Journal of the Egyptian Mathematical Society (2016) **24**, 204–209

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Simple equation method for nonlinear partial differential equations and its applications

Taher A. Nofal*

Department of Mathematics and Statistics, Faculty of Science, Taif University, Taif, Saudi Arabia
Mathematics Department, Faculty of Science, Minia University, Minia, Egypt

Received 22 March 2015; revised 17 May 2015; accepted 19 May 2015
Available online 26 June 2015

Keywords

Simple equation method;
Exact solutions;
Kodomtsev–Petviashvili equation;
Bernoulli equation;
Riccati equation

Abstract In this article, we focus on the exact solution of the some nonlinear partial differential equations (NLPDEs) such as, Kodomtsev–Petviashvili (KP) equation, the (2 + 1)-dimensional breaking soliton equation and the modified generalized Vakhnenko equation by using the simple equation method. In the simple equation method the trial condition is the Bernoulli equation or the Riccati equation. It has been shown that the method provides a powerful mathematical tool for solving nonlinear wave equations in mathematical physics and engineering problems.

2000 Mathematics Subject Classification: 34A36; 34L05; 47A10; 47A70

1. Introduction

The nonlinear partial differential equations (NLPDEs) play an important role to study many problems in physics and geometry. The effort in finding exact solutions to nonlinear equations is important for the understanding of most nonlinear physical phenomena [1,2]. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optimal fiber, biology, oceanology, solid state physics, chemical physics and geometry. In recent years, the powerful and efficient methods to find analytic solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists, such as the inverse scattering transform method [3], Backland transformation [4], Darboux transformation [5], Hirota bilinear method [6], variable separation approach [7], various tanh method [11,8–10], homogenous balance method [12], similarity reductions method [13,14], (G'/G)-expansion method [15,16], sine–cosine method [17], the exp-function method [18], the sub-ODE method [19], and so on.

In this paper, we obtain the exact solution of Kodomtsev–Petviashvili (KP) equation, the $(2+1)$-dimensional breaking soliton equation and the modified generalized Vakhnenko equation by using the simple equation method. The simple equation method is a very powerful mathematical technique for finding exact solution of nonlinear ordinary differential equations. It has been developed by Kadreyshov [20,21] and

* Address: Department of Mathematics and Statistics, Faculty of Science, Taif University, Taif, Saudi Arabia.
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.05.006

used successfully by many authors for finding exact solution of ODEs in mathematical physics [22,23]. In Section 2, we give a brief algorithm for the simple equation method. In Section 3, we apply this method to KP equation, the (2+1)-dimensional breaking soliton equation and the modified generalized Vakhnenko equation. We give the conclusion in Section 4.

2. Algorithm of the simple equation method

In this section we will describe a direct method namely simple equation method for finding the traveling wave solution of nonlinear evolution equations. Suppose that the nonlinear partial equation, in two independent variables x and t is given by:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0, \tag{2.1}$$

where $u(x, t)$ is an unknown function, P is a polynomial of $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of this method.

Step (1): Combining the independent variables x and t into one variable $\xi = x - ct$, we suppose that

$$u(x, t) = u(\xi), \quad \xi = x - ct. \tag{2.2}$$

The traveling wave transformation Eq. (2.2) permits us to reduce Eq. (2.1) to the following ordinary differential equation (ODE)

$$Q(u, u', u'', \dots) = 0 \tag{2.3}$$

where Q is a polynomial of $u(\xi)$ and its derivatives, where $u'(\xi) = \frac{du}{d\xi}$, $u''(\xi) = \frac{d^2u}{d\xi^2}$ and so on.

Step (2): We seek the solution of Eq. (2.2) in the following form:

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi). \tag{2.4}$$

which $a_i (i = 0, 1, 2, \dots, n)$ are constants to be determined later and $F(\xi)$ are the functions that satisfy the simple equations (ordinary differential equations). The simple equation has two properties, first it is lesser order than Eq. (2.2), second we know the general solution of the simple equation. In this paper, we shall use as simple equations, the Bernoulli and Riccati equations which are well known nonlinear ordinary differential equations and their solutions can be expressed by elementary functions. For the Bernoulli equation

$$F'(\xi) = cF(\xi) + dF^2(\xi). \tag{2.5}$$

Step (3): The balance number n can be determined by balancing the highest order derivative and nonlinear terms in Eq. (2.2).

Step (4): We discuss the general solutions of the simple Eq. (2.5) as following:

$$F(\xi) = \frac{c \exp[c(\xi + \xi_0)]}{1 - d \exp[c(\xi + \xi_0)]}. \tag{2.6}$$

For case $d < 0, c > 0$, here ξ_0 is a constant of the integration, and

$$F(\xi) = -\frac{c \exp[c(\xi + \xi_0)]}{1 + d \exp[c(\xi + \xi_0)]}. \tag{2.7}$$

For the Riccati equation

$$F'(\xi) = \alpha F^2(\xi) + \beta. \tag{2.8}$$

Eq. (2.8) admits the following exact solutions [23],

$$F(\xi) = -\frac{\sqrt{-\alpha\beta}}{\alpha} \tanh\left(\sqrt{-\alpha\beta}\xi - \frac{\nu \ln(\xi_0)}{2}\right), \ \xi_0 > 0, \ \nu = \pm 1, \tag{2.9}$$

where $\alpha\beta < 0$ and

$$F(\xi) = \frac{\sqrt{\alpha\beta}}{\alpha} \tan\left(\sqrt{\alpha\beta}(\xi + \xi_0)\right), \ \xi_0 \text{ is a constant}, \tag{2.10}$$

where $\alpha\beta > 0$.

Remark 1. (i) When $c = \delta, d = -1$ the Eq. (2.5) has another form of Bernoulli equation

$$F'(\xi) = \delta F(\xi) - F^2(\xi), \tag{2.11}$$

which has the exact solutions when $\delta > 0$,

$$F(\xi) = \frac{\delta}{2}\left[1 + \tanh\left(\frac{\delta}{2}(\xi + \xi_0)\right)\right], \tag{2.12}$$

and when $\delta < 0$,

$$F(\xi) = \frac{\delta}{2}\left[1 - \tanh\left(\frac{\delta}{2}(\xi + \xi_0)\right)\right], \tag{2.13}$$

(ii) When $c = 1, d = -1$ the Eq. (2.5) has another form of Riccati equation [20,21]

$$F'(\xi) - F(\xi) + F^2(\xi) = 0, \tag{2.14}$$

which has the logistic function as the exact solutions

$$F(\xi) = \frac{1}{1 + e^{-\delta}}. \tag{2.15}$$

The logistic Eq. (2.15) can be presented in the hyperbolic tangent function according to the relation

$$\frac{1}{1 + e^{-\xi}} = \frac{1}{2}\left[1 + \tanh\left(\frac{\xi}{2}\right)\right]. \tag{2.16}$$

3. Application

3.1. Kodomtsev–Petviashvili (KP) equation

The Kodomtsev–Petviashvili (KP) equation

$$u_{xt} - 6u\, u_{xx} - 6(u_x)^2 + u_{xxxx} + 3\delta^2 u_{yy} = 0 \tag{3.1}$$

or

$$(u_t - 6u\, u_x + u_{xxx})_x + 3\delta^2 u_{yy} = 0$$

is a two-dimensional generalization of the Kdv equation. Kodomtsev and Petviashvili (1970) first introduced this equation to describe slowly varying nonlinear waves in a dispersive medium [24]. Eq. (3.1) with $\delta^2 = 1$ arises in the study of weakly nonlinear dispersive waves in plasmas and also in the modulation of weakly nonlinear long water waves [3], which travel nearly in one dimension (that is, nearly in a vertical plane). The equation with $\delta^2 = -1$ arises in acoustics and admits unstable soliton solutions, whereas for $\delta^2 = 1$ the solutions are stable.

Suppose traveling wave transformation equation is

$$u(\xi) = u(x, y, t), \ \ \xi = x + y - wt. \tag{3.2}$$

The traveling wave transformation (3.2) permits us to reduce (3.1) into the following ODE

$$(-wu' - 6uu' + u''')' + 3\delta^2 u'' = 0. \tag{3.3}$$

Integrating Eq. (3.3) twice with respect to ξ, setting the integration constant to zero, we have

$$u'' + (3\delta^2 - w)u - 3u^2 = 0. \tag{3.4}$$

with balancing according procedure that be described, the balancing number $n$is a positive integer which can be determined by balancing the highest order derivative terms u'' with the highest power nonlinear terms u^2 in Eq. (3.4), i.e., $n + 2 = 2n$, hence $n = 2$. Therefore the solution of Eq. (3.4) can be expressed as follows:

$$u = \sum_{i=0}^{2} a_i (F(\xi))^i = a_0 + a_1 F + a_2 F^2. \tag{3.5}$$

where F satisfies Eq. (2.5), consequently, we have:

$$u' = a_1 cF + (2a_2 c + da_1)F^2 + 2a_2 dF^3, \tag{3.6}$$

$$u'' = a_1 c^2 F + (4a_2 c^2 + 3a_1 cd)F^2 + (10a_2 dc + 2a_1 d^2)F^3 + 6a_2 d^2 F^4,$$

$$u^2 = a_0^2 + 2a_0 a_1 F + (2a_0 a_2 + a_1^2)F^2 + 2a_1 a_2 F^3 + a_2^2 F^4.$$

Substituting Eqs. (3.5) and (3.6) into Eq. (3.4) and then equating the coefficient of F^i to zero, where $i \geq 0$, we get

$$6a_2 d^2 - 3a_2^2 = 0, \tag{3.7}$$

$$10a_2 dc + 2a_1 d^2 - 6a_1 a_2 = 0,$$
$$4a_2 c^2 + 3a_1 cd + a_2(3\delta^2 - w) - 6a_0 a_2 - 3a_1^2 = 0,$$
$$a_1(c^2 + (3\delta^2 - w) - 6a_0) = 0,$$
$$a_0(3\delta^2 - w) - 3a_0^2 = 0.$$

Solving Eqs. (3.7), we find that solution of Eq. (3.1) exists only in the following two cases:

Case (1):

$$a_0 = \frac{c^2}{3}, \ \ a_1 = 2dc, \ \ a_2 = 2d^2, \ \ w = 3\delta^2 - c^2, \ \ cd \neq 0. \tag{3.8}$$

Case (2):

$$a_0 = 0, \ \ a_1 = 2dc, \ \ a_2 = 2d^2, \ \ w = 3\delta^2 + c^2, \ \ cd \neq 0. \tag{3.9}$$

When $d < 0$ and $c > 0$ the solution of Eq. (3.1) with using case (1) is given by

$$u_1(x, y, t) = \frac{c^2}{3} + \frac{2dc^2 \exp\left[c\left((x + y) + (c^2 - 3\delta^2)t\right)\right]}{\left(1 - d\exp\left[c\left((x + y) + (c^2 - 3\delta^2)t\right)\right]\right)^2}. \tag{3.10}$$

Also solution of Eq. (3.1) with using case (2) is given by

$$u_2(x, y, t) = \frac{2dc^2 \exp\left[c\left((x + y) + (c^2 + 3\delta^2)t\right)\right]}{\left(1 - d\exp\left[c\left((x + y) + (c^2 + 3\delta^2)t\right)\right]\right)^2}. \tag{3.11}$$

When $d > 0$ and $c < 0$ the solution of Eq. (3.1) with using case (1) is given by

$$u_3(x, y, t) = \frac{c^2}{3} + \frac{2dc^2 \exp\left[c\left((x + y) + (c^2 - 3\delta^2)t\right)\right]}{\left(1 + d\exp\left[c\left((x + y) + (c^2 - 3\delta^2)t\right)\right]\right)^2}. \tag{3.12}$$

Also solution of Eq. (3.1) with using case (2) is given by

$$u_4(x, y, t) = \frac{2dc^2 \exp\left[c\left((x + y) + (c^2 + 3\delta^2)t\right)\right]}{\left(1 + d\exp\left[c\left((x + y) + (c^2 + 3\delta^2)t\right)\right]\right)^2}. \tag{3.13}$$

3.2. Application of simple equation method to the $(2 + 1)$-dimensional breaking soliton equation

In this section, we use the proposed method to find the exact solutions of following $(2 + 1)$-dimensional breaking soliton equations in [25],

$$u_t + \alpha u_{xxy} + 4(uv)_x = 0, \tag{3.14}$$

$$u_y = v_x, \tag{3.15}$$

where α is an arbitrary constant. The solutions of Eqs. (3.14) and (3.15) have been investigated using different methods, see for example [25–27]. The system of Eqs. (3.14) and (3.15) has not been solved elsewhere using the functional simple equation method. Let us now solve Eqs. (3.14) and (3.15) using the proposed method of Section 2. To this end, we apply the wave transformation $u(x, y, t) = u(\xi)$, $\xi = x + y - ct$ to reduce Eqs. (3.14) and (3.15) into the following ODE:

$$-cu' + \alpha u''' + 4(u^2)' = 0, \tag{3.16}$$

where $u = v$. Integrating Eq. (3.16) with respect to ξ, we get

$$u'' - \frac{cu}{\alpha} + \frac{4u^2}{\alpha} = 0, \ \ \alpha \neq 0 \tag{3.17}$$

with zero constant of integration. Now balancing the highest order derivative u'' and non-linear term u^2, we get $n = 2$. Now for $n = 2$, the solutions of Eq. (3.16) have the form:

$$u(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi), \tag{3.18}$$

where a_0, a_1 and a_2 are constants to be determined such that $a_2 \neq 0$, while c, d are aribtrary constants.

Substituting Eq. (3.18) into (3.17) and setting the coefficients of $F(\xi)$ to be zero, where $\xi \geq 0$, we get

$$\frac{4a_0^2}{\alpha} - \frac{ca_0}{\alpha} = 0, \tag{3.19}$$

$$-\frac{ca_1}{\alpha} + a_1 c^2 + \frac{8a_0 a_1}{\alpha} = 0,$$

$$3a_1 cd - \frac{ca_2}{\alpha} + 4a_2 c^2 + \frac{8a_0 a_2}{\alpha} + \frac{4a_1^2}{\alpha} = 0,$$

$$2a_1 d^2 + 10a_2 dc + \frac{8a_1 a_2}{\alpha} = 0,$$

$$6a_2 d^2 + \frac{4a_2^2}{\alpha} = 0.$$

Solving Eq. (3.19), we find that solution of Eq. (3.17) exists only in the following two cases:

Case (1):

$$a_0 = \frac{-1}{4\alpha},\ a_1 = \frac{3}{2}d,\ a_2 = \frac{-3}{2}d^2\alpha,\ c = \frac{-1}{\alpha}. \tag{3.20}$$

Case (2):

$$a_0 = 0,\ a_1 = \frac{-3}{2}d,\ a_2 = \frac{-3}{2}d^2\alpha,\ c = \frac{1}{\alpha}. \tag{3.21}$$

When $d < 0$ and $c > 0$ the solution of Eq. (3.17) with using case (1) is given by

$$u_1(x,y,t) = \frac{-1}{4\alpha} + \frac{3d\exp[\frac{-1}{\alpha}((x+y)+\frac{1}{\alpha}t)]}{2\alpha(1-d\exp[\frac{-1}{\alpha}((x+y)+\frac{1}{\alpha}t)])^2}. \tag{3.22}$$

Also solution of Eq. (3.17) with using case (2) is given by

$$u_2(x,y,t) = \frac{-3d\exp[\frac{1}{\alpha}((x+y)-\frac{1}{\alpha}t)]}{2\alpha(1-d\exp[\frac{1}{\alpha}((x+y)-\frac{1}{\alpha}t)])^2}. \tag{3.23}$$

When $d > 0$ and $c < 0$ the solution of Eq. (3.17) with using case (1) is given by

$$u_3(x,y,t) = \frac{-1}{4\alpha} + \frac{3d\exp[\frac{-1}{\alpha}((x+y)+\frac{1}{\alpha}t)]}{2\alpha(1+d\exp[\frac{-1}{\alpha}((x+y)+\frac{1}{\alpha}t)])^2}. \tag{3.24}$$

Also solution of Eq. (3.17) with using case (2) is given by

$$u_4(x,y,t) = \frac{3d\exp[\frac{1}{\alpha}((x+y)-\frac{1}{\alpha}t)]}{2(1+d\exp[\frac{1}{\alpha}((x+y)-\frac{1}{\alpha}t)])^2}. \tag{3.25}$$

3.3. Modified generalized Vakhnenko equation

Consider a modified generalized Vakhnenko equation (mGVE) [23,28]:

$$\frac{\partial}{\partial x}(\varphi^2 u + \frac{1}{2}pu^2 + \beta u) + q\varphi u = 0, \tag{3.26}$$

$$\varphi = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},$$

where p, q, β are arbitrary non-zero constants. Eq. (3.26) can be traced to the well-known Vakhnenko equation (VE) which was initially presented to model high-frequent wave motion in a relaxing medium [28]. Recently, Eq. (3.26) has been discussed using the (G'/G)-expansion method [15] and using the auxiliary equation method [23]. To calculate the exact solutions for Eq. (3.26) a sensible step is to transform variables. We

$$x = T + \int_{-\infty}^{x} U(X', T)dX' + x_0,\ t = X, \tag{3.27}$$

where $u(x,t) = U(X,T)$ and x_0 is a constant. We introduce a new function W defined by

$$W(X,T) = \int_{-\infty}^{x} U(X', T)dX'. \tag{3.28}$$

Then

$$W_X(X,T) = U(X,T),\ W_T(X,T) = \int_{-\infty}^{x} U_T(X', T)dX' \tag{3.29}$$

It is easy to see that

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial x}\frac{\partial x}{\partial T} + \frac{\partial}{\partial t}\frac{\partial t}{\partial T},\ \frac{\partial}{\partial X} = \frac{\partial}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial}{\partial t}\frac{\partial t}{\partial X}. \tag{3.30}$$

From Eqs. (3.27) and (3.30), we have

$$\frac{\partial}{\partial T} = (1+W_T)\frac{\partial}{\partial x},\ \frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \tag{3.31}$$

and hence $\varphi u = U_X, \varphi^2 u = U_{XX}$.

Now, Eq. (3.26) reduces to

$$W_{XXXT} + pW_X W_{XT} + q(1+W_T)W_{XX} + \beta W_{XT} = 0. \tag{3.32}$$

Assume that $W(X,T) = W(\xi)$, where $\xi = k(x - vt)$, then Eq. (3.32) reduces to the equation

$$k^2 V W''' + \frac{1}{2}(p+q)kV(W')^2 + (\beta V - q)W' = 0 \tag{3.33}$$

with zero constants of integration. Setting $r = 1$ and $W' = v$, we have $W(\xi) = \int v(\xi)d\xi + d$, where $v(\xi)$ satisfies the following ODE:

$$k^2 V v'' + \frac{1}{2}(p+q)kVv^2 + (\beta V - q)v = 0. \tag{3.34}$$

Balancing v'' with v^2 in Eq. (3.34) we get $n = 2$. Consequently, the exact solution of Eq. (3.34) can be written in the following form:

$$v(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi), \tag{3.35}$$

where $F(\xi)$ satisfies Eq. (2.5). Substituting (3.35) in (3.34) and collecting all terms with the same powers of $F^i, i = 0,1,2,3,4$ together, the left hand side of Eq. (3.34) is converted into a polynomial in F^i. Setting each coefficient of this polynomial to be zero, we get the following algebraic equations

$$(\beta V - q)a_0 + \frac{1}{2}(p+q)kVa_0^2 = 0,$$

$$(k^2V)a_1c^2 + 2a_0a_1\left(\frac{1}{2}\right)(p+q)kV + (\beta V - q)a_1 = 0,$$

$$4c^2(k^2V)a_2 + 3cdk^2V + 2\left(\frac{1}{2}\right)(p+q)kVa_0a_2$$
$$+\frac{1}{2}(p+q)kVa_1^2 + (\beta V - q)a_2 = 0,$$

$$10cdk^2Va_2 + 2d^2k^2Va_1 + 2\left(\frac{1}{2}\right)(p+q)kVa_1a_2 = 0,$$

$$6d^2k^2Va_2 + \frac{1}{2}(p+q)kVa_2^2 = 0.$$

Solving the above algebraic equations, we have the results:

Case (1):

$$a_0 = \frac{-2k^2c}{p+q},\ a_1 = \frac{-12kcd}{p+q},\ a_2 = \frac{-12kd^2}{p+q},\ v = \frac{q}{\beta - k^2c^2}. \tag{3.36}$$

Case (2):

$$a_0 = 0,\ a_1 = \frac{-12kcd}{p+q},\ a_2 = \frac{-12kd^2}{p+q},\ v = \frac{q}{\beta + k^2c^2}. \tag{3.37}$$

When $d < 0$ and $c > 0$, the solution of Eq. (3.34) with using case (1) is given by

$$v_1(\xi) = \frac{-2k^2c}{p+q} - \frac{12kdc^2\exp[c\xi]}{(p+q)(1-d\exp[c\xi])^2}, \tag{3.38}$$

and consequently, we get

$$W_1(\xi) = -\frac{12kd^2c^2}{(p+q)}\int \frac{\exp[c\xi]}{(1-d\exp[c\xi])^2}d\xi - \frac{2k^2c}{p+q}\xi + d_1 = -\frac{12kcd}{p+q}\left(\frac{1}{1-d\exp[c\xi]}\right) - \frac{2k^2c}{p+q}\xi + d_1 \quad (3.39)$$

and now the solution in this case is the soliton solution

$$u_1(x,t) = W_{1x}(\xi) = \frac{12k^2d^2c\exp[c\xi]}{(p+q)(1-d\exp[c\xi])^2} - \frac{2k^3c}{p+q}. \quad (3.40)$$

Also solution of Eq. (3.34) with using case (2) is given by

$$v_2(\xi) = -\frac{12kdc^2\exp[2c\xi]}{(p+q)(1-d\exp[c\xi])^2}. \quad (3.41)$$

and consequently, we get

$$W_2(\xi) = -\frac{12kdc^2}{(p+q)}\int \frac{\exp[c\xi]}{(1-d\exp[c\xi])^2}d\xi + d_2 = -\frac{12kc}{p+q}\left(\frac{1}{1-d\exp[c\xi]}\right) + d_2 \quad (3.42)$$

and now the solution in this case is the soliton solution

$$u_2(x,t) = W_{2x}(\xi) = \frac{-12k^2c\exp[c\xi]}{(p+q)(1-d\exp[c\xi])^2}. \quad (3.43)$$

When $d > 0$ and $c < 0$ the solution of Eq. (3.34) with using case (1) is given by

$$v_3(\xi) = \frac{-2k^2c}{p+q} + \frac{12kdc^2\exp[2c\xi]}{(p+q)(1+d\exp[c\xi])^2}. \quad (3.44)$$

and

$$W_3(\xi) = -\frac{12kc}{p+q}\left(\frac{1}{1+d\exp[c\xi]}\right) - \frac{2k^2c}{p+q}\xi + d_3 \quad (3.45)$$

and

$$u_3(x,t) = W_{3x}(\xi) = \frac{12k^2dc\exp[c\xi]}{(p+q)(1+d\exp[c\xi])^2} - \frac{2k^3c}{p+q}. \quad (3.46)$$

Also solution of Eq. (3.34) with using case (2) is given by

$$v_4(\xi) = \frac{12kdc^2\exp[2c\xi]}{(p+q)(1+d\exp[c\xi])^2}. \quad (3.47)$$

and

$$W_4(\xi) = -\frac{12kc}{p+q}\left(\frac{1}{1+d\exp[c\xi]}\right) + d_4 \quad (3.48)$$

and

$$u_4(x,t) = W_{4x}(\xi) = \frac{12k^2dc\exp[c\xi]}{(p+q)(1+d\exp[c\xi])^2}. \quad (3.49)$$

4. Conclusions

In this paper, the simple equation method has been successfully used to obtain the exact solution of KP equation, the $(2+1)$-dimensional breaking soliton equation and the modified generalized Vakhnenko equation. As the simple equation, we have used the Bernoulli and Riccati equations. For the simple equation, we have obtained a balance equation. By means of balance equation, we obtained exact solutions of the studied class of nonlinear PDEs, we have also verified that solutions we have found are indeed solutions to the original equations. Finally, we point out of either integrable or non-integrable nonlinear coupled systems.

Acknowledgments

The authors would like to express thanks to the anonymous referees for their useful and valuable comments and suggestions that enhance their presentation of the paper.

References

[1] M. Duranda, D. Langevin, Physicochemical approach to the theory of foam drainage, Eur. Phys. J. E 7 (2002) 35–44.
[2] N.A. Kudryashov, On types nonlinear nonintegrable differential equations with exact solutions, Phys. Lett. A 155 (1991) 269–275.
[3] M.J. Ablowitz, P.A. Clarkson, Solitons Nonlinear Evolution Equation and Inverse Scattering, Cambridge University Press, New York, 1991.
[4] Gu Chaohao, Soliton Theory and Its Applications, Zhejiang Science and Technology Press, 1990.
[5] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer, 1991.
[6] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, 2004.
[7] S.Y. Lou, J.Z. Lu, Special solutions from the variable separation approach: the Davey–Stewartson equation, J. Phys. A: Math. Gen. 29 (1996) 4209–4215.
[8] B.R. Duffy, E.J. Parkes, Travelling solitary wave solutions to a seventh-order generalised KdV equation, Phys. Lett. A 214 (1996) 271–272.
[9] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212–218.
[10] Zhenya Yan, New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations, J. Phys. A: Math. Gen. 292 (2001) 100–106.
[11] Y. Chen, Y. Zheng, Generalized extended tanh-function method to construct new explicit solutions for the approximate equation for long water waves, Int. J. Mod. Phys. C 14 (4) (2003) 601–611.
[12] M.L. Wang, Y.B. Zhou, Z.B. Li, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A 216 (1996) 67–75.
[13] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Appl. Math. Sci., vol. 81, Springer, New York, 1989.
[14] P.J. Olver, Applications of Lie groups to Differential Equations, SpringerVerlag, New York, 1986.
[15] E.M.E. Zayed, K.A. Gepreel, The modified (G'/G)-expansion method and its applications to construct exact solutions for nonlinear PDEs, WSEAS Trans. Math. 10 (8) (2011) 270–278.
[16] K.A. Gepreel, Exact solutions for nonlinear PDEs with the variable coefficients in mathematical physics, J. Inform. Comput. Sci. 6 (1) (2011) 3–14.
[17] A.M. Wazwaz, A sine-cosine method for handlingnonlinear wave equations, Math. Comput. Modell. 40 (5–6) (2004) 499–508.
[18] X. Hua, The exponential function rational expansion method and exact solutions to nonlinear lattice equations system, Appl. Math. Comput. 217 (2010) 1561–1565.
[19] X.Z. Li, M.L. Wang, A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with higher order nonlinear terms, Phys. Lett. A 361 (1–2) (2007) 115–118.

[20] N.A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos, Solitons & Fractals 24 (5) (2005) 1217–1231.
[21] N.A. Kudryashov, N.B. Loguinova, Extended simplest equation method for nonlinear differential equations, Appl. Math. Comput. 205 (1) (2008) 396–402.
[22] N.K. Vitanov, Z.I. Dimitrova, H. Kantz, Modified method of simplest equation and its application to nonlinear PDEs, Appl. Math. Comput. 216 (2010) 2587–2595.
[23] Y.L. Ma, B.Q. Li, C. Wang, A series of abundant exact travelling wave solutions for a modified generalized Vakhnenko equation using auxiliary equation method, Appl. Math. Comput. 211 (1) (2009) 102–107.
[24] R.S. Johnson, Water waves and Korteweg–de Vries equations, J. Fluid Mech. 97 (4) (1980) 701–719.
[25] J.F. Zhang, J.P. Men, New localized coherent structures to the $(2+1)$-dimensional breaking soliton equation, Phys. Lett. A 321 (3) (2004) 173–178.
[26] Z. Xie, H.Q. Zhang, New soliton-like solutions for $(2+1)$-dimensional breaking soliton equation, Commun. Theor. Phys. 43 (3) (2005) 401–406.
[27] K.A. Gepreel, T.A. Nofal, A.A. Al-Thobaiti, Double soliton solutions for some nonlinear partial differential equations (PDEs) in mathematical physics, Int. J. Phys. Sci. 8 (2) (2013) 57–67.
[28] V.A. Vakhnenko, Solitons in a nonlinear model medium, J. Phys. A25 (1992) 4181–4187.