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Original Article

Properties of certain subclass of *p*-valent meromorphic functions associated with certain linear operator

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Abstract We investigate several inclusion relationships of certain subclass of p-valent meromorphic functions defined in the punctured unit disc, having a pole of order p at the origin. The subclass under investigation is defined by using certain linear operator defined by combining two integral operators.

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1. Introduction

Let Σ_p denotes the subclass of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the punctured unit disc $U^* = U \backslash \{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions $f(z)$ and $g(z)$, analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ which (by definition) is analytic in U, satisfying the following conditions (see [1,2]):

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1; \qquad (z \in U)$$

such that

$$f(z) = g(\omega(z)); \quad (z \in U),$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in U) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

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In particular, if the function $g(z)$ is univalent in U, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Following the recent work of El-Ashwah [3], for a function $f(z) \in \Sigma_p$, given by (1.1), also, for $\lambda,\ \ell > 0$ and $m \in \mathbb{N}_0(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$, the integral operator $L_p^m(\lambda, \ell) : \Sigma_p \longrightarrow \Sigma_p$ is defined as follows:

$$L_p^m(\lambda,\ell)f(z) = \begin{cases} f(z); & (m=0), \\ \dfrac{\ell}{\lambda} z^{-p-\frac{\ell}{\lambda}} \displaystyle\int_0^z t^{\left(\frac{\ell}{\lambda}+p-1\right)} L_p^{m-1}(\lambda,\ell)f(t)dt; & (m = 1, 2, \ldots). \end{cases} \tag{1.2}$$

Also, following the recent work of El-Ashwah and Hassan [4], for a function $f(z) \in \Sigma_p$, given by (1.1), also, for $\mu > 0$, $a, c \in \mathbb{C}$ and $Re(c-a) \geq 0$, the integral operator $J_{p,\mu}^{a,c} : \Sigma_p \longrightarrow \Sigma_p$ is defined as follows:

$$J_{p,\mu}^{a,c} f(z) = \begin{cases} f(z); & (a = c), \\ \dfrac{\Gamma(c-p\mu)}{\Gamma(a-p\mu)\Gamma(c-a)} \displaystyle\int_0^1 t^{a-1}(1-t)^{c-a-1} f(zt^{\mu})dt; & (Re(c-a) > 0). \end{cases} \tag{1.3}$$

By iterations of the integral operators $L_p^m(\lambda, \ell)$ defined by (1.2) and $J_{p,\mu}^{a,c}$ defined by (1.3), we define the linear operator

$$I_{\lambda,\ell}^{p,m}(a, c, \mu) : \Sigma_p \longrightarrow \Sigma_p \tag{1.4}$$

for the purpose of this paper by:

$$I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z) := L_p^m(\lambda,\ell)\left(J_{p,\mu}^{a,c} f(z)\right) = J_{p,\mu}^{a,c}\left(L_p^m(\lambda,\ell)f(z)\right). \tag{1.5}$$

Now, it is easily to see that the operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$ can be expressed as follows:

$$I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z) = z^{-p} + \frac{\Gamma(c-p\mu)}{\Gamma(a-p\mu)} \times \sum_{k=1-p}^{\infty} \frac{\Gamma(a+\mu k)}{\Gamma(c+\mu k)} \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^m a_k z^k,$$
$$(\mu > 0;\ a, c \in \mathbb{C},\ Re(a) > p\mu,\ Re(c-a) \geq 0;\ \ell > 0;\ \lambda > 0;\ m \in \mathbb{N}_0;\ p \in \mathbb{N}). \tag{1.6}$$

In view of (1.2)–(1.5), it is clear that

$$I_{\lambda,\ell}^{p,0}(a,c,\mu)f(z) = J_{p,\mu}^{a,c} f(z) \quad \text{and} \quad I_{\lambda,\ell}^{p,m}(a,a,\mu)f(z) = L_p^m(\lambda,\ell)f(z). \tag{1.7}$$

Using (1.6), we can obtain the following recurrence relations of the operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$, which are necessary for our investigations

$$z\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)\right)' = \frac{a-p\mu}{\mu} I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(z) - \frac{a}{\mu} I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z). \tag{1.8}$$

and

$$z\left(I_{\lambda,\ell}^{p,m}(a,c+1,\mu)f(z)\right)' = \frac{c-p\mu}{\mu} I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z) - \frac{c}{\mu} I_{\lambda,\ell}^{p,m}(a,c+1,\mu)f(z). \tag{1.9}$$

Also

$$z\left(I_{\lambda,\ell}^{p,m+1}(a,c,\mu)f(z)\right)' = \frac{\ell}{\lambda} I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z) - \frac{\ell+\lambda p}{\lambda} I_{\lambda,\ell}^{p,m+1}(a,c,\mu)f(z). \tag{1.10}$$

The operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$ defined by (1.7) has been extensively studied by many authors with suitable restrictions on the parameters as follows:

(i) $I_{\lambda,\ell}^{1,-n}(a, c, \mu) = I_{\lambda,\ell}^{n}(a, c, \mu)f(z)(\mu>0;\ a, c\in\mathbb{C},\ Re(c-a) \geq 0,\ Re(a)>\mu;\ \ell>0;\ \lambda>0;\ n\in\mathbb{Z})$ (see El-Ashwah [5]);
(ii) $I_{\lambda,\ell}^{p,m}(p+\nu, p+1, 1) = I_{p,\nu}^{m}(\lambda, \ell)f(z)(m \in \mathbb{N}_0;\ \lambda, \ell, \nu>0;\ p \in \mathbb{N})$ (see El-Ashwah and Aouf [6]);
(iii) $I_{\nu,\lambda}^{1,m}(a+1, c+1, 1)f(z) = \Im_{\lambda,\nu}^{m}(a, c)f(z) \quad (\lambda, \nu>0;\ a \in \mathbb{C};\ c \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0)$ (see Raina and Sharma [7]);
(iv) $I_{\lambda,\ell}^{p,0}(a+p, c+p, 1)f(z) = \ell_p(a, c)f(z) \quad (a \in \mathbb{R};\ c \in \mathbb{R}\backslash \mathbb{Z}_0^-,\ \mathbb{Z}_0^- = \{0, 1, 2, \ldots\};\ p \in \mathbb{N})$ (see Liu and Srivastava [8]);
(v) $I_{1,\lambda}^{1,\beta}(\nu+1, 2, 1)f(z) = I_{\lambda,\nu}^{\beta}f(z)\ (\beta \geq 0;\ \lambda > 0;\ \nu > 0)$ (see Piejko and Sokół [9]);
(vi) $I_{1,\lambda}^{1,n}(\nu+1, 2, 1)f(z) = I_{\lambda,\nu}^{n}f(z) \quad (n \in \mathbb{N}_0;\ \lambda > 0;\ \nu > 0)$ (see Cho et al. [10]);
(vii) $I_{\lambda,\ell}^{1,0}(\nu+1, n+2, 1)f(z) = \ell_{n,\nu}f(z)\ (n > -1;\ \nu > 0)$ (see Yuan et al. [11]);
(viii) $I_{\lambda,\ell}^{p,0}(n+2p, p+1, 1)f(z) = D^{n+p-1}f(z)$ (n is an integer, $n > -p,\ p \in \mathbb{N}$) (see Uralegaddi and Somanatha [12]);
(ix) $I_{1,1}^{p,\alpha}(a, a, \mu)f(z) = P_p^{\alpha}f(z)(\alpha \geq 0;\ p \in \mathbb{N})$ (see Aqlan et al. [13]);
(x) $I_{1,\beta}^{1,\alpha}(a, a, \mu)f(z) = P_{\beta}^{\alpha}f(z)(\alpha, \beta > 0;\ p \in \mathbb{N})$ (see Lashin [14]).

Now, by the help of the linear operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$, we introduce the subclass $M_{\lambda,\ell}^{p,m}(a, c, \mu; \alpha; A, B)$ of meromorphic functions as follows:

Definition 1. For fixed parameters $A, B(-1 \leq B < A \leq 1)$ and $0 \leq \alpha < p$, the function $f(z) \in \Sigma_p$ is said to be in the class $M_{\lambda,\ell}^{p,m}(a, c, \mu; \alpha; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\alpha}\left(\frac{-z\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)} - \alpha\right) \prec \frac{1+Az}{1+Bz}(z \in U), \tag{1.11}$$

$(\mu>0;\ a, c\in\mathbb{C},\ Re(a)>p\mu,\ Re(c-a) \geq 0;\ \ell>0;\ \lambda>0;\ m\in\mathbb{N}_0;\ p\in\mathbb{N})$.

Or, equivalently

$$M_{\lambda,\ell}^{p,m}(a,c,\mu;\alpha;A,B) = \left\{ f(z)\in\Sigma_p : \left| \frac{\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)} + p}{B\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)} + [pB+(A-B)(p-\alpha)]} \right| < 1 \right\}. \tag{1.12}$$

We note that, by specializing the parameters A, B, a, c, p, λ, ℓ, μ and m we obtain the following subclasses introduced by various authors.

(i) $M^{p,m}_{\lambda,\ell}(p+\nu, p+1, 1; \alpha; 1, -1) = \Sigma S^{*m,\lambda,\ell}_{p,\nu}(\alpha)$ $(m \in \mathbb{N}_0; 0 \le \alpha < p; \lambda, \ell, \nu > 0; p \in \mathbb{N})$ (see El-Ashwah and Aouf [6]);
(ii) $M^{1,0}_{\lambda,\ell}(\nu+1, n+2, 1; \alpha; 1, -1) = S^*_{n,\nu}(\alpha)$ $(n > -1; 0 \le \alpha < 1; \nu > 0)$ (see Yuan et al. [11]);
(iii) $M^{p,0}_{\lambda,\ell}(a, a, \mu; \alpha; A, B) = Q^*(p, \alpha, A, B)$ $(-1 \le B < A \le 1; 0 \le \alpha < p; p \in \mathbb{N})$ (see Owa et al. [15]);
(iv) $M^{1,0}_{\lambda,\ell}(a, a, \mu; \alpha; \beta, -\beta) = \Sigma^*(\alpha, \beta)$ $(0 \le \alpha < p; 0 < \beta \le 1; p \in \mathbb{N})$ (see El-Ashwah et al. [16]);
(v) $M^{1,0}_{\lambda,\ell}(a, a, \mu; \alpha; 1, -1) = \Sigma^*(\alpha)$ $(0 \le \alpha < 1)$ (see Clunie [17]).

2. Preliminaries

To establish our main results, we shall need the following lemmas.

Lemma 1 ([1]). *If $-1 \le B < A \le 1$, $\beta \ne 0$ and the complex number γ satisfies $\operatorname{Re}\{\gamma\} \ge \frac{-\beta(1-A)}{1-B}$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec \frac{1+Az}{1+Bz} \quad (z \in U),$$

has a univalent solution in U given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta \int_0^z t^{\beta+\gamma-1}(1+Bz)^{\beta(A-B)/B} dt} - \dfrac{\gamma}{\beta}, & B \ne 0, \\ \dfrac{z^{\beta+\gamma}\exp(\beta A z)}{\beta \int_0^z t^{\beta+\gamma-1}\exp(\beta A t) dt} - \dfrac{\gamma}{\beta}, & B = 0. \end{cases} \tag{2.1}$$

If $\phi(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic in U and satisfies

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{2.2}$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U),$$

and q(z) is the best dominant of (2.2).

Lemma 2 ([18]). *Let ν be a positive measure on* [0, 1]. *Let h be a complex-valued function defined on $U \times [0, 1]$ such that $h(., t)$ is analytic in U for each $t \in [0, 1]$, and $h(z, .)$ is ν-integrable on* [0, 1] *for all $z \in U$. In addition, suppose that $\operatorname{Re}\{h(z, t)\} > 0$, $h(-r, t)$ is real and $\operatorname{Re}\{1/h(z, t)\} \ge 1/h(-r, t)$ for $|z| \le r < 1$ and $t \in [0, 1]$. If $h(z) = \int_0^1 h(z, t) d\nu(t)$, then $\operatorname{Re}\{1/h(z)\} \ge 1/h(-r)$.*

Lemma 3 ([19]). *For real numbers a, b, c $(c \ne 0, -1, -2, \ldots)$, we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \; {}_2F_1(a, b; c; z), \tag{2.3}$$

for $\operatorname{Re}\{c\} > \operatorname{Re}\{b\} > 0$ *and* $z \in U$. *Also, we have*

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z), \tag{2.4}$$

and

$${}_2F_1(a, b; c; z) = (1-z)^{-a} \; {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right). \tag{2.5}$$

In this paper, we find three inclusion theorems for the class $M^{p,m}_{\lambda,\ell}(a, c, \mu; \alpha; A, B)$ with respect to variations in the parameters a, c and m. In particular, we show that increasing a by one reduces the size of the class $M^{p,m}_{\lambda,\ell}(a, c, \mu; \alpha; A, B)$, but increasing the parameters c or m by one increases its size.

3. Inclusion results

Unless otherwise mentioned, we assume throughout the remainder of the paper that $-1 \le B < A \le 1$, $0 \le \alpha < p$, $\lambda > 0$, $\ell > 0$, $\mu > 0$, $a, c \in \mathbb{R}$, $a > p\mu$, $c - a \ge 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $z \in U$.

We begin with some inclusion relationships concerning the parameter a of the class $M^{p,m}_{\lambda,\ell}(a, c, \mu; \alpha; A, B)$.

Theorem 1. *(i) If $f(z) \in M^{p,m}_{\lambda,\ell}(a+1, c, \mu; \alpha; A, B)$ and*

$$\frac{a}{\mu} - \alpha \ge \frac{(p-\alpha)(1-A)}{(1-B)}, \tag{3.1}$$

then

$$\begin{aligned} &\frac{1}{p-\alpha}\left(-\frac{z\left(I^{p,m}_{\lambda,\ell}(a, c, \mu)f(z)\right)'}{I^{p,m}_{\lambda,\ell}(a, c, \mu)f(z)} - \alpha\right) \\ &\prec \frac{1}{p-\alpha}\left(\left(\frac{a}{\mu} - \alpha\right) - \frac{1}{Q_1(z)}\right) \\ &= q_1(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U), \end{aligned} \tag{3.2}$$

where

$$Q_1(z) = \begin{cases} \displaystyle\int_0^1 (u)^{\frac{a}{\mu}-p-1}\left(\frac{1+Bzu}{1+Bz}\right)^{-(p-\alpha)(A-B)/B} du, & B \ne 0, \\ \displaystyle\int_0^1 (u)^{\frac{a}{\mu}-p-1} e^{-(p-\alpha)A(u-1)z} du, & B = 0, \end{cases}$$

and $q_1(z)$ is the best dominant of (3.2). Consequently

$$M^{p,m}_{\lambda,\ell}(a+1, c, \mu; \alpha; A, B) \subseteq M^{p,m}_{\lambda,\ell}(a, c, \mu; \alpha; A, B). \tag{3.3}$$

(ii) Furthermore, if the additional constraints $0 < B < 1$ and

$$\frac{a}{\mu} > \frac{(p-\alpha)(A-B)}{B} + p - 1 \tag{3.4}$$

are satisfied. Then

$$\frac{1-|A|}{1-|B|} < \frac{1}{p-\alpha}\left(-\operatorname{Re}\left\{\frac{z\left(I^{p,m}_{\lambda,\ell}(a, c, \mu)f(z)\right)'}{I^{p,m}_{\lambda,\ell}(a, c, \mu)f(z)}\right\} - \alpha\right) < \rho_1, \tag{3.5}$$

where

$$\rho_1 = \frac{1}{p-\alpha}\left\{\left(\frac{a}{\mu}-\alpha\right) - \frac{\frac{a}{\mu}-p}{{}_2F_1\left(1, \frac{(p-\alpha)(A-B)}{B}; \frac{a}{\mu}-p+1; \frac{B}{B-1}\right)}\right\}. \tag{3.6}$$

The bound ρ_1 is the best possible.

Proof. Let $f \in M^{p,m}_{\lambda,\ell}(a+1, c, \mu; \alpha; A, B)$. Set

$$\phi(z) = \frac{1}{p-\alpha}\left(-\frac{z\left(I^{p,m}_{\lambda,\ell}(a,c,\mu)f(z)\right)'}{I^{p,m}_{\lambda,\ell}(a,c,\mu)f(z)} - \alpha\right). \tag{3.7}$$

It is clear that $\phi(z)$ is analytic in U and $\phi(0) = 1$. An application of the identity (1.8) in (3.7), yields

$$-(p-\alpha)\phi(z) + \left(\frac{a}{\mu}-\alpha\right) = \frac{a-p\mu}{\mu}\frac{I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(z)}{I^{p,m}_{\lambda,\ell}(a,c,\mu)f(z)}, \tag{3.8}$$

using the logarithmic differentiation of both sides of (3.8) with respect to z, we obtain

$$\begin{aligned}\phi(z) &+ \frac{z\phi'(z)}{-(p-\alpha)\phi(z) + \left(\frac{a}{\mu}-\alpha\right)} \\ &= \frac{1}{p-\alpha}\left(-\frac{z\left(I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(z)\right)'}{I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(z)} - \alpha\right) \\ &\prec \frac{1+Az}{1+Bz} \quad (z \in U).\end{aligned}$$

Therefore, an application of Lemma 1 with $\beta = -(p-\alpha)$ and $\gamma = \frac{a}{\mu} - \alpha$, we have

$$\phi(z) \prec q_1(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U),$$

where the best dominant $q_1(z)$ is defined by (3.2). The proof of (i) of Theorem 1 is completed. □

In order to establish (3.5) of (ii), we observe that an application of the principle of subordination in (1.11) gives

$$\frac{1-|A|}{1-|B|} < \frac{1}{p-\alpha}\left(-\text{Re}\left\{\frac{z\left(I^{p,m}_{\lambda,\ell}(a,c,\mu)f(z)\right)'}{I^{p,m}_{\lambda,\ell}(a,c,\mu)f(z)}\right\} - \alpha\right),$$

which is precisely the left hand inequality in (3.5). Also, by the principle of subordination in (3.2), we have

$$\begin{aligned}&\frac{1}{p-\alpha}\left(-\text{Re}\left\{\frac{z\left(I^{p,m}_{\lambda,\ell}(a,c,\mu)f(z)\right)'}{I^{p,m}_{\lambda,\ell}(a,c,\mu)f(z)}\right\} - \alpha\right) \\ &\le \sup_{z\in U}\text{Re}\{q_1(z)\} \\ &= \sup_{z\in U}\left[\frac{1}{p-\alpha}\left(\frac{a}{\mu}-\alpha-\text{Re}\left\{\frac{1}{Q_1(z)}\right\}\right)\right] \\ &= \frac{1}{p-\alpha}\left(\frac{a}{\mu}-\alpha-\inf_{z\in U}\text{Re}\left\{\frac{1}{Q_1(z)}\right\}\right).\end{aligned} \tag{3.9}$$

The rest of the proof is devoted to find $\inf_{z\in U}\text{Re}\{1/Q_1(z)\}$. By hypothesis $B \neq 0$, therefore by (3.2) we have

$$Q_1(z) = (1+Bz)^{\delta}\int_0^1 (u)^{\beta-1}(1-u)^{\gamma-\beta-1}(1+Bzu)^{-\delta}du,$$

where $\delta = \frac{(p-\alpha)(A-B)}{B}$, $\beta = \frac{a}{\mu} - p$ and $\gamma = \beta + 1$. Also since $\gamma > \beta > 0$, by successively using (2.3)–(2.5) of Lemma 3, we obtain

$$Q_1(z) = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\, {}_2F_1\left(1, \delta; \gamma; \frac{Bz}{Bz+1}\right). \tag{3.10}$$

Furthermore, the condition $\frac{a}{\mu} > \frac{(p-\alpha)(A-B)}{B} + p - 1$ with $0 < B < 1$ implies that $\gamma > \delta > 0$. Another application of (2.5) of Lemma 3 to (3.10) gives

$$Q_1(z) = \int_0^1 h(z,u)dv(u),$$

where

$$h(z,u) = \frac{1+Bz}{1+(1-u)Bz} \quad (0 \le u \le 1),$$

and

$$dv(u) = \frac{\Gamma(\beta)}{\Gamma(\delta)\Gamma(\gamma-\delta)}(u)^{\delta-1}(1-u)^{\gamma-\delta-1}du,$$

is a positive measure on $u \in [0,1]$. We note that $\text{Re}\{h(z,u)\} > 0$ and $h(-r,u)$ is real for $0 \le r < 1$ and $u \in [0,1]$. Therefore, using Lemma 2 implies

$$\text{Re}\left\{\frac{1}{Q_1(z)}\right\} \ge \frac{1}{Q_1(-r)} \quad (|z| \le r < 1),$$

thus

$$\begin{aligned}\inf_{z\in U}\text{Re}\left\{\frac{1}{Q_1(z)}\right\} &= \sup_{0\le r<1}\frac{1}{Q_1(-r)} \\ &= \sup_{0\le r<1}\frac{1}{\int_0^1 h(-r,u)dv} \\ &= \frac{1}{\int_0^1 h(-1,u)dv} = \frac{1}{Q_1(-1)} \\ &= \frac{\frac{a}{\mu}-p}{{}_2F_1\left(1, \frac{(p-\alpha)(A-B)}{B}; \frac{a}{\mu}-p+1; \frac{B}{B-1}\right)}.\end{aligned} \tag{3.11}$$

Hence, in view of (3.9), the right hand inequality of (3.5) follows from (3.11).

The result is the best possible as the function $q_1(z)$ is the best dominant of (3.2). This completes the proof of Theorem 1.

The next theorem gives the corresponding results due to the parameter c.

Theorem 2. *(i) If $f(z) \in M^{p,m}_{\lambda,\ell}(a, c, \mu; \alpha; A, B)$ and*

$$\frac{c}{\mu} - \alpha \ge \frac{(p-\alpha)(1-A)}{(1-B)}, \tag{3.12}$$

then

$$\frac{1}{p-\alpha}\left(-\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c{+}1,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c{+}1,\mu)f(z)}-\alpha\right)$$
$$\prec \frac{1}{p-\alpha}\left(\left(\frac{c}{\mu}-\alpha\right)-\frac{1}{Q_2(z)}\right)$$
$$= q_2(z) \prec \frac{1+Az}{1+Bz} \quad (z\in U), \tag{3.13}$$

where

$$Q_2(z)=\begin{cases}\displaystyle\int_0^1 (u)^{\frac{c}{\mu}-p-1}\left(\frac{1+Bzu}{1+Bz}\right)^{-(p-\alpha)(A-B)/B}du, & B\neq 0,\\ \displaystyle\int_0^1 (u)^{\frac{c}{\mu}-p-1}e^{-(p-\alpha)A(u-1)z}\,du, & B=0,\end{cases}$$

and $q_2(z)$ is the best dominant of (3.13). Consequently

$$M_{\lambda,\ell}^{p,m}(a,c,\mu;\alpha;A,B)\subseteq M_{\lambda,\ell}^{p,m}(a,c+1,\mu;\alpha;A,B). \tag{3.14}$$

(ii) Furthermore, if the additional constraints $0<B<1$ and

$$\frac{c}{\mu}>\frac{(p-\alpha)(A-B)}{B}+p-1 \tag{3.15}$$

are satisfied. Then

$$\frac{1-|A|}{1-|B|}<\frac{1}{p-\alpha}\left(-\mathrm{Re}\left\{\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c+1,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c+1,\mu)f(z)}\right\}-\alpha\right)<\rho_2, \tag{3.16}$$

where

$$\rho_2=\frac{1}{p-\alpha}\left\{\left(\frac{c}{\mu}-\alpha\right)-\frac{\frac{c}{\mu}-p}{{}_2F_1\left(1,\frac{(p-\alpha)(A-B)}{B};\frac{c}{\mu}-p+1;\frac{B}{B-1}\right)}\right\}. \tag{3.17}$$

The bound ρ_2 is the best possible.

Proof. Let $f\in M_{\lambda,\ell}^{p,m}(a,c,\mu;\alpha;A,B)$. Set

$$\phi(z)=\frac{1}{p-\alpha}\left(-\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c+1,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c+1,\mu)f(z)}-\alpha\right). \tag{3.18}$$

Then using (1.9) and logarithmic differentiation for (3.18), we can obtain

$$\phi(z)+\frac{z\phi'(z)}{-(p-\alpha)\phi(z)+\left(\frac{c}{\mu}-\alpha\right)}$$
$$=\frac{1}{p-\alpha}\left(-\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu)f(z)}-\alpha\right)$$
$$\prec\frac{1+Az}{1+Bz}\quad(z\in U). \tag{3.19}$$

Therefore by an application of Lemma 1,with $\beta=-(p-\alpha)$ and $\gamma=\frac{c}{\mu}-\alpha$, we have

$$\phi(z)\prec q_2(z)\prec\frac{1+Az}{1+Bz}\quad(z\in U),$$

where the best dominant $q_2(z)$ is defined by (3.13). The proof of (i) of Theorem 2 is completed.

In order to establish (3.16) of (ii), we use the same technique used in the proof of Theorem 1. Write

$$Q_2(z)=(1+Bz)^{\delta}\int_0^1 (u)^{\beta-1}(1-u)^{\gamma-\beta-1}(1+Bzu)^{-\delta}du$$
$$=\frac{\Gamma(\beta)}{\Gamma(\gamma)}\ {}_2F_1\left(1,\delta;\gamma;\tfrac{Bz}{Bz+1}\right), \tag{3.20}$$

where $\delta=\frac{(p-\alpha)(A-B)}{B}$, $\beta=\frac{c}{\mu}-p$ and $\gamma=\beta+1$.

Furthermore, the condition $\frac{c}{\mu}>\frac{(p-\alpha)(A-B)}{B}+p-1$ with $0<B<1$ implies that $\gamma>\delta>0$. Another application of (2.5) of Lemma 3 to (3.20) gives

$$Q_2(z)=\int_0^1 h(z,u)dv(u),$$

where

$$h(z,u)=\frac{1+Bz}{1+(1-u)Bz}\quad(0\le u\le 1),$$

and

$$dv(u)=\frac{\Gamma(\beta)}{\Gamma(\delta)\Gamma(\gamma-\delta)}(u)^{\delta-1}(1-u)^{\gamma-\delta-1}du.$$

Hence, by Lemma 2

$$\inf_{z\in U}\mathrm{Re}\left\{\frac{1}{Q_2(z)}\right\}=\frac{\frac{c}{\mu}-p}{{}_2F_1\left(1,\frac{(p-\alpha)(A-B)}{B};\frac{c}{\mu}-p+1;\frac{B}{B-1}\right)}. \tag{3.21}$$

The right hand inequality of (3.16) now follows from (3.21). The bound ρ_2 is sharp by the principle of subordination. The proof of Theorem 2 is thus completed. □

Also, the corresponding results due to the parameter m is established in the following theorem.

Theorem 3. *(i) If $f(z)\in M_{\lambda,\ell}^{p,m}(a,c,\mu;\alpha;A,B)$ and*

$$\frac{\ell+\lambda p}{\lambda}-\alpha\ge\frac{(p-\alpha)(1-A)}{(1-B)}, \tag{3.22}$$

then

$$\frac{1}{p-\alpha}\left(-\frac{z\left(I_{\lambda,\ell}^{p,m+1}(a,c,\mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m+1}(a,c,\mu)f(z)}-\alpha\right)$$
$$\prec\frac{1}{p-\alpha}\left(\left(\frac{\ell+\lambda p}{\lambda}-\alpha\right)-\frac{1}{Q_3(z)}\right)$$
$$=q_3(z)\prec\frac{1+Az}{1+Bz}\quad(z\in U), \tag{3.23}$$

where

$$Q_3(z)=\begin{cases}\displaystyle\int_0^1 (u)^{\frac{\ell+\lambda p}{\lambda}-p-1}\left(\frac{1+Bzu}{1+Bz}\right)^{-(p-\alpha)(A-B)/B}du, & B\neq 0,\\ \displaystyle\int_0^1 (u)^{\frac{\ell+\lambda p}{\lambda}-p-1}e^{-(p-\alpha)A(u-1)z}\,du, & B=0,\end{cases}$$

and $q_3(z)$ is the best dominant of (3.23). Consequently

$$M_{\lambda,\ell}^{p,m}(a, c, \mu; \alpha; A, B) \subseteq M_{\lambda,\ell}^{p,m+1}(a, c, \mu; \alpha; A, B). \tag{3.24}$$

(ii) Furthermore, if the additional constraints $0 < B < 1$ and

$$\frac{\ell + \lambda p}{\lambda} > \frac{(p-\alpha)(A-B)}{B} + p - 1 \tag{3.25}$$

are satisfied. Then

$$\frac{1-|A|}{1-|B|} < \frac{1}{p-\alpha}\left(-\mathrm{Re}\left\{\frac{z\left(I_{\lambda,\ell}^{p,m+1}(a, c, \mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m+1}(a, c, \mu)f(z)}\right\} - \alpha\right) < \rho_3, \tag{3.26}$$

where

$$\rho_3 = \frac{1}{p-\alpha}\left\{\left(\frac{\ell+\lambda p}{\lambda} - \alpha\right) - \frac{\frac{\ell+\lambda p}{\lambda} - p}{{}_2F_1\left(1, \frac{(p-\alpha)(A-B)}{B}; \frac{\ell+\lambda p}{\lambda} - p + 1; \frac{B}{B-1}\right)}\right\}. \tag{3.27}$$

The bound ρ_3 is the best possible.

Proof. Let $f \in M_{\lambda,\ell}^{p,m}(a, c, \mu; \alpha; A, B)$. Set

$$\phi(z) = \frac{1}{p-\alpha}\left(-\frac{z\left(I_{\lambda,\ell}^{p,m+1}(a, c, \mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m+1}(a, c, \mu)f(z)} - \alpha\right). \tag{3.28}$$

Then using (1.10) and logarithmic differentiation for (3.28), we can obtain

$$\begin{aligned}\phi(z) &+ \frac{z\phi'(z)}{-(p-\alpha)\phi(z) + \left(\frac{\ell+\lambda p}{\lambda} - \alpha\right)} \\ &= \frac{1}{p-\alpha}\left(-\frac{z\left(I_{\lambda,\ell}^{p,m}(a, c, \mu)f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a, c, \mu)f(z)} - \alpha\right) \\ &\prec \frac{1+Az}{1+Bz} \quad (z \in U).\end{aligned} \tag{3.29}$$

Therefore by an application of Lemma 1, with $\beta = -(p-\alpha)$ and $\gamma = \frac{\ell+\lambda p}{\lambda} - \alpha$, we have

$$\phi(z) \prec q_3(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U),$$

where the best dominant $q_3(z)$ is defined by (3.23). The proof of (i) of Theorem 3 is completed.

Also, in order to establish (3.26) of (ii), we use the same technique used before. Write

$$\begin{aligned}Q_3(z) &= (1+Bz)^{\delta}\int_0^1 (u)^{\beta-1}(1-u)^{\gamma-\beta-1}(1+Bzu)^{-\delta}du \\ &= \frac{\Gamma(\beta)}{\Gamma(\gamma)}\, {}_2F_1\left(1, \delta; \gamma; \frac{Bz}{Bz+1}\right),\end{aligned} \tag{3.30}$$

where $\delta = \frac{(p-\alpha)(A-B)}{B}$, $\beta = \frac{\ell+\lambda p}{\lambda} - p$ and $\gamma = \beta + 1$.

Furthermore, the condition $\frac{\ell+\lambda p}{\lambda} > \frac{(p-\alpha)(A-B)}{B} + p - 1$ with $0 < B < 1$ implies that $\gamma > \delta > 0$. Another application of (2.5) of Lemma 3 to (3.30) gives

$$Q_3(z) = \int_0^1 h(z, u)dv(u),$$

where

$$h(z, u) = \frac{1+Bz}{1+(1-u)Bz} \quad (0 \le u \le 1),$$

and

$$dv(u) = \frac{\Gamma(\beta)}{\Gamma(\delta)\Gamma(\gamma-\delta)}(u)^{\delta-1}(1-u)^{\gamma-\delta-1}du.$$

Hence, by Lemma 2

$$\inf_{z\in U} \mathrm{Re}\left\{\frac{1}{Q_3(z)}\right\} = \frac{\frac{\ell+\lambda p}{\lambda} - p}{{}_2F_1\left(1, \frac{(p-\alpha)(A-B)}{B}; \frac{\ell+\lambda p}{\lambda} - p + 1; \frac{B}{B-1}\right)}. \tag{3.31}$$

The right hand inequality of (3.26) now follows from (3.31). The bound ρ_3 is sharp by the principle of subordination. The proof of Theorem 3 is thus completed. □

Remark 2.

(i) Taking $\mu = 1$, $a = p + \nu$ and $c = p + 1$ in Theorems 1 and 3, we can obtain the inclusion relationship of El-Ashwah and Aouf [6, Theorem 2.1];

(ii) Taking $A = \mu = p = 1$, $B = -1$, $a = \nu + 1$, $c = n + 2$ and $m = 0$, in Theorems 1 and 3, we can obtain the inclusion relationship of Yuan et al. [11, Theorem 1];

(iii) By specializing the parameters in Theorems 1,2,3 we can obtain various results of different subclasses defined by operators mentioned in the introduction as special cases of the operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$.

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