Journal of the Egyptian Mathematical Society (2011) **19**, 86–87

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

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www.elsevier.com/locate/joems

SHORT COMMUNICATION

A note on discontinuous problem with a free boundary

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Available online 17 November 2011

KEYWORDS

Variational inequalities;
Uniqueness;
Free boundary

Abstract We study a nonlinear elliptic problem with discontinuous nonlinearity

$$(P)\begin{cases} -\Delta u = f(u)H(\mu - u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where H is the Heaviside's unit function, f, h are given functions and μ is a positive real parameter. We prove the existence of a unique solution and characterize the corresponding free boundary. Our methods relies on variational inequality approach combining with fixed point arguments.

In this note, we are interested to study the following free boundary problem

$$(P)\begin{cases} -\Delta u = f(u)H(\mu - u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is the unit ball of $\mathbb{R}^n$, H is the Heaviside function, f is a given function and μ is a positive real parameter.

Introduce the following assumptions. Let λ_1 be the first eigenvalue of $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions with the corresponding eigenfunction $\varphi_1 > 0$ in Ω.

(H_1) The function f is k – lipstchitzian, non-decreasing, positive and there exist two strictly positive constants k, $\beta > 0$ such that $f(s) \leqslant ks + \beta$ with $k < \min\{\lambda_1, 1\}$.

(H_2) For $\mu > 0$, we suppose

$$\delta := \frac{f(\mu)}{\mu} > \lambda_1.$$

(H_3) The function $f(s)/s$ is non increasing.

By a solution of problem (P), we mean a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla \xi dx = \int_\Omega f(u)H(\mu - u)\xi dx,$$

for $\xi \in C_0^1(\Omega)$. The main result of this note is the following theorem

Theorem 1. *Under the above assumptions, the problem (P) have a unique positive solution. Moreover, the set* $\{x \in \Omega \quad u(x) = \mu\}$ *is a ball of radius* $\rho \in (0, 1)$ *centered at the origin.*

Remark 1. In [1], the authors study the following problem

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Peer review under responsibility of Egyptian Mathematical Society.
doi:10.1016/j.joems.2011.09.002

Production and hosting by Elsevier

$$(P_0)\begin{cases} -\Delta u = f(u)H(u-\mu) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and prove that the problem (P_0) have a multiple solutions. On the other hand, we state here that the problem (P) have a unique positive solution for the same functions f. For example,

$$f(s) = ks + \beta, \quad 0 < k < \lambda_1 \quad \text{and} \quad \beta > 0,$$

satisfying

$$\frac{f(\mu)}{\mu} > \max(\lambda_1, \lambda^*),$$

where

$$\lambda^* = 2\lambda_1 \frac{||\varphi_1||_{L^1}}{||\varphi_1||^2_{L^2}}.$$

The proof of Theorem 1 will be given in several steps.

Let

$$K := \{v \in H^1_0(\Omega) : v \leqslant \mu \text{ in } \Omega\}.$$

Define

$$T(\widetilde{f}) = -(\Delta)^{-1}\widetilde{f} \quad \text{if} \quad \widetilde{f} \geqslant 0$$

and

$$-(\Delta)^{-1}\widetilde{f} \leqslant \mu \quad \text{for} \quad \mu > 0,$$

where $-(\Delta)^{-1}$ denote the inverse of Laplacian under Dirichlet conditions. Now, consider the operator $T_\delta : L^2(\Omega) \to L^2(\Omega)$ defined by

$$T_\delta(u) = T(\delta f(u)).$$

The proof consists to give the existence of a fixed point of T_δ. The fixed point of T_δ is equivalent to the solution of problem (P) by using the variational inequality approach. It easy to prove the uniqueness of solutions of (P). It remains give the characterization of the set $\{x \in \Omega, u(x) = \mu\}$. For that, we prove the symmetry of solution via the moving plane method [2] and we conclude by the following result.

Lemma 1. *Let γ be a Jordan curve in E_μ and w the interior of contour γ. Then $w \subset E_\mu$.*

Proof of lemma.

Suppose that the contrary hold, then there exists $x_0 \in w$ satisfying $u(x_0) < \mu$. Consider the set

$$w_1 := \{x \in \Omega / u(x) < \mu\}$$

which is not empty and is open since u is continuous. The function u verifies

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad w_1, \\ u = \mu & \text{on} \quad \partial w_1. \end{cases}$$

Hence, the maximum principle implies that $u \geqslant \mu$in w_1 which is a contradiction.

Proposition 1. *Lett u be a solution of problem (P), then the set E_μ is a ball of radius $r_0 \in (0,1)$.*

Proof of proposition. Since the solution is radial and non increasing, then there exists a unique r_0 such that $u(r_0) = \mu$. Hence Lemma 1 implies that

$$E_\mu = \{x \in \Omega / |x| \leqslant r_0\}.$$

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