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Original Article

ϕ-statistically quasi Cauchy sequences ☆

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Abstract Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further $P_s = \{\sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \leq s\}$, i.e. P_s is the set of those σ whose support has cardinality at most s. Let (ϕ_n) be a non-decreasing sequence of positive integers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$ and the class of all sequences (ϕ_n) is denoted by Φ. Let $E \subseteq \mathbb{N}$. The number $\delta_\phi(E) = \lim_{s\to\infty} \frac{1}{\phi_s}|\{k \in \sigma, \sigma \in P_s : k \in E\}|$ is said to be the ϕ-density of E. A sequence (x_n) of points in $\mathbb{R}$ is ϕ-statistically convergent (or S_ϕ-convergent) to a real number ℓ for every $\varepsilon > 0$ if the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$ has ϕ-density zero. We introduce ϕ-statistically ward continuity of a real function. A real function is ϕ-statistically ward continuous if it preserves ϕ-statistically quasi Cauchy sequences where a sequence (x_n) is called to be ϕ-statistically quasi Cauchy (or S_ϕ-quasi Cauchy) when $(\Delta x_n) = (x_{n+1} - x_n)$ is ϕ-statistically convergent to 0. i.e. a sequence (x_n) of points in $\mathbb{R}$ is called ϕ-statistically quasi Cauchy (or S_ϕ-quasi Cauchy) for every $\varepsilon > 0$ if $\{n \in \mathbb{N} : |x_{n+1} - x_n| \geq \varepsilon\}$ has ϕ-density zero. Also we introduce the concept of ϕ-statistically ward compactness and obtain results related to ϕ-statistically ward continuity, ϕ-statistically ward compactness, statistically ward continuity, ward continuity, ward compactness, ordinary compactness, uniform continuity, ordinary continuity, δ-ward continuity, and slowly oscillating continuity.

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1. Introduction

First of all, some definitions and notation will be given in the following. Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, … for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, … of terms in $\mathbb{R}$.

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Let us start with basic definitions from the literature. Let $K \subseteq \mathbb{N}$, the set of all natural numbers and $K_n = \{k \le n : k \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set.

Fast [1] presented the following definition of statistical convergence for sequences of real numbers. The sequence $x = (x_n)$ is said to be *statistically convergent* to L if for every $\epsilon > 0$, the set $K_\epsilon := \{n \in \mathbb{N} : |x_n - L| \ge \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{j \le n : |x_j - L| \ge \epsilon\}| = 0.$$

In this case, we write S-$\lim x = L$ or $x_n \to L(S)$ and S denotes the set of all statistically convergent sequences. Note that every convergent sequence is statistically convergent but not conversely.

Some basic properties related to the concept of statistical convergence were studied in [2,3]. In 1985, Fridy [4] presented the notion of statistically Cauchy sequence and determined that it is equivalent to statistical convergence. Caserta et al. [5] studied statistical convergence in function sapces while Caserta and Kočinac [6] investigated statistical exhaustivness. For more details on statistical convergence we refer to [7–12].

A sequence (x_n) of points in $\mathbb{R}$ is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. In [13] Burton and Coleman named these sequences as "quasi-Cauchy" and in [14] Çakallı used the term "ward convergent to 0" sequences. From now on in this paper we also prefer to using the term "quasi-Cauchy" to using the term "ward convergent to 0" for simplicity. In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e. $(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of $\mathbb{R}$ is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$.

It is known that a sequence (x_n) of points in $\mathbb{R}$, the set of real numbers, is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \le k \le [\lambda n]} |x_k - x_n| = 0$$

where $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following if $(x_m - x_n) \to 0$ whenever $1 \le \frac{m}{n} \to 1$ as $m, n \to \infty$. Using $\varepsilon > 0$ s and δ s this is also equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \ge N(\varepsilon)$ and $n \le m \le (1+\delta)n$ (see [15]).

A function f defined on a subset E of $\mathbb{R}$ is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is.

Cakalli and Hazarika [16] introduced the concept of ideal quasi Cauchy sequences and proved some results related to ideal ward continuity and ideal ward compactness.

A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$. Recently, Cakalli gave new sequential definitions of compactness and slowly oscillating compactness in [15] [17] and [18].

2. Main results

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further $P_s = \{\sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \le s\}$, i.e. P_s is the set of those σ whose support has cardinality at most s. Let (ϕ_n) be a non-decreasing sequence of positive integers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all $n \in \mathbb{N}$ and the class of all sequences (ϕ_n) is denoted by Φ. For details on class of all sequences (ϕ_n) we refer to [19–21].

Example 2.1. $\phi_n = (2n+1)$ and $\phi_n = (2n+2)$ for all $n \in \mathbb{N}$ are members in Φ, but $\phi_n = (n^2)$ and $\phi_n = (2n-1)$ for all $n \in \mathbb{N}$ are not members in Φ.

Definition 2.1. Let $E \subseteq \mathbb{N}$. The number $\delta_\phi(E) = \lim_{s\to\infty} \frac{1}{\phi_s}|\{k \in \sigma, \sigma \in P_s : k \in E\}|$ is said to be the ϕ-density of E.

Definition 2.2. A sequence (x_n) of points in $\mathbb{R}$ is ϕ-statistically convergent (or S_ϕ-convergent) to a real number ℓ for every $\varepsilon > 0$ if the set $\{n \in \mathbb{N} : |x_n - \ell| \ge \varepsilon\}$ has ϕ-density zero.

Definition 2.3. A sequence $\mathbf{x} = (x_n)$ is S_ϕ-ward convergent to a number ℓ if S_ϕ-$\lim_{n\to\infty} \Delta x_n = \ell$ where $\Delta x_n = x_{n+1} - x_n$. For the special case $\ell = 0$ we say that $\mathbf{x}$ is ϕ-statistically quasi-Cauchy, or S_ϕ-quasi-Cauchy, in place of S_ϕ-ward convergent to 0. Thus a sequence (x_n) of points of $\mathbb{R}$ is S_ϕ-quasi-Cauchy if (Δx_n) is S_ϕ-convergent to 0. We denote ΔS_ϕ the set of all ϕ-statistically quasi Cauchy sequences of points in $\mathbb{R}$.

Example 2.2. Consider a sequence $\mathbf{x} = (x_n) = (\sqrt{n})$. Then it is clear that the sequence (x_n) is ϕ-statistically quasi Cauchy.

Definition 2.4. A subset E of $\mathbb{R}$ is called S_ϕ-sequentially compact if whenever (x_n) is a sequence of points in E there is S_ϕ-convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that S_ϕ-$\lim \mathbf{y}$ is in E.

Theorem 2.5. *A subset of $\mathbb{R}$ is sequentially compact if and only if it is S_ϕ-sequentially compact.*

Proof. The proof easily follows from Corollary 3 on page 597 in [17], so is omitted. □

Definition 2.6. A function $f : E \to \mathbb{R}$ is S_ϕ-sequentially continuous at a point x_0 if, given a sequence (x_n) of points in E, S_ϕ-$\lim \mathbf{x} = x_0$ implies that S_ϕ-$\lim f(\mathbf{x}) = f(x_0)$.

Theorem 2.7. *Any S_ϕ-sequentially continuous function at a point x_0 is continuous at x_0 in the ordinary sense.*

Proof. Let f be any S_ϕ-sequentially continuous function at point x_0, Since any proper admissible ideal is a regular subsequential method, it follows from Theorem 13 on page 316 in [22] that f is continuous in the ordinary sense. □

Theorem 2.8. *Any continuous function at a point x_0 is S_ϕ-sequentially continuous at x_0.*

Corollary 2.9. *A function is S_ϕ-sequentially continuous at a point x_0 if and only if it is continuous at x_0.*

Corollary 2.10. *For any regular subsequential method G, a function is G-sequentially continuous at a point x_0, then it is S_ϕ-sequentially continuous at x_0.*

Proof. The proof follows from Theorem 13 on page 316 in [22]. □

Corollary 2.11. *Any ward continuous function on a subset E of $\mathbb{R}$ is S_ϕ-sequentially continuous on E.*

Theorem 2.12. *If a function is slowly oscillating continuous on a subset E of $\mathbb{R}$, then it is S_ϕ-sequentially continuous on E.*

Proof. Let f be any slowly oscillating continuous on E. It follows from Theorem 2.1 in [15] that f is continuous. By Theorem 2.8 we see that f is S_ϕ-sequentially continuous on E. This completes the proof. □

Theorem 2.13. *If a function is δ-ward continuous on a subset E of $\mathbb{R}$, then it is S_ϕ-sequentially continuous on E.*

Proof. Let f be any δ-ward continuous function on E. It follows from Corollary 2 on page 399 in [23] that f is continuous. By Theorem 2.8 we obtain that f is S_ϕ-sequentially continuous on E. This completes the proof. □

Corollary 2.14. *If a function is quasi-slowly oscillating continuous on a subset E of $\mathbb{R}$, then it is S_ϕ-sequentially continuous on E.*

Proof. Let f be any quasi-slowly oscillating continuous on E. It follows from Theorem 3.2 in [24] that f is continuous. By Theorem 2.8 we deduce that f is S_ϕ-sequentially continuous on E. This completes the proof. □

Now we give the definition of S_ϕ-ward compactness of a subset of $\mathbb{R}$.

Definition 2.15. A subset E of $\mathbb{R}$ is called S_ϕ-ward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ such that $S_\phi\text{-}\lim_{k\to\infty} \Delta z_k = 0$.

Theorem 2.16. *A subset E of $\mathbb{R}$ is ward compact if and only if it is S_ϕ-ward compact.*

Proof. Let us suppose first that E is ward compact. It follows from Lemma 2 on page 1725 in [25] that E is bounded. Then for any sequence (x_n), there exists a convergent subsequence (x_{n_k}) of (x_n) whose limit may be in E or not. Then the sequence (Δx_{n_k}) is a null sequence. Since S_ϕ is a regular method, (Δx_{n_k}) is S_ϕ-convergent to 0, so it is S_ϕ-quasi-Cauchy. Thus E is S_ϕ-ward compact.

To prove the converse part suppose that E is S_ϕ-ward compact. Take any sequence (x_n) of points in E. Then there exists a S_ϕ-quasi-Cauchy subsequence (x_{n_k}) of (x_n). Since S_ϕ is subsequential there exists a convergent subsequence $(x_{n_{k_m}})$ of (x_{n_k}). Therefore $(x_{n_{k_m}})$ is a quasi-Cauchy subsequence of the sequence (x_n). Thus E is ward compact. This completes the proof of the theorem. □

Theorem 2.17. *A subset E of $\mathbb{R}$ is bounded if and only if it is S_ϕ-ward compact.*

Proof. Using an idea in the proof of Lemma 2 on page 1725 in [25] and the preceding theorem the proof can be obtained easily so is omitted. □

Now we give the definition of S_ϕ-ward continuity of a real function.

Definition 2.18. A function f is called S_ϕ-ward continuous on E if $S_\phi\text{-}\lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $S_\phi\text{-}\lim_{n\to\infty} \Delta x_n = 0$, for a sequence $\mathbf{x} = (x_n)$ of terms in E.

We note that sum of two S_ϕ-ward continuous functions is S_ϕ-ward continuous but the product of two S_ϕ-ward continuous functions need not be S_ϕ-ward continuous as it can be seen by considering product of the S_ϕ-ward continuous function $f(x) = x$ with itself.

In connection with S_ϕ-quasi-Cauchy sequences and S_ϕ-convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$.

$$(\delta_{S_\phi})\ (x_n) \in \Delta S_\phi \Rightarrow (f(x_n)) \in \Delta S_\phi$$

$$(\delta_{S_\phi} c)\ (x_n) \in \Delta S_\phi \Rightarrow (f(x_n)) \in c$$

$$(c)\ (x_n) \in c \Rightarrow (f(x_n)) \in c$$

$$(c\delta_{S_\phi})\ (x_n) \in c \Rightarrow (f(x_n)) \in \Delta S_\phi$$

$$(s_\phi)\ (x_n) \in S_\phi \Rightarrow (f(x_n)) \in S_\phi$$

We see that (δs_ϕ) is S_ϕ-ward continuity of f, (s_ϕ) is a S_ϕ-continuity of f and (c) states the ordinary continuity of f. It is easy to see that $(\delta s_\phi c)$ implies (δs_ϕ), and (δs_ϕ) does not imply $(\delta s_\phi c)$; and (δs_ϕ) implies $(c\delta s_\phi)$, and $(c\delta s_\phi)$ does not imply (δs_ϕ); $(\delta s_\phi c)$ implies (c) and (c) does not imply $(\delta s_\phi c)$; and (c) is equivalent to $(c\delta s_\phi)$.

Now we give the implication (δs_ϕ) implies (s_ϕ), i.e. any S_ϕ-ward continuous function is S_ϕ-sequentially continuous.

Theorem 2.19. *If f is S_ϕ-ward continuous on a subset E of $\mathbb{R}$, then it is S_ϕ-sequentially continuous on E.*

Proof. Suppose that f is an S_ϕ-ward continuous function on a subset E of $\mathbb{R}$. Let (x_n) be an S_ϕ-quasi-Cauchy sequence of points in E. Then the sequence

$$(x_1, x_0, x_2, x_0, x_3, x_0, \ldots, x_{n-1}, x_0, x_n, x_0, \ldots)$$

is an S_ϕ-quasi-Cauchy sequence. Since f is S_ϕ-ward continuous, the sequence

$$(y_n) = (f(x_1), f(x_0), f(x_2), f(x_0), \ldots, f(x_n), f(x_0), \ldots)$$

is a S_ϕ-quasi-Cauchy sequence. Therefore $S_\phi\text{-}\lim_{n\to\infty} \Delta y_n = 0$. Hence $S_\phi\text{-}\lim_{n\to\infty}[f(x_n) - f(x_0)] = 0$. It follows that the sequence $(f(x_n))$ S_ϕ-converges to $f(x_0)$. This completes the proof of the theorem. □

The converse is not always true. It is follows for the following example.

Example 2.3. Consider the function $f(x) = x^2 + 1$ and a sequence $(x_n) = (\sqrt{n})$. Then $S_\phi - \lim_{n\to\infty} \Delta x_n = 0$, But $S_\phi\text{-}\lim_{n\to\infty} \Delta f(x_n) \neq 0$, because $(f(\sqrt{n})) = (n + 1)$.

Theorem 2.20. *If f is S_ϕ-ward continuous on a subset E of $\mathbb{R}$, then it is continuous on E in the ordinary sense.*

Proof. Let f be an S_ϕ-ward continuous function on E. By Theorem 2.19, f is S_ϕ-sequentially continuous on E. It follows from Theorem 2.7 that f is continuous on E in the ordinary sense. Thus the proof is completed. □

Theorem 2.21. *An S_ϕ-ward continuous image of any S_ϕ-ward compact subset of $\mathbb{R}$ is S_ϕ-ward compact.*

Proof. Suppose that f is an S_ϕ-ward continuous function on a subset E of $\mathbb{R}$ and E is an S_ϕ-ward compact subset of $\mathbb{R}$. Let (y_n) be a sequence of points in $f(E)$. Write $y_n = f(x_n)$ where $x_n \in E$ for each $n \in \mathbb{N}$. S_ϕ-ward compactness of E implies that there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n) with $S_\phi\text{-}\lim_{k\to\infty} \Delta z_k = 0$.

Write $(t_k) = (f(z_k))$. As f is S_ϕ-ward continuous, so we have $S_\phi - \lim_{k\to\infty} \Delta f(z_k) = 0$. Thus we have obtained a subsequenc (t_k) of the sequence $(f(x_n))$ with $S_\phi\text{-}\lim_{k\to\infty} \Delta t_k = 0$. Thus $f(E)$ is S_ϕ-ward compact. This completes the proof of the theorem. □

Corollary 2.22. *Any S_ϕ-ward continuous image of any compact subset of $\mathbb{R}$ is compact.*

Proof. The proof of this theorem follows from Theorem 2.7. □

Corollary 2.23. *Any S_ϕ-ward continuous image of any bounded subset of $\mathbb{R}$ is bounded.*

Proof. The proof follows from Theorem 2.17 and Theorem 2.20. □

Corollary 2.24. *Any S_ϕ-ward continuous image of a S_ϕ-sequentially compact subset of $\mathbb{R}$ is G-sequentially compact for any regular subsequential method G.*

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of S_ϕ-ward continuity, i.e. uniform limit of a sequence of S_ϕ-ward continuous functions is S_ϕ-ward continuous.

Theorem 2.25. *If (f_n) is a sequence of S_ϕ-ward continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is S_ϕ-ward continuous on E.*

Proof. Let $\varepsilon > 0$ and (x_n) be a sequence of points in E such that $S_\phi\text{-}\lim_{n\to\infty} \Delta x_n = 0$. By the uniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. By the definition for all $x \in E$, we have

$$\lim_{s\to\infty} \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\right\}\right| = 0. \tag{2.1}$$

As f_N is S_ϕ-ward continuous on E we have

$$\lim_{s\to\infty} \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\}\right| = 0. \tag{2.2}$$

But

$$\begin{aligned}
&\frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}\right| \\
&\quad\leq \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3}\right\}\right| \\
&\qquad+\frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\}\right| \\
&\qquad+ \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}\right|.
\end{aligned}$$

Using (2.1) and (2.2) in the above result, we have

$$\lim_{s\to\infty} \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}\right| = 0.$$

This completes the proof of the theorem. □

Theorem 2.26. *The set of all S_ϕ-ward continuous functions on a subset E of $\mathbb{R}$ is a closed subset of the set of all continuous functions on E, i.e. $\overline{\Delta s_\phi wc(E)} = \Delta s_\phi wc(E)$ where $\Delta s_\phi wc(E)$ is the set of all S_ϕ-ward continuous functions on E, $\overline{\Delta s_\phi wc(E)}$ denotes the set of all cluster points of $\Delta s_\phi wc(E)$.*

Proof. Let f be an element in $\overline{\Delta s_\phi wc(E)}$. Then there exists sequence (f_n) of points in $\Delta s_\phi wc(E)$ such that $\lim_{n\to\infty} f_n = f$. To show that f is S_ϕ-ward continuous consider a sequence (x_n) of points in E such that $S_\phi\text{-}\lim_{n\to\infty} \Delta x_n = 0$. Since (f_n) converges to f, there exists a positive integer N such that for all $x \in E$ and for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. By the definition for all $x \in E$, we have

$$\lim_{s\to\infty} \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\right\}\right| = 0. \tag{2.3}$$

As f_N is S_ϕ-ward continuous on E we have

$$\lim_{s\to\infty} \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\}\right| = 0. \tag{2.4}$$

But

$$\begin{aligned}
&\frac{1}{\phi_s}|\left\{n \in \sigma, \sigma \in P_s : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}| \\
&\quad\leq \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3}\right\}\right| \\
&\qquad+ \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\}\right| \\
&\qquad+ \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}\right|.
\end{aligned}$$

Using (2.3) and (2.4) in the above relation, we have

$$\lim_{s\to\infty} \frac{1}{\phi_s}\left|\left\{n \in \sigma, \sigma \in P_s : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}\right| = 0.$$

This completes the proof of the theorem. □

Corollary 2.27. *The set of all S_ϕ-ward continuous functions on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on E.*

Proof. The proof follows from the preceding theorem. □

Cakalli [26] introduced the concept G-sequentially connected as, a non-empty subset E of $\mathbb{R}$ is called G-sequentially connected if there are non-empty and disjoint G-sequentially closed subsets U and V such that $A \subseteq U \cup V$, and $A \cap U$ and $A \cap V$ are empty. As far as G-sequentially connectedness is considered, then we get the following results.

Theorem 2.28. *Any S_ϕ-sequentially continuous image of any S_ϕ-sequentially connected subset of $\mathbb{R}$ is S_ϕ-sequentially connected.*

Proof. The proof follows from the Theorem 1 in [26]. □

Theorem 2.29. *A subset of $\mathbb{R}$ is S_ϕ-sequentially connected if and only if it is connected in ordinary sense and so is an interval.*

Proof. The proof follows from the Corollary 1 in [26]. □

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