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Original Article

The entire sequence over Musielak *p*-metric space

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Abstract In this paper, we introduce fibonacci numbers of $\Gamma^2(F)$ sequence space over p-metric spaces defined by Musielak function and examine some topological properties of the resulting these spaces.

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1. Introduction

Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are found in Bromwich [1]. Later on it was investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy et al. [6–18], Turkmenoglu [19], Raj [20–26] and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p{-}\lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1 \; for\, some \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p{-}\lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ be the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan et al. [27,28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and obtained the α-, β-, γ-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen et al. [30–35] have independently introduced the statistical convergence and Cauchy

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for double sequences and established the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Başar [36] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Başar and Sever [37] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Recently Subramanian and Misra [38] have studied the space $\chi^2_M(p,q,u)$ of double sequences and proved some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [39] as an extension of the definition of strongly Cesàro summable sequences. Cannor [40] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In Pringsheim [41] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn}|x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by Γ^2. Let $\phi = \{all\ finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only nonzero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m, n \in \mathbb{N})$ are also continuous.

Let M and Φ be mutually complementary modulus functions. Then, we have

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young'sinequality) \tag{1.2}$$

[*See Kampthan et al.*, [42]].

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u). \tag{1.4}$$

Lindenstrauss and Tzafriri [43] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\ some\ \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - f_{mn}(u) : u \geq 0\}, m, n = 1, 2, \ldots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined by

$$t_f = \left\{x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0 as m, n \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1\right\}.$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [44] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay et al. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation: $\Delta^m x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} m \\ j \end{pmatrix} x_{m+i,n+j}$.

2. Definition and preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,
(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p)^{1/p} for 1 \leq p < \infty$; (or)
(v) $d((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) := \sup\{d_X(x_1, x_2, \ldots, x_n), d_Y(y_1, y_2, \ldots, y_n)\}$,

for $x_1, x_2, \ldots, x_n \in X, y_1, y_2, \ldots, y_n \in Y$ is called the p-product metric of the Cartesian product of n-metric spaces is the p-norm of the n-vector of the norms of the n-subspaces.

A trivial example of p-product metric of n-metric space is the p-norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p-norm:

$$\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_E = sup(|det(d_{mn}(x_{mn}, 0))|) = sup \left(\begin{vmatrix} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \ldots & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \ldots & d_{2n}(x_{1n}, 0) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \ldots & d_{nn}(x_{nn}, 0) \end{vmatrix} \right),$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p-metric. Any complete p-metric space is said to be p-Banach metric space.

Definition 2.1. Let $A = (a_{k,\ell}^{mn})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the k, ℓ-th term of Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn},$$

such transformation is said to be non-negative if $a_{k\ell}^{mn}$ is non-negative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toeplitz. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which both added an additional assumption of boundedness. This assumption was made since a double sequence which is P-convergent is not necessarily bounded.

Let λ and μ be two sequence spaces and $A = (a_{k,\ell}^{mn})$ be a four dimensional infinite matrix of real numbers $(a_{k,\ell}^{mn})$, where $m, n, k, \ell \in \mathbb{N}$. Then, we say A defines a matrix mapping from λ into μ and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_{mn}) \in \lambda$ the sequence $Ax = \{(Ax)_{k\ell}\}$, the A-transform of x, is in μ. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus $A \in (\lambda : \mu)$ if and only if the series converges for each $k, \ell \in \mathbb{N}$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

Lemma 2.2. *Matrix $A = (a_{k,\ell}^{mn})$ is regular if and only if the following three conditions hold:*

(1) *There exists $M > 0$ such that for every $k, \ell = 1, 2, \ldots$ the following inequality holds: $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{k\ell}^{mn}| \leq M$;*
(2) $\lim_{k,\ell \to \infty} a_{k\ell}^{mn} = 0$ *for every $k, \ell = 1, 2, \ldots$*
(3) $\lim_{k,\ell \to \infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} = 1$.

Let (q_{mn}) be a sequence of positive numbers and

$$Q_{k\ell} = \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} (k, \ell \in \mathbb{N}). \tag{2.1}$$

Then, the matrix $R^q = (r_{k\ell}^{mn})^q$ of the Riesz mean is given by

$$(r_{k\ell}^{mn})^q = \begin{cases} \dfrac{q_{mn}}{Q_{k\ell}} & \text{if } 0 \leq m, n \leq k, \ell, \\ 0 & \text{if } (m, n) > k\ell, \end{cases} \tag{2.2}$$

The fibonacci numbers are the sequence of numbers $f_{k\ell}^{mn} (k, \ell, m, n \in \mathbb{N})$ defined by the linear recurrence equations $f_{00} = 1$ and $f_{11} = 1, f_{mn} = f_{m-1,n-1} + f_{m-2,n-2}; m, n \geq 2$. Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. Also, some basic properties of Fibonacci numbers are the following.

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} f_{m,n} = f_{m+2,n+2} - 1; m, n \geq 1,$$

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} f_{m,n}^2 = f_{m,n} f_{m+1,n+1}; m, n \geq 1,$$

$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{f_{k\ell}^{mn}}$ *converges.*

In this paper, we define the fibonacci matrix $F = (f_{k\ell}^{mn})_{m,n=1}^{\infty}$, which differs from existing Fibonacci matrix by using Fibonacci numbers $f_{k\ell}$ and introduce some new sequence spaces χ^2 and Λ^2. Now, we define the Fibonacci matrix $F = (f_{k\ell}^{mn})_{m,n=1}^{\infty}$, by

$$(f_{k\ell}^{mn}) = \begin{cases} \dfrac{f_{k\ell}}{f_{(k+2)(\ell+2)} - 1} & \text{if } 0 \leq k \leq m; 0 \leq \ell \leq n \\ 0 & \text{if } (m, n) > k\ell \end{cases}$$

that is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0\ldots & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\ldots & \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0\ldots & \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0\ldots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It follows from Lemma 2.2 that the method F is regular.

Let M be an Musielak modulus function. We introduce the following sequence spaces based on the four dimensional infinite matrix F:

$$\left[\Lambda^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] = F_\eta(x)$$
$$= sup_{k\ell}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f^{mn}_{k\ell}|x_{mn}|^{1/m+n}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)\|_p)\right)\right] < \infty\right\}$$
$$= sup_{k\ell}\left\{\frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f^{mn}_{k\ell}|x_{mn}|^{1/m+n},\right.\right.\right.$$
$$\left.\left.\left.\|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)\|_p)\right)\right] < \infty\right\}, (k,\ell \in \mathbb{N}).$$

Consider the metric space

$[\Lambda^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *with the metric*

$$d(x,y) = sup_{k\ell}\left\{M\left(F_\eta(x) - F_\eta(y)\right) : m,n = 1,2,3,\ldots\right\}. \quad (2.3)$$

$$\left[\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] = F_\mu(x) = \lim_{k,\ell\to\infty}$$
$$\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f^{mn}_{k\ell}|x_{mn}|^{1/m+n}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)\|_p)\right)\right] = 0\right\}$$
$$= \lim_{m,n\to\infty}\left\{\frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f^{mn}_{k\ell}|x_{mn}|^{1/m+n},\right.\right.\right.$$
$$\left.\left.\left.\|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)\|_p)\right)\right] = 0\right\}, (k,\ell \in \mathbb{N}).$$

Consider the metric space

$[\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *with the metric*

$$d(x,y) = sup_{k\ell}\left\{M\left(F_\mu(x) - F_\mu(y)\right) : m,n = 1,2,3,\ldots\right\}. \quad (2.4)$$

3. Main results

Theorem 3.1. *The spaces* $[\Lambda^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *and* $[\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *are isomorphic to the spaces* $[\Lambda^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *and* $[\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$, *respectively (i.e.)* $[\Lambda^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \cong [\Lambda^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *and* $[\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \cong [\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$

Proof. Let us consider the space of Γ^2, since the four dimensional infinite matrix F is triangular, it has a unique inverse, which is also triangular. Therefore the linear operator $L_F : [\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \to [\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$, defined by $L_F(x) = F(x)$ for all $x \in [\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \cong [\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$, is bijective and is metric preserving by (2.5) in Theorem 3.1. Hence

$[\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \cong [\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$. Similarly the proof for the other space can be established. □

Theorem 3.2. *The inclusion*

$[\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \subset [\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *and* $[\Lambda^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \subset [\Lambda^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *holds.*

Proof. As F is a regular four dimensional infinite matrix, so the inclusion

$[\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \subset [\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ is obvious.

Now, let $x = (x_{mn}) \in [\Lambda^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$. Then there is a constant $M > 0$ such that $|x_{mn}|^{1/m+n} \le M$ for all $m, n \in \mathbb{N}$. Thus for each $k, \ell \in \mathbb{N}$.

$$|F_\eta(x)| \le \left\{\frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{k}\sum_{n=1}^{\ell}\left[M\left(f^{mn}_{k\ell}|x_{mn}|^{1/m+n}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)\|_p)\right)\right]\right\}$$
$$\le \left\{\frac{M}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{k}\sum_{n=1}^{\ell}\left[M\left(f^{mn}_{k\ell}|x_{mn}|^{1/m+n}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)\|_p)\right)\right]\right\}$$
$$< \infty,$$

which shows that $FX \in [\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$. Thus we conclude that $[\Gamma^2_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \subset [\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$. □

Example. Consider the sequence

$$x = (x_{mn}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \ldots.$$

Then we have for every $k, \ell \in \mathbb{N}$,

$F_\mu(x) = \frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{k}\sum_{n=1}^{\ell}[M(f^{mn}_{k\ell}|x_{mn}|^{1/m+n}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0).\|_p))] \ne 0$. This shows that $FX \in \Gamma^2$ but x is not in Γ^2. Thus the sequence x is in

$[\Gamma^{2F}_M, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$. Hence the inclusion

$$\left[\Gamma_M^2, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] \subseteq \left[\Gamma_M^{2F}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$$

is strictly holds.

Theorem 3.3. *The sequence* $x = (x_{mn}) \notin [\Gamma_M^{2F}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$ *but in* $[\Lambda_M^{2F}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$

Proof. Consider the sequence $x = (x_{mn}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1\ldots \\ 1 & 1 & 1 & 1 & 1\ldots \\ 1 & 1 & 1 & 1 & 1\ldots \\ 1 & 1 & 1 & 1 & 1\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$, for all $k, \ell \in \mathbb{N}$. Then we have for every $k, \ell \in \mathbb{N}$, $F_\mu(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1\ldots \\ 1 & 1 & 1 & 1 & 1\ldots \\ 1 & 1 & 1 & 1 & 1\ldots \\ 1 & 1 & 1 & 1 & 1\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$. This shows that $FX \notin [\Gamma_M^{2F}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$.

Again, consider the sequence $x = (x_{mn}) = \frac{(-1)^{mn}(f_{m+2n+2+m+1n+1})-1}{f_{mn}}$, for all $k, \ell \in \mathbb{N}$. Then we have for every $k, \ell \in \mathbb{N}$, $F_\eta(x) = \begin{pmatrix} (-1)^{mn} & (-1)^{mn}\ldots \\ (-1)^{mn} & (-1)^{mn}\ldots \\ (-1)^{mn} & (-1)^{mn}\ldots \\ (-1)^{mn} & (-1)^{mn}\ldots \\ \vdots & \vdots \end{pmatrix}$. This shows that $FX \in [\Lambda_M^{2F}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]$. □

References

[1] T.J.I'A. Bromwich, An Introduction to the Theory of Infinite Series, Macmillan and Co.Ltd., New York, 1965.

[2] G.H. Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc. 19 (1917) 86–95.

[3] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, Acta. Math. Hung. 57 (1–2) (1991) 129–136.

[4] F. Moricz, B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc. 104 (1988) 283–294.

[5] M. Basarir, O. Solancan, On some double sequence spaces, J. Indian Acad. Math. 21 (2) (1999) 193–200.

[6] B.C. Tripathy, S. Mahanta, On a class of vector valued sequences associated with multiplier sequences, Acta Math. Appl. Sinica (Eng. Ser.) 20 (3) (2004) 487–494.

[7] B.C. Tripathy, A.J. Dutta, On fuzzy real-valued double sequence spaces ${}_2\ell_F^p$, Math. Comput. Modell. 46 (9–10) (2007) 1294–1299.

[8] B.C. Tripathy, B. Sarma, Statistically convergent difference double sequence spaces, Acta Math. Sinica 24 (5) (2008) 737–742.

[9] B.C. Tripathy, B. Sarma, Vector valued double sequence spaces defined by Orlicz function, Math. Slovaca 59 (6) (2009) 767–776.

[10] B.C. Tripathy, A.J. Dutta, Bounded variation double sequence space of fuzzy real numbers, Comput. Math. Appl. 59 (2) (2010) 1031–1037.

[11] B.C. Tripathy, B. Sarma, Double sequence spaces of fuzzy numbers defined by Orlicz function, Acta Math. Sci. 31B (1) (2011) 134–140.

[12] B.C. Tripathy, P. Chandra, On some generalized difference paranormed sequence spaces associated with multiplier sequences defined by modulus function, Anal. Theory Appl. 27 (1) (2011) 21–27.

[13] B.C. Tripathy, A.J. Dutta, Lacunary bounded variation sequence of fuzzy real numbers, 2013 J. Intell. Fuzzy Syst., 24 185–189.

[14] B.C. Tripathy, On statistically convergent double sequences, Tamkang J. Math. 34 (3) (2003) 231–237.

[15] B.C. Tripathy, M. Sen, Characterization of some matrix classes involving paranormed sequence spaces, Tamkang J. Math. 37 (2) (2006) 155–162.

[16] B.C. Tripathy, B. Hazarika, I-convergent sequence spaces associated with multiplier sequence spaces, Math. Inequal. Appl. 11 (3) (2008) 543–548.

[17] B.C. Tripathy, P. Chandra, On some generalized difference paranormed sequence spaces associated with multiplier sequences defined by modulus function, Anal. Theory Appl. 27 (1) (2011) 21–27.

[18] B.C. Tripathy, H. Dutta, On some new paranormed difference sequence spaces defined by Orlicz functions, Kyungpook Math. J. 50 (1) (2010) 59–69.

[19] A. Turkmenoglu, Matrix transformation between some classes of double sequences, J. Inst. Math. Comp. Sci. Math. Ser. 12 (1) (1999) 23–31.

[20] K. Raj, S.K. Sharma, Some sequence spaces in 2-normed spaces defined by Musielak Orlicz function, Acta Univ. Sapientiae Math. 3 (1) (2011) 97–109.

[21] K. Raj, A.K. Sharma, S.K. Sharma, Sequence spaces defined by Musielak-Orlicz function in 2-normed spaces, J. Comput. Anal. Appl. 14 (2012).

[22] K. Raj, S.K. Sharma, Lacunary sequence spaces defined by Musielak-Orlicz function, Le Matematiche 68 (2013) 33–51.

[23] K. Raj, A.K. Sharma, S.K. Sharma, A sequence spaces defined by a Musielak-Orlicz function, Int. J. Pure Appl. Math. 67 (4) (2011) 475–484.

[24] K. Raj, A.K. Sharma, S.K. Sharma, Difference sequence spaces in n-normed spaces defined by a Musielak-Orlicz function, Armenian J. Math. 3 (2010) 127–141.

[25] K. Rajand, S.K. Sharma, Ideal convergent sequence spaces defined by a Musielak-Orlicz function, Thai J. Math. 11 (2013) 577–587.

[26] K. Raj, S.K. Sharma, Some new Lacunary strong convergent vector-valued multiplier difference sequence spaces defined by a Musielak-Orlicz function, Acta Math. Acad. Nyiregyhaziensis 28 (2012) 103–120.

[27] A. Gökhan, R. Çolak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, Appl. Math. Comput. 157 (2) (2004) 491–501.

[28] A. Gökhan, R. Çolak, Double sequence spaces ℓ_2^∞, Appl. Math. Comput. 160 (1) (2005) 147–153.

[29] M. Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.

[30] M. Mursaleen, O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (1) (2003) 223–231.

[31] M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, J. Math. Anal. Appl. 293 (2) (2004) 523–531.

[32] M. Mursaleen, O.H.H. Edely, Almost convergence and a core theorem for double sequences, J. Math. Anal. Appl. 293 (2) (2004) 532–540.

[33] M. Mursaleen, S.K. Sharma, A. Kilicman, Sequence spaces defined by Musielak-Orlicz function over n-normed space, Abstract and Applied Analysis, Article ID 364743, 2013, 10pp.

[34] M. Mursaleen, A. Alotaibi, S.K. Sharma, New classes of generalized seminormed difference sequence spaces, Abstract and Applied Analysis, Article ID 461081, 2014, 7pp.

[35] M. Mursaleen, S.K. Sharma, Entire sequence spaces defined by Musielak-Orlicz function on locally convex Hausdorf topological spaces, Iranian J. Sci. Technol. Trans. A 38 (2014).

[36] J. Altay, F. Başar, Some new spaces of double sequences, J. Math. Anal. Appl. 309 (1) (2005) 70–90.

[37] F. Başar, Y. Sever, The space $\mathcal{L}_p$ of double sequences, Math. J. Okayama Univ. 51 (2009) 149–157.

[38] N. Subramanian, U.K. Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math. 46 (2010).

[39] I.J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc. 100 (1) (1986) 161–166.

[40] J. Cannor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. 32 (2) (1989) 194–198.

[41] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, Math. Ann. 53 (1900) 289–321.

[42] P.K. Kamthan, M. Gupta, Sequence Spaces and Series, Lecture notes, Pure and Applied Mathematics, vol. 65, Marcel Dekker, Inc., New York, 1981.

[43] J. Lindenstrauss, L. Tzafriri, On Orlicz sequence spaces, Israel J. Math. 10 (1971) 379–390.

[44] H. Kizmaz, On certain sequence spaces, Canad Math. Bull. 24 (2) (1981) 169–176.