\subseteq}{\sim} v} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v)))$$
$$\ge \inf_{v\in P(Y)} \min(1, 1 - N^i_{f(x)}(v) + sN^{(i,j)}_x(f^{-1}(v)))$$
$$= [f \in SC^2_{(i,j)}]. \quad \square$$

Theorem 5.2. *Let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$ be two fuzzifying bitopological spaces and $f \in Y^X$, we set*

(1) $f \in \alpha C^1_{(i,j)} := \forall v(v \in F_i \to f^{-1}(v) \in \alpha F_{(i,j)})$;
(2) $f \in \alpha C^2_{(i,j)} := \forall x \forall v(v \in N^i_{f(x)} \to \exists u(x \in u \subseteq f^{-1}(v) \wedge u \in \alpha\tau_{(i,j)}))$;
(3) $f \in \alpha C^3_{(i,j)} := \forall x \forall v(v \in N^i_{f(x)} \to \exists u(f(u) \subseteq v \to x \in u \wedge u \in \alpha\tau_{(i,j)}))$;
(4) $f \in \alpha C^4_{(i,j)} := \forall v(cl_i(int_j(cl_i(f^{-1}(v)))) \subseteq f^{-1}(cl_i(v)))$;
(5) $f \in \alpha C^5_{(i,j)} := \forall u(f(cl_i(int_j(cl_i(u)))) \subseteq cl_i(f(u)))$;
(6) $f \in \alpha C^6_{(i,j)} := \forall v(f^{-1}(int_i(v)) \subseteq int_i(cl_j(int_i(f^{-1}(v)))))$.

Then
$\models f \in \alpha C_{(i,j)} \leftrightarrow f \in \alpha C^n_{(i,j)}, \quad n = 1, 2, 3, 4, 5, 6.$

Proof.

(a) The proof of $[f \in \alpha C_{(i,j)}] = [f \in \alpha C^n_{(i,j)}]$, $n = 1, 2, 3,$ is similar to the proof of Theorem 5.1.
(b) It is easy to show that $[f \in \alpha C^4_{(i,j)}] = [f \in \alpha C^5_{(i,j)}]$ and $[f \in \alpha C^4_{(i,j)}] = [f \in \alpha C^6_{(i,j)}]$.
(c) From part (1) of the Theorem 4.1, we have

$$[f \in \alpha C^2_{(i,j)}]$$
$$= [\forall x \forall v(v \in N^i_{f(x)} \to \exists u(x \in u \subseteq f^{-1}(v) \wedge u \in \alpha\tau_{(i,j)}))]$$
$$= \inf_{x\in X} \inf_{v\in P(Y)} (1, 1 - N^i_{f(x)}(v) + \sup_{x\in u\subseteq f^{-1}(v)} \alpha\tau_{(i,j)}(u))$$
$$= \inf_{v\in P(Y)} (1, 1 - \sup_{x\in X} N^i_{f(x)}(v) + \inf_{x\in X\sim f^{-1}(v)} \inf_{x\in f^{-1}(v)} \sup_{x\in u\subseteq f^{-1}(v)} \alpha\tau_{(i,j)}(u))$$
$$= \inf_{v\in P(Y)} (1, 1 - \sup_{x\in X} N^i_{f(x)}(v) + \inf_{x\in X\sim f^{-1}(v)} \alpha\tau_{(i,j)}(f^{-1}(v)))$$
$$= \inf_{v\in P(Y)} (1, 1 - \sup_{x\in X} N^i_{f(x)}(v) + \inf_{x\in X\sim f^{-1}(v)} \inf_{x\in f^{-1}(v)} int_i(cl_j(int_i(f^{-1}(v))))(x))$$
$$= \inf_{v\in P(Y)} (1, 1 - \sup_{x\in X} int_i(v)(f(x)) + \inf_{x\in X} int_i(cl_j(int_i(f^{-1}(v))))(x))$$
$$= \inf_{v\in P(Y)} \inf_{x\in X} (1, 1 - f^{-1}(int_i(v))(x) + int_i(cl_j(int_i(f^{-1}(v))))(x)) = [f \in \alpha C^6_{(i,j)}]. \quad \square$$

Theorem 5.3. *Let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$, and (Z, γ_1, γ_2) be three fuzzifying bitopological spaces. For any $f \in Y^X, g \in Z^Y$.*

(1) $\models SC_{(i,j)}(f) \to (C_i(g) \to SC_{(i,j)}(g \circ f))$;
(2) $\models C_i(g) \to (SC_{(i,j)}(f) \to SC_{(i,j)}(g \circ f))$;
(3) $\models \alpha C_{(i,j)}(f) \to (C_i(g) \to \alpha C_{(i,j)}(g \circ f))$;
(4) $\models C_i(g) \to (\alpha C_{(i,j)}(f) \to \alpha C_{(i,j)}(g \circ f))$.

Proof.

(1) Firstly, if $[C_i(g)] \leq [SC_{(i,j)}(g \circ f)]$, then the result holds. Secondly, if $[C_i(g)] > [SC_{(i,j)}(g \circ f)]$, then

$$\begin{aligned}
&[C_i(g)] - [SC_{(i,j)}(g \circ f)] \\
&\quad = \inf_{v \in P(Z)} \min(1, 1 - \gamma_i(v) + \sigma_i(g^{-1}(v))) \\
&\quad - \inf_{v \in P(Z)} \min(1, 1 - \gamma_i(v) + s\tau_{(i,j)}((g \circ f)^{-1}(v))) \\
&\quad \leq \sup_{v \in P(Z)} (\sigma_i(g^{-1}(v)) - s\tau_{(i,j)}((g \circ f)^{-1}(v))) \\
&\quad \leq \sup_{u \in P(Y)} (\sigma_i(u) - s\tau_{(i,j)}(f^{-1}(u))).
\end{aligned}$$

Therefore

$$\begin{aligned}
&[C_i(g) \to SC_{(i,j)}(g \circ f)] \\
&\quad = \min(1, 1 - C_i(g) + SC_{(i,j)}(g \circ f)) \\
&\quad \geq \inf_{u \in P(Y)} \min(1, 1 - \sigma_i(u) + s\tau_{(i,j)}(f^{-1}(u))) \\
&\quad = SC_{(i,j)}(f).
\end{aligned}$$

(2)

$$\begin{aligned}
&[C_i(g) \to (SC_{(i,j)}(f) \to SC_{(i,j)}(g \circ f))] \\
&\quad = [\neg(C_i(g) \dot\wedge (SC_{(i,j)}(f) \dot\wedge \neg SC_{(i,j)}(g \circ f)))] \\
&\quad = [\neg(SC_{(i,j)}(f) \dot\wedge (C_i(g) \dot\wedge \neg SC_{(i,j)}(g \circ f)))] \\
&\quad = [SC_{(i,j)}(f) \to (C_i(g) \to SC_{(i,j)}(g \circ f))].
\end{aligned}$$

The proofs of (3) and (4) are similar to (1) and (2) above. □

6. Semiopen mapping, α-open mapping and preopen mapping in fuzzifying bitopological spaces

Definition 6.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces.

(1) A unary predicate $SO_{(i,j)} \in \Im(Y^X)$, called fuzzy semiopenness, is given as follows:
$f \in SO_{(i,j)} := \forall u(u \in \tau_i \to f(u) \in s\sigma_{(i,j)})$. i.e.,
$SO_{(i,j)}(f) = \inf_{u \in P(X)} \min(1, 1 - \tau_i(u) + s\sigma_{(i,j)}(f(u)))$.

(2) A unary predicate $\alpha O_{(i,j)} \in \Im(Y^X)$, called fuzzy α-openness, is given as follows:
$f \in \alpha O_{(i,j)} := \forall u(u \in \tau_i \to f(u) \in \alpha\sigma_{(i,j)})$. i.e.,

$$\alpha O_{(i,j)}(f) = \inf_{u \in P(X)} \min(1, 1 - \tau_i(u) + \alpha\sigma_{(i,j)}(f(u))).$$

(3) A unary predicate $PO_{(i,j)} \in \Im(Y^X)$, called fuzzy preopenness, is given as follows:
$f \in PO_{(i,j)} := \forall u(u \in \tau_i \to f(u) \in p\sigma_{(i,j)})$. i.e.,

$$PO_{(i,j)}(f) = \inf_{u \in P(X)} \min(1, 1 - \tau_i(u) + p\sigma_{(i,j)}(f(u))).$$

Remark 6.1. It is clear from Theorem 4.2 and above Definition that the following implications are true:

$$O_i \to \alpha O_{(i,j)} \begin{array}{l} \nearrow SO_{(i,j)} \\ \searrow PO_{(i,j)} \end{array}$$

The following examples show that generally the reverse of these implications need not be true.

Example 6.1. For $X = \{a, b, c\}$, let τ_1, τ_2, γ_1 and γ_2 be four fuzzifying topologies, which are defined on X in Example 5.1.

Consider identity function f from (X, γ_1, γ_2) onto (X, τ_1, τ_2). Then $O_1(f) = O_2(f) = 0$, $SO_{(1,2)}(f) = 1$ and $\alpha O_{(1,2)}(f) = 1/2$. Therefore $\alpha O_{(1,2)} \nsubseteq O_1$, $SO_{(1,2)} \nsubseteq \alpha O_{(1,2)}$.

Example 6.2. For $X = \{a, b, c\}$, let τ_1, τ_2, γ_1 and γ_2 be four fuzzifying topologies, which are defined on X in Example 5.2.

Consider identity function f from (X, γ_1, γ_2) onto (X, τ_1, τ_2). Then $\alpha O_{(1,2)}(f) = 0$, $PO_{(1,2)}(f) = 1/4$. Therefore $PO_{(1,2)} \nsubseteq \alpha O_{(1,2)}$.

Theorem 6.1. *Let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$ be two fuzzifying bitopological spaces and $f \in Y^X$, we set*

(1) $f \in SO^1_{(i,j)} := \forall v \forall x(f^{-1}(v) \in N^i_x \to v \in sN^{(i,j)}_{f(x)})$;
(2) $f \in SO^2_{(i,j)} := \forall v(f^{-1}(scl_{(i,j)}(v)) \subseteq cl_i(f^{-1}(v)))$;
(3) $f \in SO^3_{(i,j)} := \forall v(int_i(f^{-1}(v)) \subseteq f^{-1}(sint_{(i,j)}(v)))$;
(4) $f \in SO^4_{(i,j)} := \forall u(f(int_i(u)) \subseteq sint_{(i,j)}(f(u)))$;
(5) $f \in \alpha O^1_{(i,j)} := \forall u(f(int_i(u)) \subseteq int_i(cl_j(int_i(f(u)))))$.
(6) $f \in \alpha O^2_{(i,j)} := \forall v(int_i(f^{-1}(v)) \subseteq f^{-1}(int_i(cl_j(int_i(v)))))$;
(7) $f \in \alpha O^3_{(i,j)} := \forall v(f^{-1}(cl_i(int_j(cl_i(v)))) \subseteq cl_i(f^{-1}(v)))$;

Then

(1) $\models f \in SO_{(i,j)} \leftrightarrow f \in SO^n_{(i,j)}$, $n = 1, 2, 3, 4$;
(2) $\models f \in \alpha O^n_{(i,j)} \to f \in \alpha O_{(i,j)}$, $n = 1, 2, 3$.

Proof.

(1) It is similar to the proof of Theorem 3.2.

(2)

(a)

$$\begin{aligned}
[f \in \alpha O_{(i,j)}] &= [\forall u(u \in \tau_i \to f(u) \in \alpha\sigma_{(i,j)})] \\
&= \inf_{u \in P(X)} [u \subseteq int_i(u) \to f(u) \\
&\subseteq int_i(cl_j(int_i(f(u))))] \\
&\geq \inf_{u \in P(X)} [f(u) \subseteq f(int_i(u)) \to f(u) \\
&\subseteq int_i(cl_j(int_i(f(u))))] \\
&= \inf_{u \in P(X)} \min(1, 1 - \inf_{y \in f(u)} f(int_i(u))(y) \\
&+ \inf_{y \in f(u)} int_i(cl_j(int_i(f(u))))(y)) \\
&\geq \inf_{u \in P(X)} \inf_{y \in Y} \min(1, 1 - f(int_i(u))(y) \\
&+ int_i(cl_j(int_i(f(u))))(y)) \\
&= [f \in \alpha O^1_{(i,j)}]
\end{aligned}$$

(b) It is easy to show that $[f \in \alpha O^1_{(i,j)}] = [f \in \alpha O^2_{(i,j)}]$ and $[f \in \alpha O^2_{(i,j)}] = [f \in \alpha O^3_{(i,j)}]$. □

Theorem 6.2. *Let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$, and (Z, γ_1, γ_2) be three fuzzifying bitopological spaces. For any $f \in Y^X, g \in Z^Y$.*

(1) $\models SO_{(i,j)}(g) \to (O_i(f) \to SO_{(i,j)}(g \circ f))$;
(2) $\models O_i(f) \to (SO_{(i,j)}(g) \to SO_{(i,j)}(g \circ f))$;
(3) $\models \alpha O_{(i,j)}(g) \to (O_i(f) \to \alpha O_{(i,j)}(g \circ f))$;
(4) $\models O_i(f) \to (\alpha O_{(i,j)}(g) \to \alpha O_{(i,j)}(g \circ f))$.

Proof. It is similar to the proof of the Theorem 5.3. □

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