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Original Article

Geometric visualization of parallel bivariate Pareto distribution surfaces

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Abstract In the present paper, the differential-geometrical framework for parallel bivariate Pareto distribution surfaces $(P,\overline{P})$ is given. Curvatures of a curve lying on $(P,\overline{P})$, are interpreted in terms of the parameters of P. Geometrical and statistical interpretations of some results are introduced and plotted.

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1. Introduction

Information geometry (Geometry and Nature) has emerged from the study of invariant properties of the manifold of probability distributions. It is regarded as mathematical sciences having vast developing areas of applications as well as giving new trends in geometrical and topological methods. Information geometry has many applications which are treated in many different branches, for instance, statistical inference, linear and nonlinear systems, time series, neural networks, linear programing, convex analysis, completely integrable dynamical systems, quantum information geometry and geometric modeling [1]. A classical and intuitive way of describing the relationship between the differential geometry and the statistics is introduced, see, for instance [2–7], but in a slightly modified manner.

Pareto distribution is named after an Italian-born Swiss professor of economics, Vilfredo Pareto (1848–1923). Pareto [8] originally used this distribution to describe the allocation of wealth among individuals since it seemed to show rather well the way that a large portion of wealth of any society is owned by a smaller percentage of the people in that society [8,9]. Pareto distribution plays an important role in socio-economic studies. It is often used as a model for analyzing areas including city population distribution, stock price fluctuations and oil field location. In addition, it has found applications in the military area. It has been found to be suitable for approximating the right tail of distribution with positive skewness [10].

Bivariate Pareto distributions are popular models in many applied areas. They are very versatile and a variety of uncertainties can be usefully modeled by them. We mention: modeling of radiation carcinogenesis, performance measures for general sys-

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tems, reliability, modeling of drought, modeling of dependent heavy tailed risks with a non-zero probability of simultaneous loss and modeling of daily exchange rate data [11].

Creation of parallel surfaces is useful in design and manufacture. Enhancing or reducing the size of free-form surfaces requires calculation of curvature and other properties of a new surface, which is parallel to the original surface. In the Riemannian framework, several authors studied parallel and semi-parallel submanifolds, and a good survey can be found in [12].

In the differential geometry of surfaces, a Darboux frame is a natural moving frame constructed on a surface. It is the analog of the Frenet–Serret frame as applied to surface geometry. A geodesic curve is intrinsic to the geometric characterization of surfaces. Geodesics are used in many fields, for example, they are used in object segmentation, multi-scale image analysis, computer vision and image processing [13].

Abdel-All et al. [14] defined the parameter space of one-dimensional Pareto distribution of the first kind using its Fisher's matrix. They calculated the Riemannian and scalar curvatures to the parameter space. The differential equations of the geodesics are obtained and solved. The J-divergence, the geodesic distance and the relations between them are found. A development of the relation between the J-divergence and the geodesic distance is illustrated. The scalar curvature of the J-space is represented.

Many different forms of bivariate Pareto distributions have been constructed in the literature [15]. The main objective of this paper is to study a bivariate Pareto distribution (two-dimensional Pareto distribution) of the first kind that was given by Mardia, cited in [15], corresponding to the one-dimensional Pareto distribution of the first kind [14], without using its Fisher's matrix.

2. Geometrical and statistical preliminaries

Let $P:\mathbf{M}=\mathbf{M}(u,v)$ be an orientable surface and let $\mathbf{N}$ be a unit normal vector field of P. We consider a surface $\overline{P}$ to be parallel to P if there is a normal geodesic congruence between P and $\overline{P}$ such that the distance between corresponding points is constant, i.e. for each $\mathbf{M}\in P$ we have

$$\overline{P}:\overline{\mathbf{M}}(u,v)=\mathbf{M}(u,v)+r\mathbf{N}(u,v), \tag{1}$$

where, $r\neq 0$ is a real constant. We can say that P and $\overline{P}$ are parallel surfaces at distance r. If K,H and $\overline{K},\overline{H}$ denote the Gaussian and mean curvatures of P and $\overline{P}$, respectively, then we have [16]:

$$\overline{K}=\frac{K}{\Omega},\quad \overline{H}=\frac{H+rK}{\Omega},\quad \Omega=1+2rH+r^2K\neq 0, \tag{2}$$

where, the relation between the principal curvatures (κ_1,κ_2) and $(\overline{\kappa}_1,\overline{\kappa}_2)$ of $(P,\overline{P})$ is given by

$$\overline{\kappa}_1=\frac{\kappa_1}{1+r\kappa_1},\quad \overline{\kappa}_2=\frac{\kappa_2}{1+r\kappa_2}.$$

Let P be a surface, and let $\boldsymbol{\beta}$ be a unit speed curve on P. At each point on $\boldsymbol{\beta}$, consider the following three vectors: the unit normal vector $\mathbf{N}$ to the surface, the unit tangent vector $\mathbf{t}$ to the curve and the tangent normal vector $\mathbf{E}=\mathbf{N}\wedge\mathbf{t}$. This vector is tangent to the surface P, but normal to the curve $\boldsymbol{\beta}$. These vectors $\{\mathbf{t};\mathbf{E};\mathbf{N}\}$ form a right-handed frame, known as the Darboux frame for $\boldsymbol{\beta}$ on P. Darboux equations for this frame are given by [16,17]

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{E}\\ \mathbf{N}\end{pmatrix}=\begin{pmatrix}0 & \kappa_g & \kappa_n\\ -\kappa_g & 0 & \tau_g\\ -\kappa_n & -\tau_g & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{E}\\ \mathbf{N}\end{pmatrix}, \tag{3}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of $\boldsymbol{\beta}$. Thus, we can write κ_g, κ_n and τ_g in the form

$$\kappa_g=(\boldsymbol{\beta}',\mathbf{N},\boldsymbol{\beta}''),\quad \kappa_n=(\boldsymbol{\beta}'',\mathbf{N}),\quad \tau_g=(\boldsymbol{\beta}',\mathbf{N},\mathbf{N}'), \tag{4}$$

and if $\boldsymbol{\beta}$ is not parameterized by arc length, the above relations take the forms

$$\kappa_g=\frac{1}{|\boldsymbol{\beta}'|^3}(\boldsymbol{\beta}',\mathbf{N},\boldsymbol{\beta}''),\quad \kappa_n=\frac{1}{|\boldsymbol{\beta}'|^2}(\boldsymbol{\beta}'',\mathbf{N}),\quad \tau_g=\frac{1}{|\boldsymbol{\beta}'|}(\boldsymbol{\beta}',\mathbf{N},\mathbf{N}'). \tag{5}$$

The bivariate distribution with joint density function for $\alpha>0$

$$f_{X,Y}\ (x,y;\gamma,\sigma,\alpha)=\alpha(\alpha+1)(\gamma\sigma)^{\alpha+1}\lambda^{-(\alpha+2)},\quad x\geq\gamma>0,\ y\geq\sigma>0, \tag{6}$$

where, $\lambda=\sigma x+\gamma y-\gamma\sigma$ may be called a bivariate Pareto distribution of the first kind [15], since the marginal distributions have density functions

$$f_{X_i}(x_i;\theta_i,\alpha)=\alpha\theta_i^{\alpha}x_i^{-(\alpha+1)},\quad x_i\geq\theta_i>0,\ i=1,2, \tag{7}$$

where, $X_1=X, X_2=Y, x_1=x, x_2=y, \theta_1=\gamma, \theta_2=\sigma$.

It can be seen that, for $\alpha>1,\alpha>2$,

$$E(X_i)=\frac{\alpha}{\alpha-1}\theta_i,\quad E(X_1X_2)=\frac{(\alpha^2-\alpha-1)}{(\alpha-1)(\alpha-2)}\theta_1\theta_2,$$
$$Var(X_i)=\frac{\alpha}{(\alpha-1)^2(\alpha-2)}\theta_i^2. \tag{8}$$

The conditional density function of Y, given $X=x$, is

$$f_{Y|X}\ (y|x)=(\alpha+1)\gamma(\sigma x)^{\alpha+1}\lambda^{-(\alpha+2)},\ y\geq\sigma>0,\gamma>0,\alpha>0. \tag{9}$$

The conditional density function of X, given $Y=y$, is

$$f_{X|Y}\ (x|y)=(\alpha+1)\sigma(\gamma y)^{\alpha+1}\lambda^{-(\alpha+2)},\quad x\geq\gamma>0,\sigma>0,\alpha>0. \tag{10}$$

Therefore, for $\alpha>1$, we also find

$$E(Y|X=x)=\sigma\left(1+\frac{x}{\gamma\alpha}\right),$$
$$Var(Y|X=x)=\left(\frac{\sigma}{\gamma}\right)^2\frac{(\alpha+1)x^2}{\alpha^2(\alpha-1)}, \tag{11}$$

$$E(X|Y=y)=\gamma\left(1+\frac{y}{\sigma\alpha}\right),$$
$$Var(X|Y=y)=\left(\frac{\gamma}{\sigma}\right)^2\frac{(\alpha+1)y^2}{\alpha^2(\alpha-1)}. \tag{12}$$

Using (8), we find

$$Cov(X,Y)=E(XY)-E(X)E(Y)=\frac{\gamma\sigma}{(\alpha-1)^2(\alpha-2)},\quad \alpha\neq 1,\ \alpha\neq 2, \tag{13}$$

and consequently, the correlation between X and Y, denoted by $R\equiv Cor(X,Y)$, is given from

$$R \equiv Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{1}{\alpha}, \quad \alpha > 0, \alpha \neq 1, \alpha \neq 2. \tag{14}$$

This shows that X and Y are positively correlated.

3. Geometry of parallel bivariate Pareto distribution surfaces

The bivariate Pareto density given in (6) is represented by an explicit form (Mong formula), i.e. $z = f(x,y)$ as a 2-dimensional surface in Euclidean 3-space [16,17]. Thus, we consider $x = u, y = v$ and the equation of the bivariate Pareto distribution surface in parametric form is given by

$$P : \mathbf{M}(u,v) = \{u, v, \alpha(\alpha+1)(\gamma\sigma)^{\alpha+1}\lambda^{-(\alpha+2)}\}, \tag{15}$$

where, $\lambda(u,v) = \sigma u + \gamma v - \gamma\sigma \neq 0$.

Thus, the metric on the surface P is given by

$$g(u,v) = 1 + (\gamma\sigma)^{2(\alpha+1)}(\gamma^2+\sigma^2)\lambda^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2, \lambda \neq 0, \tag{16}$$

where, $g = \det(g_{ij}), g_{ij}$ are the 1-st fundamental quantities of the surface P and are given by

$$g_{11} = 1 + \Phi\sigma^2, g_{22} = 1 + \Phi\gamma^2, g_{12} = \Phi\sigma\gamma \text{ and } \Phi = (\gamma\sigma)^{2(\alpha+1)}\lambda^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2.$$

The unit normal vector field of the surface P is given by

$$\mathbf{N} = \frac{1}{\sqrt{g}}\{\sigma\phi, \gamma\phi, 1\}, \tag{17}$$

where, $\phi = \alpha(\gamma\sigma)^{1+\alpha}(2 + 3\alpha + \alpha^2)\lambda^{-(\alpha+3)}, \ \lambda \neq 0$.

Thus, one can get the Gaussian curvature function K of P is equal to zero and consequently, from the relation (2), the Gaussian curvature function $\overline{K}$ of its parallel surface $\overline{P}$ is equal to zero also. Therefore, we have the following:

Corollary 1. *The points on bivariate Pareto distribution surface P and their images on $\overline{P}$ are parabolic points. Consequently, the parabolic points on bivariate Pareto distribution surface P and their images on $\overline{P}$, are in one to one correspondence.*

Hence, and using (2) one can obtain the following:

Corollary 2. *The mean curvature functions of parallel bivariate Pareto distribution surface $\overline{P}$ and its original surface P are given, respectively, by*

$$\overline{H} = \frac{H}{1+2rH}, \quad H \neq -\frac{1}{2r}, \tag{18}$$

and

$$H(u,v) = -\frac{1}{2g^{\frac{3}{2}}}\{(\gamma\sigma)^{\alpha+1}\lambda^{-(\alpha+10)}\prod_{i=0}^{3}(\alpha+i)[2(\gamma\sigma)^{2(\alpha+2)}\prod_{i=0}^{2} \times (\alpha+i)^2 + \sigma^2\lambda^{2(\alpha+3)}(1+\gamma^2\eta) + \gamma^2\lambda^{2(\alpha+3)}(1+\sigma^2\eta)]\}, g \neq 0, \tag{19}$$

where, $\eta(u,v) = (\gamma\sigma)^{2(\alpha+1)}\lambda^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2, \lambda \neq 0$.

From the above two corollaries and using (2), one can compute the principal curvatures (κ_1,κ_2) and $(\overline{\kappa}_1,\overline{\kappa}_2)$ of $(P,\overline{P})$ as the following:

$$(\kappa_1,\kappa_2) = (2H,0) \text{ and } (\overline{\kappa}_1,\overline{\kappa}_2) = (2\overline{H},0).$$

3.1. Curvatures of the curves lying on P and $\overline{P}$

Here, the geodesic, normal curvatures and geodesic torsion on P and $\overline{P}$ surfaces in terms of the parameters of P are obtained. The rest of this subsection is an attempt to find the necessary and sufficient conditions for curves on the surfaces P and $\overline{P}$ to be geodesic, asymptotic lines and lines of curvature. For this purpose we recall the following definitions [16,17]:

(1) $\boldsymbol{\beta}$ is a geodesic curve if the geodesic curvature κ_g vanishes identically.
(2) $\boldsymbol{\beta}$ is an asymptotic line if the normal curvature κ_n vanishes identically.
(3) $\boldsymbol{\beta}$ is a line of curvature if the geodesic torsion τ_g vanishes identically.

Making use of the bivariate Pareto distribution surface P, given from (15), let the curve $\boldsymbol{\beta}(u(v))$ lying on P. Therefore, we consider for simplicity, $u = v$. Taking (5) into account and considering that $|\boldsymbol{\beta}'| \neq 0, \varphi_1 \neq 0$ and $\mu \neq 0$, one can get the following:

Corollary 3. *The geodesic curvature, normal curvature and geodesic torsion of $\boldsymbol{\beta}$ on P are given from*

$$k_g = -\frac{(\gamma-\sigma)}{|\boldsymbol{\beta}'|^3\varphi_1^{\frac{1}{2}}(v)} \times \left[(\gamma\sigma)^{2(\alpha+1)}(\gamma+\sigma)^2\alpha^2\mu^{-(2\alpha+7)}\prod_{i=1}^{3}(\alpha+i)\right], \tag{20}$$

$$k_n = \frac{1}{|\boldsymbol{\beta}'|^2\varphi_1^{\frac{1}{2}}(v)}\left[(\gamma\sigma)^{(\alpha+1)}(\gamma+\sigma)^2\mu^{-(\alpha+4)}\prod_{i=0}^{3}(\alpha+i)\right], \tag{21}$$

$$\tau_g = -\frac{(\gamma-\sigma)}{2|\boldsymbol{\beta}'|\varphi_1^2(v)}\left[(\gamma\sigma)^{(\alpha+1)}(\gamma+\sigma)\mu^{(\alpha+2)}\prod_{i=0}^{3}(\alpha+i)\right], \tag{22}$$

respectively, where $|\boldsymbol{\beta}'|, \varphi_1(v)$ and $\mu(v)$ are given from

$$|\boldsymbol{\beta}'| = \sqrt{1+\varphi_1(v)}, \varphi_1(v) = 1 + (\gamma\sigma)^{2\alpha+2}(\gamma^2+\sigma^2)\mu^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2,$$
$$\mu(v) = \sigma v + \gamma v - \gamma\sigma. \tag{23}$$

Making use of the parallel bivariate Pareto distribution surfaces $(P,\overline{P})$, given from (1) and (15), let the curve $\overline{\boldsymbol{\beta}}(u(v))$ lying on $\overline{P}$ be the image, in parallel correspondence, of the curve $\boldsymbol{\beta}(u(v))$ lying on P. Therefore, for simplicity, we consider $u = v$. Then, the relation between $\overline{\boldsymbol{\beta}}(v)$ and $\boldsymbol{\beta}(v)$ can be expressed in the following way:

$$\overline{\boldsymbol{\beta}}(v) = \boldsymbol{\beta}(v) + r\mathbf{N}(v), \tag{24}$$

where, $\overline{\boldsymbol{\beta}}(v), \boldsymbol{\beta}(v)$ and $\mathbf{N}(v)$ are given from (1), (15) and (17), respectively.

Thus, using (5) and taking into account that $|\overline{\boldsymbol{\beta}}'| \neq 0$ and $\varphi_1 \neq 0$, one can get the following:

Corollary 4. *The image of geodesic curvature, normal curvature and geodesic torsion of $\overline{\boldsymbol{\beta}}$ on $\overline{P}$ are given from*

$$\overline{\kappa}_g = -\frac{(\gamma-\sigma)}{\left|\overline{\boldsymbol{\beta}}'\right|^3 \varphi_1^{\frac{1}{2}}(v)} \Psi_1(v;\gamma,\sigma,\alpha), \tag{25}$$

$$\overline{\kappa}_n = \frac{(\gamma\sigma)^{\alpha}\alpha\mu^2}{\left|\overline{\boldsymbol{\beta}}'\right|^2 \varphi_1^{\frac{1}{2}}(v)} \Psi_2(v;\gamma,\sigma,\alpha), \tag{26}$$

$$\overline{\tau}_g = -\frac{(\gamma-\sigma)}{2\left|\overline{\boldsymbol{\beta}}'\right| \varphi_1^{2}(v)} \Psi_3(v;\gamma,\sigma,\alpha), \tag{27}$$

respectively, and $|\overline{\boldsymbol{\beta}}'|$ *is given from*

$$\left|\overline{\boldsymbol{\beta}}'(v)\right| = \sqrt{\begin{array}{l}(\gamma\sigma)^{2(1+\alpha)}(\gamma+\sigma)^2\alpha^2\mu^{-(4\alpha+14)}\left\{-(\alpha+1)(\alpha+2)\mu^{(\alpha+4)} - \frac{\lambda_1}{\lambda_2}\right\}^2 \\ +\left\{\frac{\gamma\lambda_3}{\lambda_2} + \frac{\gamma\lambda_4}{\sqrt{\lambda_2}} - 1\right\}^2 + \left\{\frac{\sigma\lambda_3}{\lambda_2} + \frac{\sigma\lambda_4}{\sqrt{\lambda_2}} - 1\right\}^2,\end{array}} \tag{28}$$

where, $\lambda_1,\lambda_2,\lambda_3$ *and* λ_4 *are given from*

$$\begin{aligned}
\lambda_1 &= -(\gamma\sigma)^{\alpha+1}(\gamma^2+\sigma^2)\alpha(\alpha+3)\lambda^2 r,\\
\lambda_2 &= 1+(\gamma\sigma)^{2(\alpha+1)}\alpha^2\lambda^2\mu^{-2(\alpha+3)}\left(\gamma^2+\sigma^2\mu^{-(\alpha+3)}\right)r,\\
\lambda_3 &= -(\gamma\sigma)^{3(\alpha+1)}(\gamma+\sigma)(\gamma^2+\sigma^2)\alpha^3(\alpha+3)\lambda^3\mu^{-2(3\alpha+10)}r,\\
\lambda_4 &= (\gamma\sigma)^{(\alpha+1)}(\gamma+\sigma)\alpha(\alpha+3)\lambda\mu^{-(\alpha+4)}r,
\end{aligned}$$

respectively, and since the values of polynomials functions Ψ_1,Ψ_2 *and* Ψ_3 *of different degrees in the variable* v *and the parameters* γ,σ *and* α *, are so long, they have been omitted.*

According to the above results one can see that, the normal curvature and its image of $(\boldsymbol{\beta},\overline{\boldsymbol{\beta}})$ on $(P,\overline{P})$ do not vanish. Thus, we have the following :

Corollary 5. *There are no asymptotic lines on the bivariate Pareto distribution surface and its parallel.*

In view of (20), (22), (25) and (27) we reach the important theorem.

Theorem 1. *The curves* $(\boldsymbol{\beta},\overline{\boldsymbol{\beta}})$ *on the parallel bivariate Pareto distribution surfaces* $(P,\overline{P})$ *are geodesic curves and lines of curvatures if and only if the following condition is satisfied:*

$$\gamma = \sigma. \tag{29}$$

In other words, the curves $(\boldsymbol{\beta},\overline{\boldsymbol{\beta}})$ *have a dual property for the geodesics and lines of curvatures.*

3.2. *Curvatures of the parametric curves lying on* P *and* $\overline{P}$

Here, we want to shed light on the geodesic curves and asymptotic lines and lines of curvatures of the u/v-parametric curves $(v=v_0)/(u=u_0)$, respectively on $(P,\overline{P})$.

By a manner similar to the previous subSection 3.1. for the parallel bivariate Pareto distribution surface $(P,\overline{P})$, given from (1) and (15), let the curve $\boldsymbol{\xi}_1(u,v_0)$ and its image $\overline{\boldsymbol{\xi}_1}(u,v_0)$ lying on $(P,\overline{P})$, respectively. Thus, the relation between $\overline{\boldsymbol{\xi}_1}$ and $\boldsymbol{\xi}_1$ takes the form

$$\overline{\boldsymbol{\xi}_1}(u,v_0) = \boldsymbol{\xi}_1(u,v_0) + r\mathbf{N}(u,v_0), \tag{30}$$

where, $\overline{\boldsymbol{\xi}_1}(u,v_0),\boldsymbol{\xi}(u,v_0)$ and $\mathbf{N}(u,v_0)$ are given from (1), (15) and (17), respectively. Thus, using (5) and for $|\boldsymbol{\xi}_1'| \neq 0$ and $\varphi_2 \neq 0$, one can get the following:

Corollary 6. *The geodesic curvature, normal curvature and geodesic torsion of the u-parametric curves* $(v=v_0)$ *of* $\boldsymbol{\xi}_1$ *on* P *are given from*

$$(\kappa_g)_{v=v_0} = -\frac{1}{|\boldsymbol{\xi}_1'|^3\varphi_2^{\frac{1}{2}}(u)}\left[\sigma(\gamma\sigma)^{(2\alpha+3)}(\alpha+3)\omega_1^{-(5\alpha+7)}\prod_{i=0}^{2}(\alpha+i)^2\right], \tag{31}$$

$$(\kappa_n)_{v=v_0} = \frac{1}{|\boldsymbol{\xi}_1'|^2\varphi_2^{\frac{1}{2}}(u)}\left[\sigma^2(\gamma\sigma)^{(\alpha+1)}\omega_1^{-(5\alpha+4)}\prod_{i=0}^{3}(\alpha+i)\right], \tag{32}$$

$$(\tau_g)_{v=v_0} = -\frac{1}{2|\boldsymbol{\xi}_1'|\varphi_2^{2}(u)}\left[(\gamma\sigma)^{(\alpha+2)}\omega_1^{(\alpha+2)}\prod_{i=0}^{3}(\alpha+i)\right], \tag{33}$$

respectively, where $|\boldsymbol{\xi}_1'|$ *,* φ_2 *and* ω_1 *are given from*

$$\begin{aligned}
|\boldsymbol{\xi}_1'| &= \sqrt{1+\left\{\sigma^2(\gamma\sigma)^{2(\alpha+1)}\omega_1^{-2(\alpha+3)}\right\}\prod_{i=0}^{2}(\alpha+i)^2},\\
\varphi_2(u) &= 1+(\gamma\sigma)^{2(\alpha+1)}(\gamma^2+\sigma^2)\omega_1^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2,\\
\omega_1(u) &= \sigma u + \gamma v_0 - \gamma\sigma.
\end{aligned}$$

Thus, we have $(\tau_g)_{v=v_0} = 0 \Leftrightarrow \omega_1 = 0$ *, and this implies*

$$u = \frac{\gamma(\sigma - v_0)}{\sigma}, \quad \sigma \succ v_0,\ \gamma,\sigma \succ 0. \tag{34}$$

Alternatively, one could use Eq. (30) approach by letting the curve $\boldsymbol{\xi}_2(u_0,v)$ and its parallel image $\overline{\boldsymbol{\xi}_2}(u_0,v)$ lying on $(P,\overline{P})$, respectively. Thus, the relation between $\boldsymbol{\xi}_2$ and $\overline{\boldsymbol{\xi}_2}$ takes the form

$$\overline{\boldsymbol{\xi}_2}(u_0,v) = \boldsymbol{\xi}_2(u_0,v) + r\mathbf{N}(u_0,v), \tag{35}$$

where, $\overline{\boldsymbol{\xi}_2}(u_0,v),\boldsymbol{\xi}_2(u_0,v)$ and $\mathbf{N}(u_0,v)$ are given from (1), (15) and (17), respectively. Thus, using (5) and for $|\boldsymbol{\xi}_2'| \neq 0$ and $\varphi_3 \neq 0$, we obtain the following:

Corollary 7. *The geodesic curvature, normal curvature and the geodesic torsion of the v -parametric curves* $(u=u_0)$ *of* $\boldsymbol{\xi}_2$ *on* P *are given from*

$$(\kappa_g)_{u=u_0} = -\frac{1}{|\boldsymbol{\xi}_2'|^3\varphi_3^{\frac{1}{2}}(v)}\left[\gamma(\gamma\sigma)^{(2\alpha+3)}(\alpha+3)\omega_2^{-(5\alpha+7)}\prod_{i=0}^{2}(\alpha+i)^2\right], \tag{36}$$

$$(\kappa_n)_{u=u_0} = \frac{1}{|\boldsymbol{\xi}_2'|^2\varphi_3^{\frac{1}{2}}(v)}\left[\gamma^2(\gamma\sigma)^{(\alpha+1)}\omega_2^{-(5\alpha+4)}\prod_{i=0}^{3}(\alpha+i)\right], \tag{37}$$

$$(\tau_g)_{u=u_0} = -\frac{1}{2|\boldsymbol{\xi}_2'|\varphi_3^{2}(v)}\left[(\gamma\sigma)^{(\alpha+2)}\omega_2^{(\alpha+2)}\prod_{i=0}^{3}(\alpha+i)\right], \tag{38}$$

respectively, where, $|\boldsymbol{\xi}_2'|$, φ_3 *and* ω_2 *are given from*

$$\begin{aligned}
|\boldsymbol{\xi}_2'| &= \sqrt{1+\left\{\gamma^2(\gamma\sigma)^{2(\alpha+1)}\omega_2^{-2(\alpha+3)}\right\}\prod_{i=0}^{2}(\alpha+i)^2},\\
\varphi_3(v) &= 1+(\gamma\sigma)^{2(\alpha+1)}(\gamma^2+\sigma^2)\omega_2^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2,\\
\omega_2(v) &= \sigma u_0 + \gamma v - \gamma\sigma.
\end{aligned}$$

Thus, we have $(\tau_g)_{u=u_0} = 0 \to \omega_2 = 0$, and this implies

$$v = \frac{\sigma(\gamma - u_0)}{\gamma}, \quad \gamma \succ u_0, \gamma, \sigma \succ 0. \tag{39}$$

After long straight-forward computations analogous to (25)–(27), we get the parallel images of the curvatures given in Eqs. (31)–(33), (36), (37) and (38). According to the above results one can see that the geodesic curvatures and normal curvatures of $(\xi_1, \overline{\xi}_1)$ and $(\xi_2, \overline{\xi}_2)$ on $(P, \overline{P})$ are undefined for the values $(\omega_1(u), \omega_2(v)) = (0,0)$. Thus, we have:

Corollary 8. *The u/v-parametric curves $(v = v_0)/(u = u_0)$ of $(\xi_1, \overline{\xi}_1)$ and $(\xi_2, \overline{\xi}_2)$ on $(P, \overline{P})$ cannot be geodesic curves and asymptotic lines on $(P, \overline{P})$.*

In view of (33), (34), (38) and (39) we find that the geodesic torsions are defined for the values $(\omega_1(u), \omega_2(v)) = (0,0)$, i.e. the geodesic torsions are defined at the points $(v_0, \frac{\gamma(\sigma - v_0)}{\sigma})$ and $(u_0, \frac{\sigma(\gamma - u_0)}{\gamma})$. Hence we reach the following corollary:

Corollary 9. *The u /v -parametric curves $(v = v_0)/(u = u_0)$ of $(\xi_1, \overline{\xi}_1)$ and $(\xi_2, \overline{\xi}_2)$ on $(P, \overline{P})$ are not lines of curvatures on $(P, \overline{P})$.*

The curves $(\beta, \overline{\beta}), (\xi_1, \overline{\xi}_1)$ and $(\xi_2, \overline{\xi}_2)$ on $(P, \overline{P})$ are shown in Fig. 1.

4. Geometrical and statistical interpretations of some results

This section is considered the most important section, where the effect of some geometrical results on moments of P is illustrated. Moreover, the effect of the weak, moderate and strong positive linear relationships between two variables on some curvatures of/on P is shown.

4.1. Effect of some geometrical results on moments

Remark 1. [18]. Taking (7) into account, one can easily verify, the standard form of (6) $(\gamma = \sigma = 1)$, like its univariate version, is characterized by form invariance in the context of size biased sampling.

Combining the above statistical remark and the geometrical condition (29), we have the following:

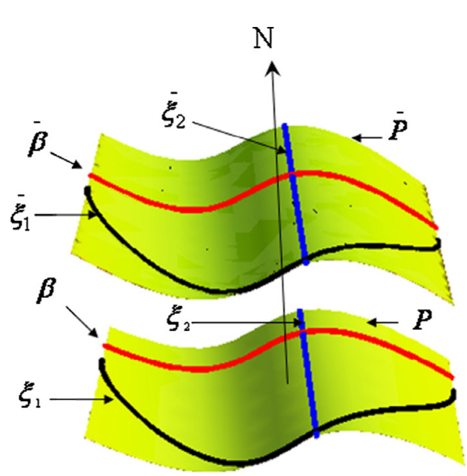


Fig. 1 The curves $(\beta, \overline{\beta}), (\xi_1, \overline{\xi}_1)$ and $(\xi_2, \overline{\xi}_2)$ on $(P, \overline{P})$.

Corollary 10. *At the geodesic curves and lines of curvatures on the bivariate Pareto distribution surface P and its parallel $\overline{P}$, the mean curvature functions of $\overline{P}$ and its original P are given by (18) and*

$$H(v)|_{\gamma=\sigma=1} = -\frac{1}{2g^{\frac{3}{2}}(v)}\left[2\mu^{2(\alpha+3)}(1+\eta) + \mu^{-(\alpha+10)}\prod_{i=0}^{3}(\alpha+i)\left[2\prod_{i=0}^{2}(\alpha+i)^2\right]\right], \tag{40}$$

respectively, where, $\mu(v)$ is given from (23) and

$$g(v)|_{\gamma=\sigma=1} = 1 + 2\mu^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2, \quad \eta(v)|_{\gamma=\sigma=1} = \mu^{-2(\alpha+3)}\prod_{i=0}^{2}(\alpha+i)^2. \tag{41}$$

Corollary 11. *At the geodesic curves and lines of curvatures on the bivariate Pareto distribution surface P and its parallel $\overline{P}$, the moments in (8) take the following forms:*

The means and variances of the marginal distributions are given from

$$E(X) = E(Y) = \frac{\alpha}{\alpha - 1}, \quad \alpha > 0, \alpha \neq 1, \tag{42}$$

$$Var(X) = Var(Y) = \frac{\alpha}{(\alpha-1)^2(\alpha-2)}, \quad \alpha \neq 1, \quad \alpha \neq 2. \tag{43}$$

The conditional density functions in (9) and (10) are given by

$$f_{Y|X}(y|x) = (\alpha+1)(x)^{\alpha+1}\lambda^{-(\alpha+2)}, \quad x > 0, y > 0, \alpha > 0,$$
$$f_{X|Y}(x|y) = (\alpha+1)(y)^{\alpha+1}\lambda^{-(\alpha+2)}, \tag{44}$$

Thus, taking into account that $\alpha > 0, \alpha \neq 1$ and $\gamma = \sigma = 1$, we find that there are symmetrical relations of the conditional means and variances as the following:

$$E(Y|X{=}x) = \left(1 + \frac{x}{\alpha}\right), \; Var(Y|X = x) = \frac{(\alpha+1)}{\alpha^2(\alpha-1)}x^2, \tag{45}$$

$$E(X|Y = y) = \left(1 + \frac{y}{\alpha}\right), \; Var(X|Y = y) = \frac{(\alpha+1)}{\alpha^2(\alpha-1)}y^2. \tag{46}$$

Using (13), we find

$$Cov(X, Y) = \frac{1}{(\alpha-1)^2(\alpha-2)}, \quad \alpha \neq 1, \alpha \neq 2. \tag{47}$$

Geometrically, we can give a geometric visualization of the relations (45) and (46). The conditional means are represented linear functions and the conditional variances are represented parabolic curves in the plane.

4.2. The effect of the correlation $R \equiv Cor(X, Y)$ on some of geometrical results

The ultimate goal of every research or scientific analysis is finding relationships between variables. The philosophy of science teaches us that there is no other way of representing "meaning" except in terms of relationships between some quantities or qualities. Thus, the advancement of science must always involve finding new relations between variables. Statistics help us evaluate the relation between variables. Before we embark upon studying the relationship between variables let us see the basic definition of correlation.

Linear correlation coefficient is a measure which determines the strength and direction of a linear relationship between two variables. The value of linear correlation coefficient always lies between -1 and $+1$, i.e. $-1 \le R \le +1$. The + and − signs are used for positive (+) linear correlation and negative (−) linear correlation, respectively.

Positive values indicate that if X values are increasing (decreasing) then the values of Y are also increasing (decreasing). If X and Y have a strong positive linear correlation, R is close to +1. When R is exactly +1 it indicates a perfect positive fit.

Negative values indicate that if X values are increasing then the values of Y are decreasing and vice versa. If X and Y have a strong negative linear correlation, R is close to -1. When R is exactly -1 it indicates a perfect negative fit.

If there is no linear correlation, R is equal 0. Thus, the values of the correlation coefficient show the relation between the two variables according to the following diagram [19,20]:

Actually, it is a remarkable fact that to calculate the correlation coefficient of X and Y, we have to compute the moments given in (8). Hence, the correlation coefficient has an effective role in classifying the relation between X and Y that appears in the above diagram. That is why, here, we study the effect of the weak, moderate and strong positive linear relationships between two variables on the mean curvature function of $(P,\overline{P})$.

From the following diagram, using (14) and for $\lambda \neq 0$, we have three cases.

(i) Case $\alpha = 3$. We have a weak positive linear relationship. Thus, we get the following:

Corollary 12. *At a weak positive linear relationship, the mean curvature functions of $\overline{P}$ and its original P are given by (2) and*

$$H(u,v)|_{\alpha=3} = -\frac{1}{g^{\frac{3}{2}}}\left[180(\gamma\sigma)^4\lambda^{-13}\left[14400(\gamma\sigma)^{10} + \lambda^{12}\left(\sigma^2 + \gamma^2\right)\right]\right], \tag{48}$$

respectively, where, λ is given from (15) and $g(u,v) = 1 + 3600(\gamma\sigma)^8\lambda^{-12}(\gamma^2 + \sigma^2)$.

(ii) Case $\alpha = 1.5$. We have a moderate positive linear relationship. Thus, we get the following:

Corollary 13. *At a moderate positive linear relationship, the mean curvature functions of $\overline{P}$ and its original P are given by (2) and*

$$H(u,v)|_{\alpha=1.5} = -\frac{1}{g^{\frac{3}{2}}}\left[(29.53)(\gamma\sigma)^2\lambda^{-(11.5)}\left[(689.06)(\gamma\sigma)^7 + \lambda^9(\sigma^2 + \gamma^2)\right]\right], \tag{49}$$

respectively, where, $g(u,v) = 1 + (172.27)(\gamma\sigma)^5\lambda^{-9}(\gamma^2 + \sigma^2)$.

(iii) Case $\alpha = 1.3$. We have a strong positive linear relationship. Thus, we get the following:

Corollary 14. *At a strong positive linear relationship, the mean curvature functions of $\overline{P}$ and its original P are given by (2) and*

$$H(u,v)|_{\alpha=1.3} = -\frac{1}{g^{\frac{3}{2}}}\left[(21.21)(\gamma\sigma)^{2.3}\lambda^{-(11.3)}\left[(389.43)(\gamma\sigma)^{(6.6)} + \lambda^{(8.6)}\left(\sigma^2 + \gamma^2\right)\right]\right], \tag{50}$$

respectively, where, $g(u,v) = 1 + (97.36)(\gamma\sigma)^{(4.6)}\lambda^{-(8.6)}(\gamma^2 + \sigma^2)$.

Finally, we have to check the effect of the weak, moderate and strong positive linear relationship on the geodesic curvature of $\boldsymbol{\beta}$ on P. Using the above three cases of α and for $|\boldsymbol{\beta}'| \neq 0$, $\varphi_1 \neq 0$ and $\mu \neq 0$, we obtain the following:

Corollary 15. *At a weak positive linear relationship, the geodesic curvature of $\boldsymbol{\beta}$ on P is given from*

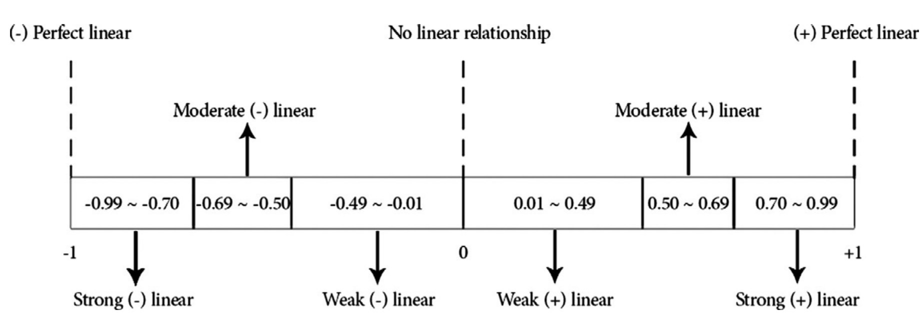
$$k_g|_{\alpha=3} = -\frac{(\gamma - \sigma)}{|\boldsymbol{\beta}'|^3\varphi_1^{\frac{1}{2}}(v)}\left[1080(\gamma\sigma)^8(\gamma+\sigma)^2\mu^{-13}\right], \tag{51}$$

where, $|\boldsymbol{\beta}'|$ and $\mu(v)$ are given from (23) and $\varphi_1(v) = 1 + (3600)(\gamma\sigma)^8(\gamma^2 + \sigma^2)\mu^{-12}$.

Corollary 16. *At a moderate positive linear relationship, the geodesic curvature of $\boldsymbol{\beta}$ on P is given from*

$$k_g|_{\alpha=1.5} = -\frac{(\gamma - \sigma)}{|\boldsymbol{\beta}'|^3\varphi_1^{\frac{1}{2}}(v)}\left[(88.61)(\gamma\sigma)^5(\gamma+\sigma)^2\mu^{-10}\right], \tag{52}$$

where, $\varphi_1(v) = 1 + (172.27)(\gamma\sigma)^5(\gamma^2 + \sigma^2)\mu^{-9}$.



Corollary 17. *At a strong positive linear relationship, the geodesic curvature of* $\boldsymbol{\beta}$ *on P is given from*

$$k_g|_{\alpha=1.3} = -\frac{(\gamma-\sigma)}{|\boldsymbol{\beta}'|^3 \varphi_1^{\frac{1}{2}}(v)}\left[(55.16)(\gamma\sigma)^{(4.6)}(\gamma+\sigma)^2\mu^{-(9.6)}\right], \quad (53)$$

where, $\varphi_1(v) = 1 + (97.36)(\gamma\sigma)^{(4.6)}(\gamma^2+\sigma^2)\mu^{-(8.6)}$.

As a similar way to Eqs. (51)–(53) one can study the effect of the weak, moderate and strong positive linear relationship on the normal curvature and the geodesic torsion of $(\boldsymbol{\beta},\overline{\boldsymbol{\beta}})$ on $(P,\overline{P})$. Also, the same study on the geodesic curvature, normal curvature and the geodesic torsion of a u /v-parametric curves $(v = v_0)/(u = u_0)$ of $(\boldsymbol{\xi}_1,\overline{\boldsymbol{\xi}_1})$ and $(\boldsymbol{\xi}_2,\overline{\boldsymbol{\xi}_2})$ on $(P,\overline{P})$ can be done using the same technique as in the above equations.

Having the above obtained results in mind, we can conclude that from Eqs. (48)–(50) we obtain the relations between the mean curvature functions H and the two variables u and v of P at the week, moderate and strong positive linear relationships that are shown in Figs. 2–4, respectively.

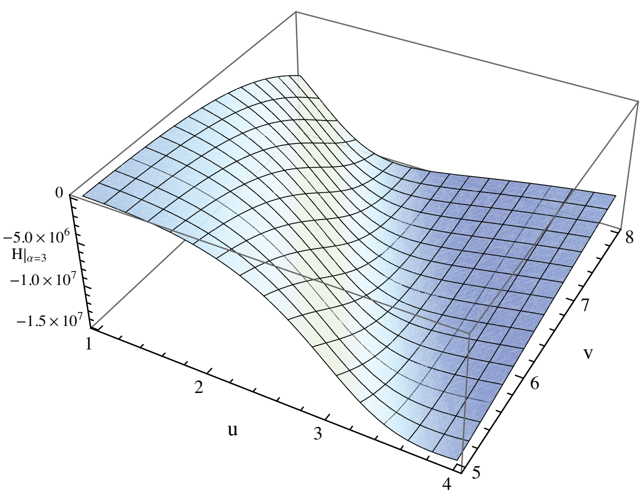


Fig. 2 The mean curvature function of P at a weak positive linear relationship.

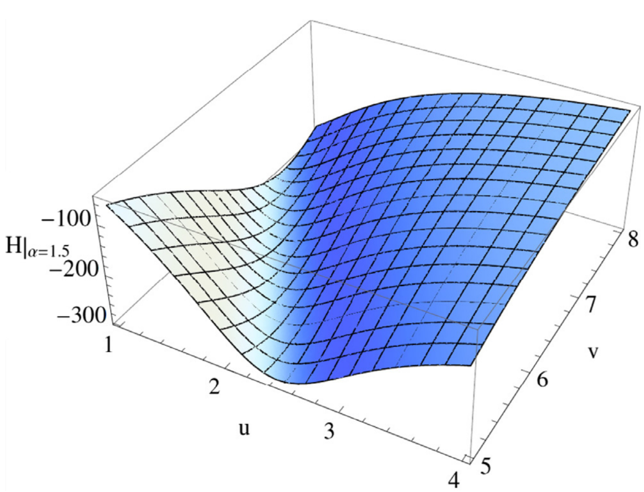


Fig. 3 The mean curvature function of P at a moderate positive linear relationship.

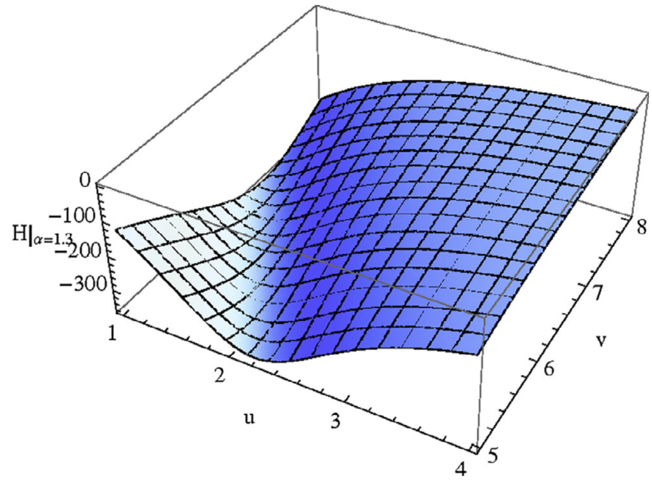


Fig. 4 The mean curvature function of P at a strong positive linear relationship.

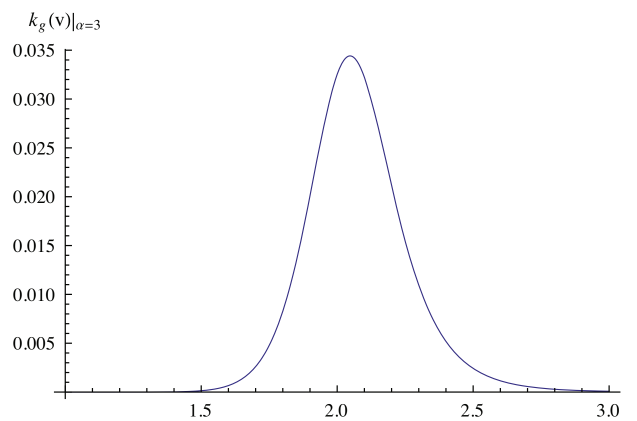


Fig. 5 The geodesic curvature of $\boldsymbol{\beta}$ on P at a weak positive linear relationship.

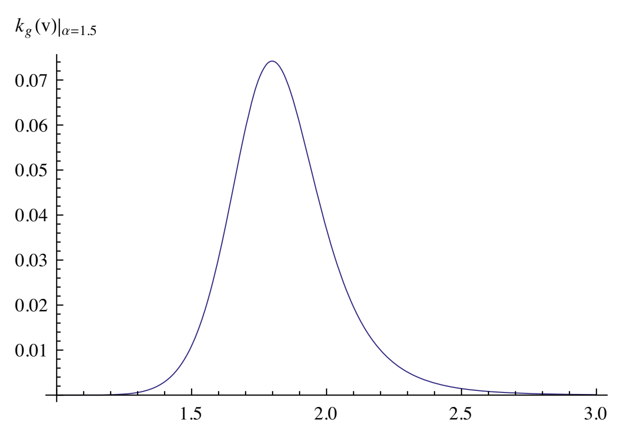


Fig. 6 The geodesic curvature of $\boldsymbol{\beta}$ on P at a moderate positive linear relationship.

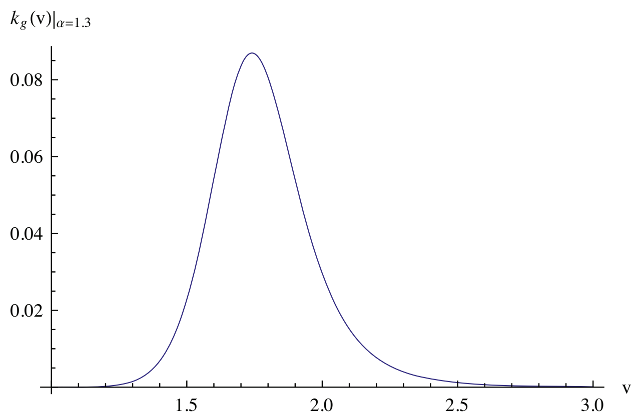


Fig. 7 The geodesic curvature of $\boldsymbol{\beta}$ on P at a strong positive linear relationship.

From Eqs. (51)–(53) we obtain the relations between the geodesic curvature k_g and the variable v on P at the week, moderate and strong positive linear relationships that are given through the Figs. 5–7, respectively.

From the Figs. 5–7, it is easy to see that the geodesic curvatures are continuous (differentiable) functions on the intervals [1.4,3] ((1.4,3)),[1.2,2.8] ((1.2,2.8)) and [1.1,1.9] ((1.1,1.9)), respectively, and also have a relative maximum at the points (2.1,0.034),(1.8,0.072) and (1.7,0.085), respectively.

5. Conclusion

These results show the usefulness of the present geometric framework. The Mong formula is a geometrical concept introduced in a theory of statistical manifold and has been discussed to be useful for finding geometrical and statistical results. It should again be noted that the new geometric results were first introduced in a manifold of statistical models, and have already been proved to play an essential role in the theory of statistical manifold. The present paper aims to study invariant differential geometric structures inherent in a bivariate Pareto distribution manifold. We have shown that the geometrical results can be obtained using statistical concepts, and that help in establishing several dependency properties of this model. We have established several new results of this distribution manifold. The field is developing rapidly, and there are a lot of problems to be solved and more work is needed to establish different results of new distributions.

Numerical computations of the correlation coefficient R given in Eq. (14) for different values of the parameter α are obtained. We have been able to construct some bounds for the parameter α to give the weak, moderate and strong positive linear relationships that are given, respectively, by $2 < \alpha \le 65, 1.45 \le \alpha < 2$ and $1 < \alpha \le 1.4$. Therefore, there are no more relationships rather than the weak, moderate and strong positive linear relationships.

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