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ORIGINAL ARTICLE

Solutions of some class of nonlinear PDEs in mathematical physics

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Abstract In this work, the modified simple equation (MSE) method is applied to some class of nonlinear PDEs, namely, a system of nonlinear PDEs, a (2 + 1)-dimensional nonlinear model generated by the Jaulent–Miodek hierarchy, and a generalized KdV equation with two power nonlinearities.

As a result, exact traveling wave solutions involving parameters have been obtained for the considered nonlinear equations in a concise manner. When these parameters are chosen as special values, the solitary wave solutions are derived. It is shown that the proposed technique provides a more powerful mathematical tool for constructing exact solutions for a broad variety of nonlinear PDEs in mathematical physics.

Mathematics Subject Classification: 35C05; 35L05; 35Q99

1. Introduction

Recently, several methods have been used to extract exact solutions of nonlinear partial differential equations (NPDEs) such as inverse scattering method [1,2], the Darboux transform [3,4], the Hirota bilinear method [5,6], the Backlund transformation method [7–9], the Exp-function method [10–12], the (G'/G)-expansion method [13–15], the projective Riccati equation method [16], first integral method [17–20], exp $(-\Phi(\xi))$-expansion method [21–23], the functional variable method [24,25], modified simple equation method [26–33] and others. However, up to now, a unified method that can be used to deal with all types of nonlinear PDEs has not been found yet. To obtain more different types of exact solution, the enhancement of these methods is a challenge topic.

This paper is organized as follows: a description of the modified simple equation (MSE) method is presented first [26–33]. This is followed by an application of this method to three distinct model equations, two of them are (2 + 1)-dimensional nonlinear types, namely, a system of nonlinear PDEs and a nonlinear model generated by Jaulent–Miodek hierarchy, and the third is the generalized KdV nonlinear model with two power nonlinearities.

Finally, some conclusions are given.

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2. The MSE method

Consider a general nonlinear PDE in the form of

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \dots) = 0. \tag{1}$$

where P is a polynomial in its arguments.

Step 1. Seek solitary wave solutions of Eq. (1) by taking

$$u(x,t) = U(z), \quad z = x - ct + \zeta, \tag{2}$$

where ζ is an arbitrary constant, and transform (1) to a nonlinear ordinary differential equation (ODE) as

$$Q(U, U', U'', U''', \dots) = 0. \tag{3}$$

where the prime denotes the derivation with respect to z.

Step 2. We suppose that (3) has the formal solution as

$$U(z) = \sum_{i=0}^{M} \alpha_i \left(\frac{\Psi'(z)}{\Psi(z)}\right)^i, \tag{4}$$

where $\alpha_i, (i = 0, 1, \dots, M)$ are constants to be determined such that $\alpha_M \neq 0$. The function $\Psi(z)$ is an unknown function to be determined later such that $\Psi'(z) \neq 0$.

Step 3. We determine the positive integer M in (4) by balancing the highest order derivatives and the nonlinear terms in (3).

Step 4. We substitute (4) into (3), we calculate all the necessary derivatives $U', U'', U''', \dots$ and then we account the function $\Psi(z)$.
As a result of this substitution, we get a polynomial of $\Psi^{-j}(z)(j = 0, 1, \dots)$, with the derivatives of $\Psi(z)$. Equating all the coefficients of $\Psi^{-j}(z)(j = 0, 1, \dots)$, to zero yields a system of equations which can be solved to obtain α_i and $\Psi(z)$. Finally, substituting the values of α_i and $\Psi(z)$ and its derivative $\Psi'(z)$ into (4) leads to exact solutions of (1).

3. Applications

Example 1. A system of nonlinear PDEs given by [34–36] is

$$iu_t + n(u_{xx} + \alpha_1 u_{yy}) + \beta_1 |u|^2 u + \gamma_1 uv = 0 \tag{5}$$

$$\alpha_2 v_{tt} + (v_{xx} - \beta_2 v_{yy}) + \gamma_2 \left(|u|^2\right)_{xx} = 0 \tag{6}$$

where $n, \alpha_i, \beta_i, \gamma_i (i = 1, 2)$ are real constants and $n \neq 0, \beta_1 \neq 0, \gamma_1 \neq 0, \gamma_2 \neq 0$. The important cases of (5) and (6) are shown in [36–38] that are the nonlinear Schrodinger equation [36], the Davey–Stewartson (DS) equations [37], and the generalized Zakharov (GZ) equations [38].

At first we are going to extract exact solutions for (5) and (6), for this purpose, using the transformation formula

$$\begin{aligned} &u(x,y,t) = e^{i\theta}\varphi(\xi), v(x,y,t) = \Gamma(\xi), \\ &\theta = px + qy + \varepsilon t, \xi = x + cy + dt + \xi_0 \end{aligned} \tag{7}$$

where $p, q, \varepsilon, c, \xi_0$ and d are real constants.

Substituting (7) into (5) and (6), we can know that $d = -2n(p + \alpha_1 qc)$, and φ, Γ satisfy the equations

$$\begin{aligned} &-(\omega + p^2 n + n\alpha_1 q^2)\varphi(\xi) + (n + n\alpha_1 c^2)\varphi''(\xi) + \beta_1 \varphi^3(\xi) \\ &\quad + \gamma_1 \varphi(\xi)\Gamma(\xi) = 0, \end{aligned} \tag{8}$$

$$(\alpha_2 d^2 - \beta_2 c^2 + 1)\Gamma''(\xi) + \gamma_2 (\varphi^2(\xi))'' = 0. \tag{9}$$

Integrating (9) twice with respect to ξ and considering the constant of integration equal to zero, we find

$$v(x,y,t) = \Gamma(\xi) = -\frac{\gamma_2}{(\alpha_2 d^2 - \beta_2 c^2 + 1)}\varphi^2(\xi). \tag{10}$$

Substituting (10) into (8), gives

$$\varphi''(\xi) - \lambda\varphi(\xi) - \mu\varphi^3(\xi) = 0, \tag{11}$$

where

$$\lambda = \frac{\omega + p^2 n + n\alpha_1 q^2}{n + n\alpha_1 c^2}, \quad \mu = \frac{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2}{(n + n\alpha_1 c^2)(\alpha_2 d^2 - \beta_2 c^2 + 1)}. \tag{12}$$

Balancing $\varphi''(\xi)$ with $\varphi^3(\xi)$ yields $M = 1$. Consequently, we have the formal solution as

$$\varphi(\xi) = A_0 + A_1\left(\frac{\Psi'(\xi)}{\Psi(\xi)}\right), \tag{13}$$

where A_0 and A_1 are constants to be determined with $A_1 \neq 0$. Also the function $\Psi(\xi)$ has to be determined where $\Psi'(\xi) \neq 0$.

It is easy to see that

$$\varphi'(\xi) = A_1\left(\frac{\Psi''}{\Psi} - \frac{\Psi'^2}{\Psi^2}\right), \tag{14}$$

$$\varphi''(\xi) = A_1\left(\frac{\Psi'''}{\Psi} - 3\frac{\Psi'\Psi''}{\Psi^2} + 2\frac{\Psi'^3}{\Psi^3}\right). \tag{15}$$

Substituting (13)–(15) into (11) and equating all the coefficients of $\Psi^0, \Psi^{-1}, \Psi^{-2}, \Psi^{-3}$, to zero, we respectively obtain

$$\Psi^0: \ -\lambda A_0 - \mu A_0^3 = 0, \tag{16}$$

$$\Psi^{-1}: \ A_1\Psi''' - \lambda A_1\Psi' - 3A_0^2 A_1\mu\Psi' = 0, \tag{17}$$

$$\Psi^{-2}: \ -3A_1\Psi'\Psi'' - 3A_0 A_1^2\mu\Psi'^2 = 0, \tag{18}$$

$$\Psi^{-3}: \ 2A_1\Psi'^3 - \mu A_1^3\Psi'^3 = 0, \tag{19}$$

From (16) and (19), we deduce that

$$A_0 = 0, \quad A_0 = \pm\sqrt{\lambda/\mu}, \quad A_1 = \pm\sqrt{2/\mu} \tag{20}$$

From (17) and (18), we obtain

$$\Psi'''/\Psi'' = -\ell_1, \tag{21}$$

where

$$\ell_1 = (\lambda + 3A_0^2\mu)/(A_0 A_1\mu). \tag{22}$$

Integrating (21), we obtain

$$\Psi'' = C_1 e^{-\ell_1\xi}, \tag{23}$$

where C_1 is a constant of integration.

And from (18) and (23), we obtain

$$\Psi' = -m_1 e^{-\ell_1\xi} \tag{24}$$

where

$$m_1 = C_1/(A_0 A_1\mu). \tag{25}$$

Integrating (24) with respect to ξ, we get

$$\Psi(\zeta) = C_2 + (m_1/\ell_1)e^{-\ell_1\xi}, \tag{26}$$

where C_2 is the constant of integration.

Substituting the values of Ψ and Ψ' into (13), we obtain

$$\varphi(\xi) = A_0 - A_1\left(\left(m_1 e^{-\ell_1\xi}\right)/\left(C_2 + \frac{m_1}{\ell_1}e^{-\ell_1\xi}\right)\right). \tag{27}$$

Case 1. When $A_0 = 0$, solution (27) collapses and hence this case is rejected.

Case 2. When $A_0 = \pm\sqrt{\lambda/\mu}, A_1 = \pm\sqrt{2/\mu}$, where λ and μ are given in (12), substituting the values of A_0 and A_1 into (27) and simplifying, we obtain the exact solution as follows:

$$\varphi(x,y,t) = \pm\sqrt{\frac{(\alpha_2 d^2 - \beta_2 c^2 + 1)(\omega + p^2 n + n\alpha_1 q^2)}{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2}} \mp \frac{(n + n\alpha_1 c^2)\sqrt{\alpha_2 d^2 - \beta_2 c^2 + 1}C_1}{\sqrt{(-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2)(\omega + p^2 n + n\alpha_1 q^2)}} \times \left(\frac{e^{-\ell_1(x+cy+dt+\xi_0)}}{C_2 + \frac{(n+n\alpha_1 c^2)C_1}{2(\omega+p^2 n+n\alpha_1 q^2)}e^{-\ell_1(x+cy+dt+\xi_0)}}\right). \tag{28}$$

Since C_1 and C_2 are arbitrary constants, therefore, if we set $C_1 = [2(\omega + p^2 n + n\alpha_1 q^2)]/(n + n\alpha_1 c^2)$, we obtain the exact solution of the system. (5) and (6) and can be written as

$$u(x,y,t) = \pm\sqrt{\frac{(\alpha_2 d^2 - \beta_2 c^2 + 1)(\omega + p^2 n + n\alpha_1 q^2)}{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2}} \times \left[1 \mp 2\frac{e^{-\ell_1(x+cy+dt+\xi_0)}}{C_2 + e^{-\ell_1(x+cy+dt+\xi_0)}}\right]. \times \exp[i(px + qy + \varepsilon t)], \tag{29}$$

$$v(x,y,t) = \frac{\gamma_2(\omega + p^2 n + n\alpha_1 q^2)}{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2} \times \left[1 \mp 2\frac{e^{-\ell_1(x+cy+dt+\xi_0)}}{C_2 + e^{-\ell_1(x+cy+dt+\xi_0)}}\right]^2. \tag{30}$$

Therefore, the exact solution (29) and (30) turns out to the following solitary wave solution:

$$u_1(x,y,t) = \pm\sqrt{\frac{(\alpha_2 d^2 - \beta_2 c^2 + 1)(\omega + p^2 n + n\alpha_1 q^2)}{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2}} \times \tanh\left(\frac{1}{2}\sqrt{\frac{\omega + p^2 n + n\alpha_1 q^2}{2(n + n\alpha_1 c^2)}}(x + cy + dt + \xi_0)\right) \times \exp[i(px + qy + \varepsilon t)], \tag{31}$$

$$v_1(x,y,t) = \frac{\gamma_2(\omega + p^2 n + n\alpha_1 q^2)}{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2} \times \tanh^2\left(\frac{1}{2}\sqrt{\frac{\omega + p^2 n + n\alpha_1 q^2}{2(n + n\alpha_1 c^2)}}(x + cy + dt + \xi_0)\right), \tag{32}$$

when $C_2 = 1$.

On the other hand, if we set $C_2 = -1$, solution (29) and (30) reduces to the solitary wave solution as follows:

$$u_2(x,y,t) = \pm\sqrt{\frac{(\alpha_2 d^2 - \beta_2 c^2 + 1)(\omega + p^2 n + n\alpha_1 q^2)}{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2}} \times \coth\left(\frac{1}{2}\sqrt{\frac{\omega + p^2 n + n\alpha_1 q^2}{2(n + n\alpha_1 c^2)}}(x + cy + dt + \xi_0)\right) \times \exp[i(px + qy + \varepsilon t)]. \tag{33}$$

$$v_2(x,y,t) = \frac{\gamma_2(\omega + p^2 n + n\alpha_1 q^2)}{-\beta_1(\alpha_2 d^2 - \beta_2 c^2 + 1) + \gamma_1\gamma_2} \times \coth^2\left(\frac{1}{2}\sqrt{\frac{\omega + p^2 n + n\alpha_1 q^2}{2(n + n\alpha_1 c^2)}}(x + cy + dt + \xi_0)\right). \tag{34}$$

Comparing these results with the results obtained in [34–36], it can be seen that the solutions here are new.

Example 2. A (2 + 1)-dimensional nonlinear model generated by the Jaulent–Miodek hierarchy [39] is

$$u_t = -\frac{1}{4}\left(u_{xx} - 2u^3\right)_x - \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1}u_{yy} + u_x\partial_x^{-1}u_y\right), \tag{35}$$

where ∂_x^{-1} is the inverse of ∂_x with $\partial_x\partial_x^{-1} = \partial_x^{-1}\partial_x = 1$ and $(\partial_x^{-1}f)(x) = \int_{-\infty}^{x} f(t)dt$, under the decaying condition at infinity.

This model was studied by Liu and Yan in [40] using qualitative analysis method. Wazwaz in [39] has derived the multiple kink solutions and multiple singular kink solutions for this model. In addition, in [41],Taha and Noorani have obtained the exact solutions of (35) by using the $(G'/G)-$ expansion method.

Now, we conduct our analysis to this model to extract its exact solutions. For this purpose, we introduce the potential

$$u(x,y,t) = v_x(x,y,t), \tag{36}$$

that transforms (35) to

$$v_{xt} + \frac{1}{4}v_{xxxx} - \frac{3}{2}v_x^2 v_{xx} + \frac{3}{16}v_{yy} + \frac{3}{4}v_{xx}v_y = 0 \tag{37}$$

Let $v(x,y,t) = v(\xi)$, $\xi = x + y + ct + \xi_0$, where c is the wave speed and ξ_0 is an arbitrary constant. Putting the wave variable $\xi = x + y + ct + \xi_0$ into (37) yields

$$c\,v'' + \frac{1}{4}v'''' - \frac{3}{2}(v')^2 v'' + \frac{3}{16}v'' + \frac{3}{4}v''v' = 0. \tag{38}$$

Integrating (38) with respect to ξ once and considering the constant of integration to be zero, we find

$$16\,c\,v' + 4v''' - 8(v')^3 + 3v' + 6(v')^2 = 0 \tag{39}$$

Assuming that

$$v' = w, \tag{40}$$

and substituting (40) into (39), we obtain

$$16\,c\,w + 4w'' - 8w^3 + 3w + 6w^2 = 0 \tag{41}$$

As in the previous example, we assume the following solution formula:

$$w\,(\xi) = A_0 + A_1\left(\frac{\Psi'(\xi)}{\Psi(\xi)}\right), \tag{42}$$

where A_0 and A_1 are constants to be determined and $A_1 \neq 0$. Thus we obtain the following system of equations

$$\Psi^0: \ 16cA_0 - 8A_0^3 + 3A_0 + 6A_0^2 = 0, \tag{43}$$

$$\Psi^{-1}: \ 16cA_1\Psi' + 4A_1\Psi''' - 24A_0^2A_1\Psi' + 3A_1\Psi' + 12A_0A_1\Psi' = 0, \tag{44}$$

$$\Psi^{-2}: \ -12A_1\Psi'\Psi'' - 24A_0A_1^2\Psi'^2 + 6A_1^3\Psi'^2 = 0, \tag{45}$$

$$\Psi^{-3}: \ 8A_1\Psi'^3 - 8A_1^3\Psi'^3 = 0 \tag{46}$$

From (43) and (46), we find

$$A_0 = 0, \quad 8A_0^2 - 6A_0 - (16c+3) = 0, \quad A_1 = \pm 1, \tag{47}$$

From (44) and (45), we get

$$\Psi'''/\Psi'' = -\ell_2, \tag{48}$$

where

$$\ell_2 = \left(16\,c - 24\,A_0^2 + 3 + 12\,A_0\right)/\left(2\,A_1^3 - 8\,A_0\,A_1^2\right). \tag{49}$$

Integrating (48), we obtain

$$\Psi'' = C_1 e^{-\ell_2\xi}, \tag{50}$$

where C_1 is a constant of integration.

From (45) and (50), we obtain

$$\Psi' = m_2 e^{-\ell_2\xi} \tag{51}$$

where

$$m_2 = 2C_1/\left(A_1^2 - 4A_1A_0\right). \tag{52}$$

Integrating (51) with respect to ξ, we get

$$\Psi(\xi) = C_2 - (m_2/\ell_2)e^{-\ell_2\xi}, \tag{53}$$

where C_2 is the constant of integration.

Substituting the values of Ψ and Ψ' into (42), we obtain

$$w(\xi) = A_0 + A_1\left((m_2e^{-\ell_2\xi})/\left(C_2 - \frac{m_2}{\ell_2}e^{-\ell_2\xi}\right)\right). \tag{54}$$

Case 1. When $A_0 = 0$, $A_1 \neq 0$, and $\Psi' \neq 0$, then the exact solution of (35) has the following formula:

$$w(\xi) = \left(\frac{3+16c}{2}\right)\left(\frac{e^{-\ell_2\xi}}{C_2 - e^{-\ell_2\xi}}\right). \tag{55}$$

If $C_2 = \pm 1$, the exact solutions (55) are transformed to the following formulas:

$$w_1(\xi) = \left(\frac{3+16c}{4}\right)\left(1 - \coth\left(\pm\frac{3+16c}{4}\xi\right)\right), \tag{56}$$

$$w_2(\xi) = -\left(\frac{3+16c}{4}\right)\left(1 - \tanh\left(\pm\frac{3+16c}{4}\xi\right)\right), \tag{57}$$

respectively.

But $w = dv/d\xi$, then $v = \int w d\xi$, and $u = \partial v/\partial x$, we find the exact solutions of (35) as

$$u_1(x,y,t) = \left(\frac{3+16c}{4}\right) \times \left[1 - \coth\left(\left(\frac{3+16c}{4}\right)(x+y+ct+\xi_0)\right)\right] \tag{58}$$

$$u_2(x,y,t) = \left(\frac{3+16c}{4}\right) \times \left[-1 \pm \tanh\left(\left(\frac{3+16c}{4}\right)(x+y+ct+\xi_0)\right)\right] \tag{59}$$

Case 2. When $A_0 \neq 0$, $A_1 \neq 0$, and $\Psi' \neq 0$, then we deduce from (44) and (42) that

$$\left(16c - 24A_0^2 + 3 + 12A_0\right)\Psi' + 4\Psi''' = 0 \tag{60}$$

$$(4\,A_0 - A_1)A_1\Psi' + 2\Psi'' = 0 \tag{61}$$

With the help of (47), we can simplify (60) to take the form

$$(3 - 8\,A_0)A_0\Psi' + 2\Psi'' = 0. \tag{62}$$

From (61) and (62), we have

$$\Psi'''/\Psi'' = \ell_3 \tag{63}$$

where $\ell_3 = (8A_0^2 - 3A_0)/[A_1(A_1 - 4A_0)]$. Integrating (63) and using (61), we deduce that

$$\Psi' = m_3 e^{\ell_3\xi} \tag{64}$$

where $m_3 = 2C_1/[A_1(A_1 - 4A_0)]$, and then

$$\Psi = C_2 + (m_3/\ell_3)e^{\ell_3\xi}, \tag{65}$$

where C_1 and C_2 are constants of integration. From (42), (64) and (65), we have the exact solution

$$w(\xi) = A_0\left[1 + \left(\frac{8A_0 - 3}{A_1 - 4A_0}\right)\frac{e^{\ell_3\xi}}{C_2 + e^{\ell_3\xi}}\right], \tag{66}$$

where $A_0 = \frac{3\pm\sqrt{9+8(3+16c)}}{8}$.

If $C_2 = \pm 1$, the solution (66) turns out to the following formulas:

$$w_3(\xi) = A_0\left[1 + \left(\frac{8A_0 - 3}{2A_1 - 8A_0}\right)\left\{1 + \tanh\left(\frac{A_0(8A_0-3)}{(2A_1^2 - 8A_0A_1)}\xi\right)\right\}\right] \tag{67}$$

$$w_4(\xi) = A_0\left[1 + \left(\frac{8A_0 - 3}{2A_1 - 8A_0}\right)\left\{1 + \coth\left(\frac{A_0(8A_0-3)}{(2A_1^2 - 8A_0A_1)}\xi\right)\right\}\right] \tag{68}$$

But $w = dv/d\xi$, then $v = \int w d\xi$, and $u = \partial v/\partial x$, we then find the exact solutions of (35) and can be written as

$$u_3(x,y,t) = A_0 + A_0\left(\frac{8A_0 - 3}{\pm 2 - 8A_0}\right) \times \left\{1 + \tanh\left(\frac{A_0(8A_0-3)}{(2 \mp 8A_0)}(x+y+ct+\xi_0)\right)\right\} \tag{69}$$

(66)

$$u_4(x,y,t) = A_0 + A_0\left(\frac{8A_0 - 3}{\pm 2 - 8A_0}\right) \times \left\{1 + \coth\left(\frac{A_0(8A_0-3)}{(2 \mp 8A_0)}(x+y+ct+\xi_0)\right)\right\} \tag{70}$$

respectively, where $A_0 = \frac{3\pm\sqrt{33+128c}}{8}$, as given in (47).

Eqs. (58), (59) and (69), (70), are new types of exact traveling wave solutions to the (2 + 1)-dimensional nonlinear model generated by the Jaulent–Miodek hierarchy (35). It could not be obtained by the methods presented in [39–41].

Example 3. A generalized KdV equation with two power nonlinearities [42] is

$$u_t + (au^n - bu^{2n})u_x + u_{xxx} = 0 \tag{71}$$

This equation describes the propagation of nonlinear long acoustic-type waves.

If the amplitude is not supposed to be small, (71) serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves along a characteristic direction [42,43].

The wave variable $\xi = x - ct + \xi_0$, and integrating once, (71) is converted to the ODE

$$-cu + \frac{a}{n+1}u^{n+1} - \frac{b}{2n+1}u^{2n+1} + u'' = 0, \tag{72}$$

Balancing u^{2n+1} with u'' in (72), we find that $M = 1/n$.

We use the following transformation:

$$u = v^{1/n} \tag{73}$$

and using (73) into (72) gives

$$-cn^2(2n+1)(n+1)v^2 + an^2(2n+1)v^3 - bn^2(n+1)v^4 + n(2n+1)(n+1)vv'' + (1-n^2)(2n+1)v'^2 = 0. \tag{74}$$

Balancing vv'' with v^4 gives $M = 1$. The modified simple equation method admits the use of the finite expansion

$$v(\xi) = A_0 + A_1\left(\frac{\Psi'(\xi)}{\Psi(\xi)}\right), \tag{75}$$

where A_0 and A_1 are constants to be determined and $A_1 \neq 0$.

With similar procedures made in previous examples, we have

$$\Psi^0: \ -cn^2(2n+1)(n+1)A_0 + an^2(2n+1)A_0^3 - bn^2(n+1)A_0^4 = 0 \tag{76}$$

$$\Psi^{-1}: \ -2A_0A_1cn^2(2n+1)(n+1)\Psi' + 3A_0^2A_1an^2(2n+1)\Psi' - 4A_0^3A_1bn^2(n+1)\Psi' + nA_0A_1(2n+1)(n+1)\Psi''' = 0 \tag{77}$$

$$\Psi^{-2}: \ -cn^2(2n+1)(n+1)A_1^2\Psi'^2 + 3A_0A_1^2an^2(2n+1), \\ \Psi'^2 - 6A_0^2A_1^2bn^2(n+1), \\ \Psi'^2 - 3A_0A_1n(2n+1)(n+1)\Psi'\Psi'' + n(2n+1)(n+1)A_1^2\Psi'\Psi''' + (1-n^2)(2n+1)A_1^2\Psi''^2 = 0 \tag{78}$$

$$\Psi^{-3}: \ an^2(2n+1)A_1^3\Psi'^3 - 4A_0A_1^3bn^2(n+1)\Psi'^3 + 2A_0A_1n(2n+1)(n+1)\Psi'^3 - 3A_1^2n(2n+1)(n+1)\Psi'^2\Psi'' - 2A_1^2(1-n^2)(2n+1)\Psi''\Psi'^2 = 0 \tag{79}$$

$$\Psi^{-4}: \ -bn^2(n+1)A_1^4\Psi'^4 + 2A_1^2n(2n+1)(n+1)\Psi'^4 + A_1^2(1-n^2)(2n+1)\Psi'^4 = 0 \tag{80}$$

Thus from (76) and (80), we get

$$A_0 = 0, bn^2(n+1)A_0^2 - an^2(2n+1)A_0 + cn^2(2n+1)(n+1) = 0, \\ A_1 = \pm\frac{1}{n}\sqrt{\frac{2n^2+3n+1}{b}} \tag{81}$$

Let us consider here the case $A_0 = 0$, $A_1 = \pm\frac{1}{n}\sqrt{\frac{2n^2+3n+1}{b}}$, and only left the case of $A_0 \neq 0$, for the reader.

Therefore from (77), we deduce that

$$2cn\Psi' - \Psi''' = 0 \tag{82}$$

The general solution of this equation is given by

$$\Psi(\xi) = C_0 + C_1e^{r_1\xi} + C_2e^{r_2\xi}, \tag{83}$$

where $r_{1,2} = \pm\sqrt{2cn}$ and $C_i (i = 0, 1, 2)$, are arbitrary constants.

Consequently, the exact solutions of (71) have the form of

$$u(x,t) = \left[\pm\frac{1}{n}\sqrt{\frac{2n^2+3n+1}{b}} \times \left(\frac{C_1r_1e^{r_1(x-ct+\xi_0)} + C_2r_2e^{r_2(x-ct+\xi_0)}}{C_0 + C_1e^{r_1(x-ct+\xi_0)} + C_2e^{r_2(x-ct+\xi_0)}}\right)\right]^{\frac{1}{n}} \tag{84}$$

Comparing the solutions (84) with those obtained in [42], we see that our solutions are new.

Remark. For $n = 1, a = 6, b = \pm 6$, the solutions (84) are the solutions for the two models for the Gardner equation:

$$u_t + 6uu_x \pm 6u^2u_x + u_{xxx} = 0,$$

that describes internal solitary waves in shallow seas. These models are the positive and negative Gardner equation depending on the sign of the cubic nonlinear term.

4. Conclusion

In this paper, we have successfully implemented and applied the modified simple equation (MSE) method to find the exact and solitary wave solutions for some class of nonlinear PDEs, via a system of nonlinear PDEs, a (2 + 1)-dimensional nonlinear model generated by the Jaulent–Miodek hierarchy, and a generalized KdV equation with two power nonlinearities. The method offers solutions with free parameters that may be important for explaining some physical phenomena in mathematical physics. The solitary wave solutions are obtained by setting appropriate values to the free parameters. The MSE method has been achieved without using any symbolic computations, since the method is simple, effective and can be applied to many other nonlinear PDEs, especially for equations with distinct power law nonlinearities that can be done in a future work.

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