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REVIEW PAPER

New perspectives in algebraic logic, from neat embeddings to Erdos graphs

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Abstract The notion of neat reducts is an old venerable notion in cylindric algebra theory invented by Henkin. This notion is regaining momentum. In this paper we explain why. This notion is discussed in connection to the algebraic notions of representability and complete representability, and the corresponding metalogical ones of completeness and omitting types, particularly for finite variable fragments. Also it is shown how such a notion has found intersection with non-trivial topics in model theory (like finite forcing) and set theory (forcing).

An important central concept introduced in [54] is that of neat reducts, and the related one of neat embeddings. The notion of neat reducts is due to Henkin, and one can find that the discussion of this notion is comprehensive and detailed in [54] (closer to the end of the book). This notion proved useful in at least two respects. Analyzing the number of variables appearing in proofs of first order formulas [53], and characterizing the class of representable algebras; those algebras that are isomorphic to genuine algebras of relations. In fact, several open problems that appeared in [54,55] are on neat reducts, some of which appeared in part 1, and (not yet resolved) appeared again in part 2. This paper, among other things, surveys the status of these problems 40 years after they first appeared. Long proofs are omitted, except for one, which gives the gist of techniques used to solve such kind of problems.

All the open problems in [54,55] on neat reducts are now solved. The most recent one was solved by the present author. This is problem 2.3 in [54]. A solution of this problem on neat embeddings is presented in [19]. But the present paper also poses new problems related to this key notion in the representation theory of cylindric algebras, that have emerged in recent years.

Our notation is in conformity with the two monographs on the subject [54,55]. Cylindric set algebras are algebras whose elements are relations of a certain pre-assigned arity, endowed with set-theoretic operations that utilize the form of elements of the algebra as sets of sequences. Our $\mathcal{B}(X)$ denotes the Boolean set algebra $(\wp(X), \cup, \cap, \sim, \emptyset, X)$. Let U be a set and α an ordinal. α will be the dimension of the algebra. For $s, t \in {}^{\alpha}U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq {}^{\alpha}U$ and $i, j < \alpha$, let

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : \exists t \in X(t \equiv_i s)\}$$

and

$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

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$(\mathcal{B}(^{\alpha}U), \mathsf{C}_i, \mathsf{D}_{ij})_{i,j<\alpha}$ is called the full cylindric set algebra of dimension α with unit (or greatest element) $^{\alpha}U$. Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed if $\mathfrak{M}$ is a first order structure in a first order language L with α many variables, and one sets

$$\phi^{\mathfrak{M}} = \{s \in {}^{\alpha}\mathfrak{M} : \mathfrak{M} \models \phi[s]\}$$

(here $\mathfrak{M} \models \phi[s]$ means that s satisfies ϕ in $\mathfrak{M}$), then the set $\{\phi^{\mathfrak{M}} : \phi \in Fm^L\}$ is a cylindric set algebra of dimension α. Indeed

$$\phi^{\mathfrak{M}} \cap \psi^{\mathfrak{M}} = (\phi \wedge \psi)^{\mathfrak{M}},$$

and

$$^{\alpha}\mathfrak{M} \sim \phi^{M} = (\neg\phi)^{\mathfrak{M}},$$

$$\mathsf{C}_i(\phi^{\mathfrak{M}}) = \exists v_i \phi^{\mathfrak{M}},$$

and finally

$$\mathsf{D}_{ij} = (x_i = x_j)^{\mathfrak{M}}.$$

$\mathbf{Cs}_{\alpha}$ denotes the class of all subalgebras of full set algebras of dimension α. Let $\mathbf{RCA}_{\alpha}$ denote the variety generated by the class $\mathbf{Cs}_{\alpha}$. Algebras in $\mathbf{RCA}_{\alpha}$ are said to be representable. An old problem in algebraic logic is [51]: Can we describe $\mathbf{RCA}_{\alpha}$ by a simple schema of equations? In other words, is there a simple set of equations Σ such that $\mathcal{A} \models \Sigma$ if and only if $\mathcal{A}$ is representable? Let us call this problem the *representation problem*. Andréka, Németi and Sain refer to the related problem of modifying the class of representable cylindric algebras to get a new variety that is still adequate for the algebraization of first order logic, but is finitely axiomatizable, as the *finitizability* problem in algebraic logic [66]. Both of these problems are discussed at length in [6]. The representation problem, and for that matter the finitizability problem, have provoked extensive research and are still, in some sense, open! An approximation $\mathbf{CA}_{\alpha}$ was introduced by Tarski. $\mathbf{CA}_{\alpha}$ is defined by an indeed simple finite set of equations Σ that aims at capturing the essential algebraic essence of existential quantifiers and equality. But early on in the investigations of $\mathbf{CA}$'s it turned out that there are cylindric algebras that are not representable. The choice of Σ was motivated by the fact that it works in some special cases that are significant (like for example locally finite cylindric algebras). The locally finite cylindric algebras correspond in an exact sense [55] 4.3.28 (ii) to Lindenbaum–Tarski algebras of formulas in (ordinary) first order logic.

If $\mathcal{C} \in \mathbf{Cs}_{\beta}$ with base U, then for any $\alpha < \beta$, the elements of $\mathcal{C}$ that are fixed by C_i, $i \geqslant \alpha$, can be thought of as representations of α ary relations on U. In fact if we keep only these elements and those operations whose indices are all in α, then the resulting algebra is obviously isomorphic to a $\mathbf{Cs}_{\alpha}$ (and in fact to one with base U). This observation carries over to abstract $\mathbf{CA}_{\beta}$'s in general yielding the concept of neat reducts.

A reduct of an algebra $\mathcal{A}$ is another algebra $\mathcal{B}$ obtained from $\mathcal{A}$ by dropping some of the operations. $\mathcal{B}$ thus has the same universe of $\mathcal{A}$ but the operations defined on these elements constitute only a part of the original operations. In cylindric algebras, reducts are important because certain reducts of cylindric algebras are cylindric algebras (of a different dimension, though).

Let $\mathcal{A} = (A, +, \cdot, -, \mathsf{c}_i, \mathsf{d}_{ij}) \in \mathbf{CA}_{\beta}$[1] and $\rho : \alpha \to \beta$ be one to one. Then $\mathfrak{Rd}^{\rho}\mathcal{A} = (A, +, \cdot, -, \mathsf{c}_{\rho(i)}, \mathsf{d}_{\rho(i),\rho(j)})_{i,j<\alpha}$ is a $\mathbf{CA}_{\alpha}$ [54] 2.6.1.

Here a reduct is defined by renaming the operations. However, when $\alpha \subseteq \beta$ and ρ is the inclusion map then $\mathfrak{Rd}_{\alpha}\mathcal{A}$ is just the algebra obtained by discarding the operations indexed by ordinals in $\beta \sim \alpha$. For $x \in \mathcal{A}$, let $\Delta x = \{i \in \beta : \mathsf{c}_i x \neq x\}$. Then for i, $j < \alpha$ we have $\Delta \mathsf{d}_{ij} \subseteq \alpha$ and if $\Delta x \subseteq \alpha$ and $i < \alpha$ then $\Delta \mathsf{c}_i x \subseteq \alpha$. Also $\Delta(x + y) \subseteq \Delta x \cup \Delta y$ and $\Delta(-x) = \Delta x$.

The set $Nr_{\alpha}\mathcal{B} = \{x \in B : \Delta x \subseteq \alpha\}$ is a subuniverse of $\mathfrak{Rd}_{\alpha}\mathcal{B}$. The algebra $\mathfrak{Nr}_{\alpha}\mathcal{B} \in \mathbf{CA}_{\alpha}$ with universe $Nr_{\alpha}\mathcal{B}$ is called the neat α reduct of $\mathcal{B}$ [54] 2.6.28.

If there is an embedding $e : \mathcal{C} \to \mathfrak{Nr}_{\alpha}\mathcal{B}$ then we say that $\mathcal{C}$ neatly embeds in $\mathcal{B}$. For $K \subseteq \mathbf{CA}_{\beta}$ and $\alpha < \beta$, $\mathfrak{Nr}_{\alpha}K = \{\mathfrak{Nr}_{\alpha}\mathcal{B} : \mathcal{B} \in K\}$. Now if $\alpha < \beta$ and $\theta : \wp(^{\alpha}U) \to \wp(^{\beta}U)$ is defined by

$$X \mapsto \{s \in {}^{\beta}U : (s{\restriction}\alpha) \in X\},$$

then θ maps $\wp(^{\alpha}U)$ into $\mathfrak{Nr}_{\alpha}\wp(^{\beta}U)$. Thus set algebras can be neatly embedded into algebras in arbitary extra dimensions. But the converse is strikingly true. If $\mathcal{A} \in \mathbf{CA}_{\alpha}$ and there exists an embedding $e : \mathcal{A} \to \mathfrak{Nr}_{\alpha}\mathcal{B}$, with $\mathcal{B} \in \mathbf{CA}_{\alpha+\omega}$, then $\mathcal{A}$ is representable. So we have the following (neat) Neat Embedding theorem, or *NET* for short, of Henkin: $\mathbf{RCA}_{\alpha} = S\mathfrak{Nr}_{\alpha}\mathbf{CA}_{\alpha+\omega}$ for any α. Here S stands for the operation of forming subalgebras. This is fully proved in [55], cf. Theorem 3.2.10. This theorem has several incarnations in the literature, see e.g [33,47,71,73], some of which are quite sophisticated. Following the conventions of [54], algebras in the class $S\mathfrak{Nr}_{\alpha}\mathbf{CA}_{\alpha+\omega}$ are said to have the Neat Embedding property (*NEP*). Monk proved that $\mathbf{RCA}_{\alpha} \subset S\mathfrak{Nr}_{\alpha}\mathbf{CA}_{\alpha+n}$ for every $\alpha > 2$ and $n \in \omega$, so that all ω extra dimensions are needed to enforce representability. However, we can dispense with some of the $\mathbf{CA}$ axiom, when we get to ω-extra dimensions as illustrated by Ferenczi [47]. We will return to such issues in some depth at the end of the paper. The non-finite axiomatizability result for $\mathbf{RCA}_{\alpha}$ when $\alpha > 2$ is finite, follows from Monk's result. Indeed, let $\mathcal{A}_k \in S\mathfrak{Nr}_{\alpha}\mathbf{CA}_{\alpha+k} \sim \mathbf{RCA}_{\alpha}$, then any non trivial ultraproduct of the $\mathcal{A}_k$'s will be representable.

But why is the notion of neat reducts so important in cylindric algebra theory and related structures; in a nut shell: due to its intimate connection to the notion of representability, via Henkin's Neat Embedding Theorem.

But this is not the end of the story, in fact this is where the fun begins. A new unexpected viewpoint can yield dividends, and indeed the notion of neat reducts has been revived lately, to mention a few references: [1–4,6,11–13,17,18,20–24,33,39, 44,47–49,53,73]. Indeed there has been a rise of interest in the study of neat embeddings for cylindric algebras, and related structures with pleasing progress. In this paper we intend to survey (briefly) such results on neat reducts putting them in a wider perspective.

Our first family of results will concern the class of neat reducts proper, that is the class $\mathfrak{Nr}_{\alpha}\mathbf{CA}_{\beta}$, e.g. is it closed under homomorphic images, products; is it a variety, if not, is it perhaps an elementary class? But why address such question on neat reducts. There are (at least) three possible answers to this

[1] *We follow the conventions of [54], so that operations of abstract algebras are denoted by $+$, $\cdot$, $-$, c_i, d_{ij}, standing for join, meet, complementation, etc., while in set algebras these operations are denoted by $\cup$, $\cap$, $\sim$, C_i, D_{ij}.*

question. First there are aesthetic reasons. Motivated by intellectual curiosity, the investigation of such questions is likely to lead to nice mathematics. The second reason concerns definability or classification. Now that we have the class of neat reducts in front of us, the most pressing need is to try to classify it. Classifying is a kind of defining. Most mathematical classification is by axioms (preferably first order) or, even better, equations (if the class in question is a variety.) Now we come to the third reason, for studying such questions on neat reducts. Here we do not address neat embeddings as an end itself but rather discuss such notion in connection to the so called amalgamation property and the notion of complete representations.

Accordingly, the rest of the paper is divided into four parts. In the first part (Section 1), we discuss results on the class of neat reducts proper. In the second part (Section 2) we discuss neat embeddings in connection to the amalgamation property. Two open questions in the problem session paper of [43] are answered. In Section 3 we discuss neat embeddings in connection to complete representations. In Section 4, we go back to the classical *NET* of Henkin and review several variations on this deep theorem introduced by Ferenczi. In the final section we comment on related results concerning relation algebras. We note that many other classes of algebras studied in algebraic logic enjoy a *NET*, like Pinter's substitution algebras, Halmos quasipolyadic algebras and Halmos' polyadic algebras. To keep the paper as short as possible, we discuss those very briefly.

1. The class of neat reducts is not elementary

Problems 2.11 and 2.12 in the monograph [54] are on neat reducts. Problem 2.12 is solved by Hirsch et al. [53]. Hirsch et al. show that the sequence $\langle S\mathfrak{N}r_n\mathbf{CA}_{n+k} : k \in \omega\rangle$ is strictly decreasing for $n > 2$ with respect to inclusion. Problem 2.11 which is relevant to our later discussion asks: For which pair of ordinals $\alpha < \beta$ is the class $\mathfrak{N}r_\alpha\mathbf{CA}_\beta$ closed under forming subalgebras? Németi [65] proves that for any $1 < \alpha < \beta$ the class $\mathfrak{N}r_\alpha\mathbf{CA}_\beta$ though closed under forming homomorphic images and products is not a variety, i.e. it is not closed under forming subalgebras. The next natural question is whether this class is elementary? Andréka and Németi prove that the class $\mathfrak{N}r_2\mathbf{CA}_\beta$ for $\beta > 2$ is not elementary. Their remarkable proof appears in [56]. Not resolved for higher dimensions, this problem reappears in [55] problem 4.4. Since this class is closed under ultraproducts this is equivalent to asking whether it is closed under forming elementary subalgebras. In [13] it is proved that for any $2 < \alpha < \beta$, the class $\mathfrak{N}r_\alpha\mathbf{CA}_\beta$ is not elementary. Here we give a model theoretic proof of this result that has appeared in [2].

Definition 1.1.

(i) Let L be a signature and $\mathcal{D}$ an L structure. The *age* of $\mathcal{D}$ is the class $\mathbf{K}$ of all finitely generated structures that can be embedded in $\mathcal{D}$.
(ii) A class $\mathbf{K}$ is the *age* of $\mathcal{D}$ if the structures in $\mathbf{K}$ are *up to isomorphism*, exactly the finitely generated substructures of $\mathcal{D}$.
(iii) Let $\mathbf{K}$ be a class of structures.$\mathbf{K}$ has the *Hereditary Property, HP for short,* if whenever $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B}$ is a finitely generated substructure of $\mathcal{A}$ then $\mathcal{B}$ is isomorphic to some structure in $\mathbf{K}$.$\mathbf{K}$ has the *Joint Embedding Property, JEP for short,* if whenever $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ then there is a $\mathcal{C} \in \mathbf{K}$ such that both $\mathcal{A}$ and $\mathcal{B}$ are embeddable in $\mathcal{C}$.$\mathbf{K}$ has *Amalgamation Property, or AP for short,* if $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ and $e : \mathcal{A} \to \mathcal{B}, f : \mathcal{A} \to \mathcal{C}$ are embeddings, then there are $\mathcal{D}$ in $\mathbf{K}$ and embeddings $g : \mathcal{B} \to \mathcal{D}$ and $h : \mathcal{C} \to \mathcal{D}$ such that $g \circ e = h \circ f$.
(iv) A structure $\mathcal{D}$ is weakly homogeneous if it has the following property if $\mathcal{A}, \mathcal{B}$ are finitely generated substructures of $\mathcal{D}, \mathcal{A} \subseteq \mathcal{B}$ and $f : \mathcal{A} \to \mathcal{D}$ is an embedding, then there is an embedding $g : \mathcal{B} \to \mathcal{D}$ which extends f.
(v) We call a structure $\mathcal{D}$ homogeneous if every isomorphism between finitely generated substructures extends to an automorphism of $\mathcal{D}$.

Note that if $\mathcal{D}$ is homogeneous, then it is weakly homogeneous. We recall Theorem 7.1.2 from [57], a theorem of Fraisse that puts the above pieces together.

Theorem 1.2. *Let L be a countable signature and let $\mathbf{K}$ be a nonempty finite or countable set of finitely generated L-structures which has HP, JEP and AP. Then there is an L structure $\mathcal{D}$, unique up to isomorphism, such that*

(i) *$\mathcal{D}$ has cardinality $\leqslant\omega$,*
(ii) *$\mathbf{K}$ is the age of D, and*
(iii) *$\mathcal{D}$ is homogeneous.*

Using Theorem 1.2, we shall construct an algebra $\mathcal{A} \in \mathfrak{N}r_3\mathbf{CA}_\beta$ that has an elementary equivalent algebra $\mathcal{B} \notin \mathfrak{N}r_3\mathbf{CA}_\beta$. The proof for the finite dimensional case is the same. For infinite dimensions we refer to [13].

Notation. S_3 denotes the set of all permutations of 3. ${}^XY$ denotes the set of functions from X to Y. For $u, v \in {}^33$, $i < 3$ we write u_i for $u(i) < 3$, and we write $u \equiv_i v$ if u and v agree off i, i.e. if $u_j = v_j$ for all $j \in 3\backslash\{i\}$. For a symbol R of the signature of $\mathfrak{M}$ we write $R^{\mathfrak{M}}$ for the interpretation of R in $\mathfrak{M}$.

Our algebras will be based on the model proven to exist in the next lemma.

Lemma 1.3. *Let L be a signature consisting of the unary relation symbols P_0, P_1, P_2 and uncountably many 3-ary predicate symbols. For $u \in {}^33$, let χ_u be the formula $\bigwedge_{i<3}P_{u_i}(x_i)$. Then there exists an L-structure $\mathfrak{M}$ with the following properties:*

(i) *$\mathfrak{M}$ has quantifier elimination, i.e. every L-formula is equivalent in $\mathfrak{M}$ to a Boolean combination of atomic formulas.*
(ii) *The sets $P_i^{\mathfrak{M}}$ for $i < 3$ partition M.*
(iii) *$\mathfrak{M} \models \forall x_0x_1x_2(R(x_0, x_1x_2) \to \bigvee_{u\in S_3}\chi_u)$, for all $R \in L$.*
(iv) *$\mathfrak{M} \models \exists x_0x_1x_2(\chi_u \wedge R(x_0, x_1, x_2) \wedge \neg S(x_0, x_1, x_2))$ for all distinct ternary R, $S \in L$, and $u \in S_3$.*
(v) *For $u \in S_3, i < 3, \mathfrak{M} \models \forall x_0x_1x_2(\exists x_i\chi_u \leftrightarrow \bigvee_{v\in{}^33, v\equiv_i u}\chi_v)$.*
(vi) *For $u \in S_3$ and any L-formula $\phi(x_0, x_1, x_2)$, if $\mathfrak{M} \models \exists x_0x_1x_2(\chi_u \wedge \phi)$ then $\mathfrak{M} \models \forall x_0x_1x_2(\exists x_i\chi_u \leftrightarrow \exists x_i(\chi_u \wedge \phi))$ for all $i < 3$.*

Sketch of Proof. The proof of this lemma is model theoretic. Let $\mathcal{L}$ be the relational signature containing unary relation symbols $P_0, \ldots, P_3$ and a 4-ary relation symbol X. Let $\mathbf{K}$ be the class of all finite $\mathcal{L}$-structures $\mathcal{D}$ satisfying

(1) The P_i's *are disjoint* : $\forall x \bigvee_{i<j<4} \left(P_i(x) \wedge \bigwedge_{j\neq i} \neg P_j(x) \right)$.

(2) $\forall x_0 \cdots x_3 \left(X(x_0, \cdots, x_3) \to P_3(x_3) \wedge \bigvee_{u\in s_3} \chi_u \right)$. □

Then **K** contains countably many isomorphism types, because for each $n \in \omega$, there are countably many isomorphism types of finite L structures (satisfying (1) and (2)) having cardinality $\leqslant n$. Also it is easy to check that **K** is closed under substructures and that **K** has the *AP*. From the latter it follows that it has the *JEP*, since **K** contains the one element structure that is embeddable in any structure in **K**.[2] By Theorem 1.2 there is a countably infinite homogeneous $\mathcal{L}$-structure $\mathfrak{N}$ with age **K**. $\mathfrak{N}$ has quantifier elimination, and obviously, so does any elementary extension of $\mathfrak{N}$. **K** contains structures with arbitrarily large P_3-part, so $P_3^{\mathfrak{N}}$ is infinite. Let $\mathfrak{N}^*$ be an elementary extension of $\mathfrak{N}$ such that $|P_3^{\mathfrak{N}^*}| = |L|$, and fix a bijection * from the set of ternary relation symbols of L to $P_3^{\mathfrak{N}^*}$. Define an L-structure $\mathfrak{M}$ with domain $P_0^{\mathfrak{N}^*} \cup P_1^{\mathfrak{N}^*} \cup P_2^{\mathfrak{N}^*}$, by: $P_i^{\mathfrak{M}} = P_i^{\mathfrak{N}^*}$ for $i < 3$ and for ternary $R \in L$,

$$\mathfrak{M} \models R(a_0, a_1, a_2) \quad \text{iff } \mathfrak{N}^* \models X(a_0, a_1, a_2, R^*).$$

If $\phi(\bar{x})$ is any L-formula, let $\phi^*(\bar{x}, \bar{R})$ be the $\mathcal{L}$-formula with parameters $\bar{R}$ from $\mathfrak{N}^*$ obtained from ϕ by replacing each atomic subformula $R(x, y, z)$ by $X(x, y, z, R^*)$ and relativizing quantifiers to $\neg P_3$, that is replacing $(\exists x)\phi(x)$ and $(\forall x)\phi(x)$ by $(\exists x)(\neg P_3(x) \to \phi(x))$ and $(\forall x)(\neg P_3(x) \to \phi(x))$, respectively. A straightforward induction on complexity of formulas gives that for $\bar{a} \in \mathfrak{M}$

$$\mathfrak{M} \models \phi(\bar{a}) \quad \text{iff } \mathfrak{N}^* \models \phi^*(\bar{a}, \bar{R}).$$

Then $\mathfrak{M}$ is as required.

Now we are going to prove that the class $\mathfrak{N}r_\alpha \mathbf{CA}_\beta$ is not elementary for $3 \leqslant \alpha < \beta$. We prove the result for $\alpha = 3$. The proof for higher finite dimensions is the same. For the infinite dimensional case, we refer to [13].

Theorem 1.4. *For $\beta > 3$, the class $\mathfrak{N}r_3\mathbf{CA}_\beta$ is not elementary.*

Proof. Fix L and $\mathfrak{M}$ as in Lemma 1.3. Let $A_\omega = \{\phi^M : \phi \in L\}$ and $A = \{\phi^M : \phi \in L_3\}$ with operations defined as for set algebras. Then $\mathcal{A} \cong \mathfrak{N}r_3\mathcal{A}_\omega$, the isomorphism is given by

$$\phi^{\mathfrak{M}} \mapsto \phi^{\mathfrak{M}}. \qquad \square$$

Quantifier elimination in M guarantees that this map is onto. For $u \in {}^33$, let $\mathcal{A}_u$ denote the relativization of $\mathcal{A}$ to $\chi_u^{\mathfrak{M}}$ i.e.

$$\mathcal{A}_u = \{x \in A : x \leqslant \chi_u^M\}.$$

$\mathcal{A}_u$ is a Boolean algebra. Also $\mathcal{A}_u$ is uncountable for every $u \in S_3$ because by property (iv) of Lemma 1.3 the sets $(\chi_u \wedge R(x_0, x_1, x_2))^{\mathfrak{M}}$, for $R \in L$ are distinct elements of $\mathcal{A}_u$. Define a map $f : \mathcal{A} \to \prod_{u\in {}^33}(\mathcal{A}_u)$, by

$$f(a) = \langle a \cdot \chi_u \rangle_{u\in {}^33}.$$

We will expand the language of the Boolean algebra $\prod_{u\in {}^33}\mathcal{A}_u$ in such a way that the cylindric algebra $\mathcal{A}$ becomes interpretable in the expanded structure. For this we need the following definition:

Let $\mathfrak{P}$ denote the following structure for the signature of Boolean algebras expanded by constant symbols 1_u for $u \in {}^33$ and d_{ij} for $i, j \in 3$:

(1) The Boolean part of $\mathfrak{P}$ is the Boolean algebra $\prod_{u\in {}^33}\mathcal{A}_u$.
(2) $1_u^{\mathfrak{P}} = f(\chi_u^{\mathfrak{M}}) = \langle 0, \cdots 0, 1, 0, \cdots \rangle$ (with the 1 in the uth place) for each $u \in {}^33$.
(3) $\mathsf{d}_{ij}^{\mathfrak{P}} = f(\mathsf{d}_{ij}^{\mathcal{A}})$ for $i, j < 3$.

We now show that $\mathcal{A}$ is interpretable in $\mathfrak{P}$ [57]. For this it is enough to show that f is one to one and that *Rng*(f) (Range of f) and the f-images of the graphs of the cylindric algebra functions in $\mathcal{A}$ are definable in $\mathfrak{P}$. Since the $\chi_u^{\mathfrak{M}}$ partition the unit of $\mathcal{A}$, each $a \in A$ has a unique expression in the form $\sum_{u\in {}^33}(a \cdot \chi_u^{\mathfrak{M}})$, and it follows that f is Boolean isomorphism: $bool(\mathcal{A}) \to \prod_{u\in {}^33}\mathcal{A}_u$. So the f-images of the graphs of the Boolean functions on $\mathcal{A}$ are trivially definable. f is bijective so *Rng*(f) is definable, by $x = x$. For the diagonals, $f(\mathsf{d}_{ij}^{\mathcal{A}})$ is definable by $x = \mathsf{d}_{ij}$. Finally we consider cylindrifications. For $S \subseteq {}^33$, $i < 3$, let t_S be the closed term

$$\sum\{1_v : v \in {}^33, v \equiv_i u \text{ for some } u \in S\}.$$

Let

$$\eta_i(x, y) = \bigwedge_{S\subseteq {}^33} \left(\bigwedge_{u\in S} x \cdot 1_u \neq 0 \wedge \bigwedge_{u\in {}^33\setminus S} x \cdot 1_u = 0 \to y = t_S \right).$$

We claim that for all $a \in A$, $b \in P$, we have

$$\mathfrak{P} \models \eta_i(f(a), b) \quad \text{iff } b = f(\mathsf{c}_i^{\mathcal{A}}a).$$

To see this, let $f(a) = \langle a_u \rangle_{u\in {}^33}$, say. So in $\mathcal{A}$ we have $a = \sum_u a_u$. Let u be given; a_u has the form $(\chi_i \wedge \phi)^{\mathfrak{M}}$ for some $\phi \in L^3$, so $\mathsf{c}_i^{\mathcal{A}}(a_u) = (\exists x_i(\chi_u \wedge \phi))^{\mathfrak{M}}$. By property (vi) of Lemma 1.3, if $a_u \neq 0$, this is $(\exists x_i \chi_u)^M$; by property *5*, this is $(\bigvee_{v\in {}^33, v\equiv_i u}\chi_v)^{\mathfrak{M}}$. Let $S = \{u \in {}^33 : a_u \neq 0\}$. By normality and additivity of cylindrifications we have,

$$\mathsf{c}_i^{\mathcal{A}}(a) = \sum_{u\in {}^33} \mathsf{c}_i^{\mathcal{A}} a_u = \sum_{u\in S} \mathsf{c}_i^{\mathcal{A}} a_u = \sum_{u\in S} \left(\sum_{v\in {}^33, v\equiv_i u} \chi_v^{\mathfrak{M}} \right)$$
$$= \sum \{\chi_v^{\mathfrak{M}} : v \in {}^33, v \equiv_i u \text{ for some } u \in S\}.$$

So $\mathfrak{P} \models f(\mathsf{c}_i^{\mathcal{A}}a) = t_S$. Hence $\mathfrak{P} \models \eta_i(f(a), f(\mathsf{c}_i^{\mathcal{A}}a))$. Conversely, if $\mathfrak{P} \models \eta_i(f(a), b)$, we require $b = f(\mathsf{c}_i a)$. Now S is the unique subset of ^{33} such that

$$\mathfrak{P} \models \bigwedge_{u\in S} f(a) \cdot 1_u \neq 0 \wedge \bigwedge_{u\in {}^33\setminus S} f(a) \cdot 1_u = 0.$$

So we obtain

$$b = t_S = f(\mathsf{c}_i^{\mathcal{A}}a).$$

We have proved that $\mathcal{A}$ is interpretable in $\mathfrak{P}$. Furthermore it is easy to see that the interpretation is one dimensional and quantifier free. Next we extract an algebra $\mathcal{B}$ elementary equivalent to $\mathcal{A}$ that is not a neat reduct i.e. not in $\mathfrak{N}r_3\mathbf{CA}_4$. Let $Id \in {}^33$ be the identity map on 3. Choose any countable Boolean elementary subalgebra of $\mathcal{A}_{Id}$, $\mathcal{B}_{Id}$ say. Thus $\mathcal{B}_{Id} \preceq \mathcal{A}_{Id}$. Then

[2] It is not always true that *AP* implies *JEP*; think of fields.

$$Q = \left(\left(B_{Id} \times \prod_{u \in {}^3 3 \setminus Id} \mathcal{A}_u\right), 1_u, d_{ij}\right)_{u \in {}^3 3, i,j<3}$$
$$\equiv \left(\left(\prod_{u \in {}^3 3} \mathcal{A}_u\right), 1_u, \mathsf{d}_{ij}\right)_{u \in {}^3 3, i,j<3} = P.$$

Let $\mathcal{B}$ be the result of applying the interpretation given above to Q. Then $\mathcal{B} \equiv \mathcal{A}$ as cylindric algebras. Now we show that $\mathcal{B}$ cannot be a neat reduct, in fact we show that $\mathcal{B} \notin \mathfrak{Nr}_3\mathbf{CA}_\beta$ for any $\beta > 3$. Assume for contradiction that $\mathcal{B} = \mathfrak{Nr}_3\mathcal{D}$ for some $\mathcal{D} \in \mathbf{CA}_\beta$; with $\beta > 3$. Note that $\mathcal{D}$ may not be representable. It is only here that we deal with possibly non-representable algebras. Now $\chi_u^M \in B$ for each $u \in {}^3 3$. Identifying functions with sequences we let $v = \langle 1,0,2 \rangle \in {}^3 3$. Let $t(x)$ be the $\mathbf{CA}_2$ term $\mathsf{s}_1^0\mathsf{c}_1 x \cdot \mathsf{s}_0^1\mathsf{c}_0 x$, where $\mathsf{s}_i^j(x) = \mathsf{c}_i(\mathsf{d}_{ij} \cdot x)$, for $i \neq j$. Then we claim that $t^B(\chi_v^{\mathfrak{M}}) = \chi_{Id}^{\mathfrak{M}}$. For the sake of brevity, denote $\chi_v^{\mathfrak{M}}$ by 1_{10} and $\chi_{Id}^{\mathfrak{M}}$ by 1_{01}. Then, by definition, we have

$$t^B(1_{01}) = \mathsf{c}_0(\mathsf{d}_{01} \cdot \mathsf{c}_1 1_{10}) \cdot \mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0 1_{10}).$$

Computing we get

$$\mathsf{c}_0(\mathsf{d}_{01} \cdot \mathsf{c}_1 1_{10}) = \mathsf{c}_0\left(\mathsf{d}_{01} \cdot \left(\sum\{1_u : u \equiv_1 1_{10}\}\right)\right)$$
$$= \mathsf{c}_0(\mathsf{d}_{01} \cdot 1_{112}) = 1_{01} + 1_{112}.$$

Here 1_{112} denotes $\chi_{\langle 1,1,2 \rangle}$. Note that we are using that the evaluation of the term $\mathsf{c}_1 1_{10}$ in $\mathcal{B}$ is equal to its value in $\mathcal{A}$. This is so, because $\mathcal{B}$ inherits the interpretation given to $\prod A_u$. A similar computation gives

$$\mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0 1_{01}) = 1_{002} + 1_{01},$$

where 1_{002} denotes $\chi_{\langle 0,0,2 \rangle}$. Therefore as claimed

$$t^B(1_{10}) = 1_{01}.$$

Now let ${}_3\mathsf{s}(0,1)$ be the unary substitution term as defined in [54] 1.5.12, that is

$${}_3\mathsf{s}(0,1)x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1(x).$$

Then for any $\beta > 3$ we have

$$\mathbf{CA}_\beta \models {}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leqslant t(\mathsf{c}_3 x).$$

Indeed by [54] 1.5.12, 1.5.8 and 1.5.10 (ii), we get

$${}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leqslant {}_3\mathsf{s}(0,1)\mathsf{c}_1\mathsf{c}_3 x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1\mathsf{c}_1\mathsf{c}_3 x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_3 x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_3\mathsf{c}_1 x$$
$$= \mathsf{s}_0^3\mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1 x = \mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1 x = \mathsf{s}_0^1\mathsf{c}_1\mathsf{c}_3 x.$$

Similarly

$${}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leqslant \mathsf{s}_1^0\mathsf{c}_0\mathsf{c}_3 x.$$

Therefore

$${}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leqslant t(\mathsf{c}_3 x).$$

It thus follows that

$$\mathcal{D} \models {}_3\mathsf{s}(0,1)(\chi_u^M) \leqslant \mathsf{s}_1^0\mathsf{c}_1(\chi_u^M) \cdot \mathsf{s}_0^1\mathsf{c}_0(\chi_u^M) = \chi_{Id}^M.$$

Now ${}_3\mathsf{s}(0,1)$ preserves $\leqslant$ and is one to one $\mathfrak{Nr}_3\mathcal{D}$. By [54], 1.5.12 and 1.5.1, we have:

$${}_3\mathsf{s}(0,1)\mathsf{c}_3 x = \mathsf{s}_0^n\mathsf{s}_1^0\mathsf{s}_3^1\mathsf{c}_3 x = \mathsf{c}_3(\mathsf{d}_{30} \cdot \mathsf{c}_0(\mathsf{d}_{01} \cdot \mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_1(\mathsf{d}_{13} \cdot \mathsf{c}_3 x)))).$$

By [54], 1.3.8, $0 < x$, implies $0 < \mathsf{d}_{ij} \cdot \mathsf{c}_j x$, for all $i, j \in \beta$. We have shown that if $x > 0 \in Nr_3 D$, then ${}_3\mathsf{s}(0,1)x > 0$, i.e. that ${}_3\mathsf{s}(0,1)$, being a Boolean endomorphism, is one to one. Since $B_v = A_v$ it follows (by condition (iv) in Lemma 1.3) that $B_v = \{b \in B : b \leqslant \chi_v^M\}$ is uncountable. Since ${}_3\mathsf{s}(0,1)$ is one to one, it follows that ${}_3\mathsf{s}(0,1)B_u$ is also uncountable. But by the above we have

$${}_3\mathsf{s}(0,1)B_u \subseteq B_{Id} = \{b \in B : b \leqslant \chi_{Id}^B\},$$

and so B_{Id} is also uncountable. But by construction, we have $B_{Id} = \{b \in B : b \leqslant \chi_{Id}^M\}$ is countable. This contradiction shows that $\mathcal{B} \notin \mathfrak{Nr}_3\mathbf{CA}_\beta$ for any $\beta > 3$.

We formulate a (new) theorem that further indicates that the class of neat reducts is really hard to characterize. But first some set-theoretic preparations. Let $\mathfrak{M}$ denote the universe of sets and let $\mathcal{C} \in M$ be a complete Boolean algebra. (Note that $\mathcal{C}$ is a Boolean algebra "from the outside as well" but not necessarily complete.) Form the Boolean valued extension $\mathfrak{M}^{\mathcal{C}}$ of $\mathfrak{M}$ and let $\|\phi\|$ be the Boolean value of a sentence ϕ of set theory containing parameters from $\mathfrak{M}^{\mathcal{C}}$. ϕ is valid in $\mathfrak{M}$ if $\|\phi\| = 1$ in symbols $\mathfrak{M}^{\mathcal{C}} \models \phi$. Write $\mathcal{C} : \mathcal{A} \cong \mathcal{B}$ if $\mathfrak{M}^{\mathcal{C}} \models \bar{\mathcal{A}} \cong \bar{\mathcal{B}}$. Here $\breve{s}$ is the canonical name of s in $\mathfrak{M}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are Boolean isomorphic if there is such $\mathcal{C}$, and (in which case) we write $\mathcal{A} \cong_b \mathcal{B}$. It turns out that Boolean isomorphism lies somewhere between $\equiv$ (elementary equivalence) and $\cong$ (isomorphism). Such an equivalence relation, as it turns out, is purely structural and can characterized by games. (The idea is to look at isomorphisms between a finite number of elements at a time. In model theory this is expressed by back- and-forth systems.) Of course if $\mathcal{A} \cong \mathcal{B}$ then trivially $\mathcal{A} \cong_b \mathcal{B}$. Call a class of algebras K *Boolean closed* if whenever $\mathcal{A} \in K$ and $\mathcal{B} \cong_b \mathcal{A}$ in some Boolean valued extension of the universe of sets, then $\mathcal{B} \in K$. We now have the following theorem proved in [35].

Theorem 1.5. *Let $1 < \alpha < \beta$. Then the following hold:*

(i) *The class $\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$ is not Boolean closed.*
(ii) *The classes $\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$ regarded as concrete categories are not finitely complete (that is closed under finite limits).*
(iii) *Let L denote the first order language of $\mathbf{CA}_\alpha$. There is no sentence $\sigma \in L_{\infty\omega}$ that characterizes $\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$.*

Here $L_{\infty\omega}$ is the logic obtained from first order logic by allowing infinite conjunctions without any restrictions on cardinality. Examples of finite limits are products and equalizers. In [20] it is shown that there are $\mathcal{A}, \mathcal{B} \in \mathfrak{Nr}_\alpha\mathbf{CA}_\beta$ and morphisms f, g from $\mathcal{A}$ to $\mathcal{B}$ such that $\{x \in A : f(x) = g(x)\}$ is not the universe of an algebra in $\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$.

The closure of the class of neat reducts for other algebras under forming (elementary) subalgebras is investigated in [1,8,10].

2. Amalgamation and neat reducts

Let K be a class of algebras having a Boolean reduct. $\mathcal{A}_0 \in K$ is in the amalgamation base of K if for all $\mathcal{A}_1, \mathcal{A}_2 \in K$ and monomorphisms $i_1 : \mathcal{A}_0 \to \mathcal{A}_1$, $i_2 : \mathcal{A}_0 \to \mathcal{A}_2$ there exist $\mathcal{D} \in K$ and monomorphisms $m_1 : \mathcal{A}_1 \to \mathcal{D}$ and $m_2 : \mathcal{A}_2 \to \mathcal{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. If in addition, $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leqslant m_k(y) \Rightarrow (\exists z \in A_0)(x \leqslant i_j(z) \wedge i_k(z) \leqslant y))$ where $\{j,k\} = \{1,2\}$, then we say that $\mathcal{A}_0$ lies in the super amalgamation base of K. Here $\leqslant$ is the Boolean order. K has the (super) amalgamation property ((SUP)AP), if the (super) amalgamation base of K coincides with K.

The amalgamation property (for classes of models), since its discovery, has played a dominant role in algebra and model theory [57]. Algebraic logic is the natural interface between universal algebra and logic (in our present context a variant of first order logic). Indeed, in algebraic logic amalgamation properties in classes of algebras are proved to be equivalent to interpolation results in the corresponding logic. Pigozzi [68], is a milestone for working out such equivalences for cylindric algebras, see also [59]. The super amalgamation property was introduced by Maksimova [63] (for expansions of Boolean algebras) and it is studied extensively by Madarász in e.g. [62] and more recently (for cylindric algebras) by Sági and Shelah [69] and by the present author [15]. The super amalgamation property for a class of algebras corresponds to a strong form of interpolation in the corresponding logic [5,59,62].

It is usually not an easy matter to characterize the amalgamation base (or for that matter the super amalgamation base) of K when K does not have the $AP(SUPAP)$. An example is the case when $K = \mathbf{RCA}_\alpha$ with $\alpha > 1$ [46,68]. We set out to determine both the amalgamation base and super amalgamation base of $\mathbf{RCA}_\alpha$ for any ordinal α answering a question in the problem session paper in [43]. (This question, formulated as problem 45 in [43] addresses only the amalgamation base case).

We start by giving a natural sufficient condition for an algebra $\mathcal{A}$ to belong to the (super) amalgamation base of $\mathbf{RCA}_\alpha$. The conditions are formulated in terms of *neat embeddings*. This is indeed expected since we have a *NET*. For a cylindric algebra $\mathcal{A}$ and $X \subseteq A$, $\mathfrak{Sg}^{\mathcal{A}}X$ denotes the subalgebra of $\mathcal{A}$ generated by X. $\mathfrak{Ig}^{\mathcal{A}}X$ is the ideal generated by X.

Definition 2.1. Let $\mathcal{A} \in \mathbf{RCA}_\alpha$. Then $\mathcal{A}$ has the *UNEP* (short for unique neat embedding property) if for all $\mathcal{A}' \in \mathbf{CA}_\alpha, \mathcal{B}, \mathcal{B}' \in \mathbf{CA}_{\alpha+\omega}$, isomorphism $i : \mathcal{A} \to \mathcal{A}'$, embeddings $e_A : \mathcal{A} \to \mathfrak{Nr}_\alpha\mathcal{B}$ and $e_{A'} : \mathcal{A}' \to \mathfrak{Nr}_\alpha\mathcal{B}'$ such that $\mathfrak{Sg}^{\mathcal{B}}e_A(A) = \mathcal{B}$ and $\mathfrak{Sg}^{\mathcal{B}'}e_{A'}(A)' = \mathcal{B}'$, there exists an isomorphism $\bar{i} : \mathcal{B} \to \mathcal{B}'$ such that $\bar{i} \circ eA = e_{A'} \circ i$.

Definition 2.2. Let $\mathcal{A} \in RCA_\alpha$. Then $\mathcal{A}$ has the *NS* property (short for neat reducts commuting with forming subalgebras) if for all $\mathcal{B} \in CA_{\alpha+\omega}$ if $\mathcal{A} \subseteq \mathfrak{Nr}_\alpha\mathcal{B}$ then for all $X \subseteq A$, $\mathfrak{Sg}^{\mathcal{A}}X = \mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathcal{B}}X$.

Let us examine closely the above conditions. At first glance Definition 2.1 might seem complicated, but in fact it is a slight generalization of a very simple and indeed ''natural'' property. Let $\mathcal{A} \in CA_\alpha$. Let $\mathcal{A} \subseteq \mathfrak{Nr}_\alpha\mathcal{B}$ with $\mathcal{B} \in CA_{\alpha+\omega}$. Call $\mathcal{B}$ an ω dilation of $\mathcal{A}$. If further A generates $\mathcal{B}$ (using the $\alpha + \omega$ operations of $\mathcal{B}$) call $\mathcal{B}$ a *minimal* ω dilation of $\mathcal{A}$. In this case, one might expect that $\mathcal{A}$ has some control of $\mathcal{B}$. In fact, Definition 2.1 implies that any two minimal ω dilations of $\mathcal{A}$ are in fact *isomorphic*. Furthermore this isomorphism can be chosen to fix A. This follows from the special case when $\mathcal{A} = \mathcal{A}'$, and i and $e_A = e_{A'}$ are the inclusion maps. So, roughly, Definition 2.1 says that $\mathcal{A}$ determines essentially the structure of its minimal ω dilations.

Now for Definition 2.2. Again let $\mathcal{B}$ be an ω dilation of $\mathcal{A} \in CA_\alpha$ so that $\mathcal{A} \subseteq \mathfrak{Nr}_\alpha\mathcal{B}$, with $\mathcal{B} \in CA_{\alpha+\omega}$. Let $X \subseteq A$. Form the subalgebra of $\mathcal{A}$ generated by X and form the subalgebra of $\mathcal{B}$ generated by X. Then, in principal, in the second process of generation, new α dimensional elements can be generated. Definition 2.2 excludes this possibility. It says that if we take the set of α dimensional elements of $\mathfrak{Sg}^{\mathcal{B}}X$ (i.e. we form $\mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathcal{B}}X$), then we come back exactly to where we started namely to $\mathfrak{Sg}^{\mathcal{A}}X$ (and not to a bigger algebra). No new α dimensional elements are generated (even in the presence of ω extra dimensions). Note that in this case, we have

$$\mathfrak{Sg}^{\mathcal{A}}X = \mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathcal{B}}X = \mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathcal{B}}X.$$

Here two operations commute. Forming the subalgebra of the neat reduct is the same as taking the neat reduct of the subalgebra.

Let $APbase(K)$ be the class of algebras that lie in the amalgamation base of K and $SUPAPbase(K)$the class of algebras that lie in the super amalgam base of K. Then the following is known $\mathbf{RCA}_\alpha = APbase(\mathbf{RCA}_\alpha) = SUPAPbase(\mathbf{RCA}_\alpha)$ if and only if $\alpha \leqslant 1$, cf. [46,59,68]. Now we have:

Theorem 2.3. *Let α be an ordinal. Let $\mathcal{A} \in RCA_\alpha$.*

(i) *If $\mathcal{A}$ has UNEP, then $\mathcal{A} \in APbase(RCA_\alpha)$.*
(ii) *If $\mathcal{A}$ has UNEP and NS, then $\mathcal{A} \in SUPAPbase(RCA_\alpha)$.*

We omit the proof that can be found in [35]. Let $Dc_\alpha = \{\mathcal{A} \in CA_\alpha : \Delta x \neq \alpha, \text{for all } x \in \mathcal{A}\}$. These algebras are referred to as dimension complemented cylindric algebras. It is not hard to show for $\alpha \geqslant \omega$, $\mathbf{Dc}_\alpha$ has *NS* and *UNEP*. Hence $\mathbf{Dc}_\alpha \subseteq SUPAPbase(\mathbf{RCA}_\alpha)$. $\mathbf{Mn}_\alpha$ denotes the class of minimal cylindric algebras of dimension α. Since $\mathbf{Mn}_\alpha \subseteq \mathbf{Dc}_\alpha$ the latter class is also contained in the *SUPAPbase* of $\mathbf{RCA}_\alpha$. However, for $1 < n < \omega$, $\mathbf{Mn}_n \not\subseteq APbase(\mathbf{RCA}_n)$ a result of Comer [46]. Expressed differently $\mathbf{RCA}_n$ does not have the embedding property. Another comprehensive class of algebras that is contained in $SUPAPbase(\mathbf{CA}_\alpha)$ for $\alpha > 1$ is the class of cylindric algebras of positive characteristic [30]. It seems likely that the class of algebras having the unique neat embedding property coincides with $APbase(\mathbf{RCA}_\alpha)$, and that for infinite α, $SUPAPbase(\mathbf{RCA}_\alpha) = \mathbf{Dc}_\alpha$ but further research is needed. The following theorem will be used to confirm some conjectures of Tarski on neat reducts. First we need:

For a cardinal $\beta > 0$, $L \subseteq \mathbf{CA}_\alpha$ and $\rho : \beta \to \wp(\alpha)$, $\mathfrak{Fr}^{\rho}_{\beta}L$ stands for the dimension restricted L free algebra on β generators [54] 2.5.31. The sequence $\langle \eta/Cr^{\rho}_{\beta}L : \eta < \beta\rangle$ L-freely generates $\mathfrak{Fr}^{\rho}_{\beta}L$, cf. [54] Theorem 2.5.35.

Theorem 2.4. *If $\alpha < \beta$ are any ordinals and $L \subseteq \mathbf{CA}_\beta$, then, in the sequence of conditions (i)–(v) below, (i)–(iv) implies the immediately following one:*

(i) *For any $\mathcal{A} \in L$ and $\mathcal{B} \in CA_\beta$ with $\mathcal{A} \subseteq \mathfrak{Nr}_\alpha\mathcal{B}$, for all $X \subseteq A$ we have $\mathfrak{Sg}^{\mathcal{A}}X = \mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathcal{B}}X$.*
(ii) *For any $\mathcal{A} \in L$ and $\mathcal{B} \in CA_\beta$ with $\mathcal{A} \subseteq \mathfrak{Nr}_\alpha\mathcal{B}$, if $\mathfrak{Sg}^{\mathcal{B}}A = \mathcal{B}$, then $\mathcal{A} = \mathfrak{Nr}_\alpha\mathcal{B}$.*
(iii) *For any $\mathcal{A} \in L$ and $\mathcal{B} \in CA_\beta$ with $\mathcal{A} \subseteq \mathfrak{Nr}_\alpha\mathcal{B}$, if $\mathfrak{Sg}^{\mathcal{B}}A = \mathcal{B}$, then for any ideal I of $\mathcal{B}$, $\mathfrak{Ig}^{\mathcal{B}}(A \cap I) = I$.*
(iv) *If whenever $\mathcal{A} \in L$, there exists $x \in {}^{|A|}A$ such that if $\rho = \langle \Delta x_i : i < |A|\rangle$, $\mathcal{D} = \mathfrak{Fr}^{\rho}_{|A|}CA_\beta$ and $g_\xi = \xi/Cr^{\rho}_{|A|}CA_\beta$, then $\mathfrak{Sg}^{\mathfrak{Rd}_\alpha\mathcal{D}}\{g_\xi : \xi < |A|\} \in L$, then the following UNEP hold: For $\mathcal{A}, \mathcal{A}' \in L$, $\mathcal{B}, \mathcal{B}' \in CA_\beta$ with embeddings $e_A : \mathcal{A} \to \mathfrak{Nr}_\alpha\mathcal{B}$ and $e_{A'} : \mathcal{A}' \to \mathfrak{Nr}_\alpha\mathcal{B}'$ such that $\mathfrak{Sg}^{\mathcal{B}}e_A(A) = \mathcal{B}$ and $\mathfrak{Sg}^{\mathcal{B}'}e_{A'}(A) = \mathcal{B}'$, whenever $i : \mathcal{A} \to \mathcal{A}'$ is an isomorphism, then there exists an isomorphism $\bar{i} : \mathcal{B} \to \mathcal{B}'$ such that $\bar{i} \circ e_A = e_{A'} \circ i$.*
(v) *Assume that $\beta = \alpha + \omega$. Then $L \subseteq APbase(\mathbf{RCA}_\alpha)$.*

Proof. (i) Implies (ii) is trivial. Now we prove (ii) implies (iii). The proof is similar to [54] 2.6.71. From the premise that $\mathcal{A}$ is a generating subreduct of $\mathcal{B}$ we easily infer that $|\Delta x \backslash \alpha| < \omega$ for all $x \in B$. We now have $\mathcal{A} = \mathfrak{Nr}_\alpha \mathcal{B}$. Now clearly $\mathfrak{Ig}^{\mathcal{B}}(I \cap A) \subseteq I$. Conversely let $x \in I$. Then $\mathsf{c}_{(\Delta x \backslash \alpha)} x$ is in $\mathfrak{Nr}_\alpha \mathcal{B}$, hence in $\mathcal{A}$. Therefore $\mathsf{c}_{(\Delta x \backslash \alpha)} x \in A \cap I$. But $x \leqslant \mathsf{c}_{(\Delta x \backslash \alpha)} x$, hence the required. We now prove (iii) implies (iv). The proof is a generalization of the proof of [54] 2.6.72. Let $\mathcal{A}, \mathcal{A}' \in L, \mathcal{B}, \mathcal{B}' \in CA_\beta$ and assume that $e_A, e_{A'}$ are embeddings from $\mathcal{A}, \mathcal{A}'$ into $\mathfrak{Nr}_\alpha \mathcal{B}, \mathfrak{Nr}_\alpha \mathcal{B}'$, respectively, such that $\mathfrak{Sg}^{\mathcal{B}}(e_A(A)) = \mathcal{B}$ and $\mathfrak{Sg}^{\mathcal{B}'}(e_{A'}(A')) = \mathcal{B}'$, and let $i : \mathcal{A} \to \mathcal{A}'$ be an isomorphism. We need to "lift" i to β dimensions. Let $\mu = |A|$. Let x be a bijection from μ onto A that satisfies the premise of *(4)*. Let y be a bijection from μ onto A', such that $i(x_j) = y_j$ for all $j < \mu$. Let $\rho = \langle \Delta^{(\mathcal{A})} x_j : j < \mu \rangle, \mathcal{D} = \mathfrak{Fr}_\mu^{(\rho)} CA_\beta, g_\xi = \xi / Cr_\mu^{(\rho)} CA_\beta$ for all $\xi < \mu$ and $\mathcal{C} = \mathfrak{Sg}^{\mathfrak{Rd}_\alpha \mathcal{D}} \{g_\xi : \xi < \mu\}$. Then $\mathcal{C} \subseteq \mathfrak{Nr}_\alpha \mathcal{D}$, C generates $\mathcal{D}$ and by hypothesis $\mathcal{C} \in L$. There exist $f \in Hom(\mathcal{D}, \mathcal{B})$ and $f' \in Hom(\mathcal{D}, \mathcal{B}')$ such that $f(g_\xi) = e_A(x_\xi)$ and$f'^{(g_\xi)} = e_{A'}(y_\xi)$ for all $\xi < \mu$. Note that f and f' are both onto. We now have $e_A \circ i^{-1} \circ e_{A'}^{-1} \circ (f' \restriction \mathcal{C}) = f \restriction \mathcal{C}$. Therefore $Kerf' \cap \mathcal{C} = Kerf \cap \mathcal{C}$. Hence $\mathfrak{Ig}(Kerf' \cap \mathcal{C}) = \mathfrak{Ig}(Kerf \cap \mathcal{C})$. So by (iii), $Ker\ f' = Ker\ f$. Let $y \in B$, then there exists $x \in D$ such that $y = f(x)$. Define $\hat{\imath}(y) = f'(x)$. The map is well defined and is as required. The proof of $(iv) \Rightarrow (v)$ follows from Theorem 2.3. □

Since for $\alpha \geqslant \omega$, $\mathbf{RCA}_\alpha$ does not have AP, a classical result of Pigozzi, it follows that *we cannot replace* $\mathbf{Dc}_\alpha$ *in 2.6.67 (ii), 2.6.71–72 of [54] by* $\mathbf{RCA}_\alpha$ *when* $\alpha \geqslant \omega$. This answers a question of Monk and Henkin mentioned in the introduction of [55]. That this replacement cannot be made was mentioned in [54] with the proof deferred to the second part, but in the second the proof never appeared. Actually the co-authors Henkin and Monk admit in [55] p. (iv) that they could not reconstruct Tarski's proof. So the above theorem confirms three conjectures of Tarski, the proof of which could not be recovered by his co-authors Henkin and Monk. The first of those conjectures is confirmed more directly in [32].

Let $APbase(K)$ denote the amalgamation base of K and $SUPAPbase(K)$ denote the super amalgamation base of K.

We say that $\mathcal{A}_0 \in K$ *is in the strong amalgamation base of K, briefly* $\mathcal{A}_0 \in SAPbase(K)$ *if for all* $\mathcal{A}_1, \mathcal{A}_2 \in K$ *and monomorphisms* $i_1 : \mathcal{A}_0 \to \mathcal{A}_1 i_2 : \mathcal{A}_0 \to \mathcal{A}_2$ *there exist* $\mathcal{D} \in K$ *and monomorphisms* $m_1 : \mathcal{A}_1 \to \mathcal{D}$ *and* $m_2 : \mathcal{A}_2 \to \mathcal{D}$ *such that* $m_1 \circ i_1 = m_2 \circ i_2$ *and* $m_1(A_1) \cap m_2(A_2) = m_2 \circ i_2(A_0)$.

Then, it is easy to see that

$$SUPAPbase(K) \subseteq SAPbase(K) \subseteq APbase(K).$$

In [72], using the remarkable technique of twisting, it is shown that $\mathbf{Mn}_\omega \not\subseteq APbase(\mathbf{CA}_\omega)$. In particular, we have

$$AP(RCA_\omega) \nsubseteq APbase(CA_\omega)$$

and

$$SAPbase(RCA_\omega) \nsubseteq SAPbase(CA_\omega).$$

This answers two questions in the problem session paper of [43]. The problem of finding the AP base for classes of algebras that does not have AP originates with Bjarni Jónsson. (This is mentioned in the problem session paper of [43]). In [19] a solution to problem 2.3 (using neat embeddings and the amalgamation property) in [54] is presented. Further results connecting neat embeddings to various amalgamation properties can be found in [16,23–27,60,61].

3. Complete representations

Unless otherwise specified, we assume that $n \leqslant \omega$. If $\mathcal{A} \in RCA_n = SPCs_n$, then for all non-zero $a \in A$ there exist $\mathcal{C}_a \in Cs_n$ with base M and a homomorphism $f : \mathcal{A} \to \mathcal{C}_a$ such that $f(a) \neq 0$. (If $\mathcal{A} \in Cs_n$ has greatest element ${}^nM$, then M is called its base). This, can be easily proved to be equivalent to the fact that $\mathcal{A}$ has a representation on some set (in the sense of the coming definition).

Definition 3.1. A representation of $\mathcal{A} \in CA_n$ on a set V of n ary sequences, is an injective Boolean homomorphism $h : \mathcal{A} \to \wp(V)$ (the power set of V) such that

(i) $h(1) = V = \bigcup_{i \in I}^n X_i$ where the X_i's are disjoint. Here 1 is the greatest element of the Boolean reduct of $\mathcal{A}$ and I is an arbitrary non-empty set.
(ii) For all $i, j < n$, $\bar{x} \in h(\mathsf{d}_{ij})$ iff $x_i = x_j$
(iii) For all $i < n$, $a \in A$ and $\bar{x} \in h(\mathsf{c}_i a)$ iff $\bar{x}_y^i \in h(a)$ for some $y \in X$. Here $\bar{x}_y^i$ is the sequence that agrees with $\bar{x}$ except for i where its value is y.

In this case $\mathcal{A} \cong h(\mathcal{A})$, and $h(\mathcal{A})$ is a $\mathbf{Gs}_n$ in the sense of [55] with greatest element $V = \bigcup_{i \in I}^n X_i$. Then $h(\mathcal{A}) \subseteq (\wp(V), \cup, \cap, \sim V, \emptyset, \mathsf{C}_i, \mathsf{D}_{ij})_{i,j<n}$ with the C_i's and D_{ij}'s defined as in set algebras. In this case we say that (h, V) is a representation of $\mathcal{A}$. Let $\mathcal{A} \in RCA_n$ and (h, V) a representation of $\mathcal{A}$. If $s \in V$, we let

$$f^{-1}(s) = \{a \in \mathcal{A} : s \in f(a)\}.$$

Clearly $f^{-1}(s)$ is a Boolean ultrafilter in $\mathcal{A}$.

Definition 3.2.

(i) An atomic representation $f : \mathcal{A} \to \wp(V)$ is an (injective cylindric) representation such that for each $s \in V$, the ultrafilter $f^{-1}(s)$ is principal. Equivalently s is in the image of some atom of $\mathcal{A}$ (Recall that an atom is a minimal non-zero element.).
(ii) A complete representation of $\mathcal{A}$ is an injective representation $f : \mathcal{A} \to \wp(V)$ satisfying

$$f\left(\prod X\right) = \bigcap f[X]$$

whenever $X \subseteq \mathcal{A}$ and $\prod X$ is defined. Equivalently, $f(\sum X) = \bigcup f[X]$ whenever $\sum X$ is defined.

Lemma 3.3. *Let* $\mathcal{A} \in RCA_n$. *A representation f of* $\mathcal{A}$ *is atomic if and only if it is complete.*

Proof. This is proved for Boolean algebras in [50]. The proof lifts to the cylindric case with no modifications. □

Lemma 3.4. *Assume that* $\mathcal{A}$ *has a complete representation. Then* $\mathcal{A}$ *is atomic. That is every non-zero element contains an atom.*

Proof. [50]. Let f be a complete representation. Then it is atomic. Let a be a non-zero element of $\mathcal{A}$. Let $s \in f(a)$. (Here we are using that f is injective). So there is an atom $b \in A$ with $s \in f(b)$. Therefore $b \wedge a \neq 0$. Thus $b \leqslant a$. Hence $\mathcal{A}$ is atomic. □

For an algebra $\mathcal{A}$ with a Boolean reduct, we write $At\mathcal{A}$ for the set of atoms of $\mathcal{A}$. By Lemma 3.4 a necessary condition for existence of complete representations is atomicity. However representable atomic cylindric algebras may not be completely representable. In fact, the class of completely representable algebras is not even elementary [36,50]. But again we can characterize the class of completely representable algebras using neat embeddings: We start with a definition.

Definition 3.5. For K a class with a Boolean reduct we define

$$S_cK = \{\mathcal{A} : \exists \mathcal{B} \in K : A \subseteq B, \text{ and whenever } \sum X = 1 \text{ in } \mathcal{A} \text{ then} \sum X = 1 \text{ in } \mathcal{B} \text{ for all } X \subseteq A\}.$$

We sometimes refer to algebras in the class $S_c\mathfrak{Nr}_nCA_{n+\omega}$ as algebras having the *strong* neat embedding property. We now have:

Theorem 3.6. [3] *Let $n < \omega$. Let $\mathcal{A} \in CA_n$ be countable. Then $\mathcal{A}$ is completely representable if and only if $\mathcal{A}$ is atomic and $\mathcal{A} \in S_c\mathfrak{Nr}_nCA_\omega$.*

One implication follows from Lemma 3.4. We sketch a proof of the non-trivial implication, the if part. However this follows from the stronger:

(*). If $\mathcal{A} \in S_c\mathfrak{Nr}_nCA_{n+\omega}$ is countable, $n \leqslant \omega$ (note that here n is allowed to be infinite) and $\{X_i : i < \omega\}$ is a family of subsets of $\mathcal{A}$ such that $\prod X_i = 0$ for all $i < \omega$, then for every non-zero $a \in A$ there exists $\mathcal{C} \in Ws_n$, with countable base, and $f : \mathcal{A} \to \mathcal{C}$ a homomorphism such that $f(a) \neq 0$ and for all $i \in \omega$ we have $\bigcap_{x \in X_i} f(x) = \emptyset$.

Here $\mathbf{Ws}_n$ stands for the class of weak set algebras of dimension n. (*) is proved in [3]. The proof is a Baire category argument at heart hence the condition of countability cannot be omitted [11]. We recall that a weak set algebra has unit of the form ${}^n U^{(p)} = \{s \in {}^nU : |\{i \in n : s_i \neq p_i\}| < \omega\}$, for some $p \in {}^nU$. U is called its base. To emphasize the connection with the omitting types Theorem, we refer to the X_i's as *non-principal types* and to $\mathcal{B}$ as a representation *omitting* these types. Note that for $n < \omega$ we have $\mathbf{Ws}_n = \mathbf{Cs}_n$, so that a unit of a $\mathbf{Ws}_n$ is simply of the form ${}^n U$.

Sketch of proof of the non-trivial implication of Theorem 3.6. We show how the only if part of Theorem 3.6 follows from (*). Assume that (*) is proved and let $n < \omega$. Let $\mathcal{A} \in S_c\mathfrak{Nr}_nCA_\omega$ be countable and atomic. We will assume, to simplify matters, that $\mathcal{A}$ is simple. The general case is not much harder. Then taking $X_i = Y = \{-b : b \text{ is an atom of } \mathcal{A}\}$, and applying (*) for any non-zero a in A, upon noting that $\prod Y = 0$ since $\mathcal{A}$ is atomic, we get an atomic representation, hence a complete representation of $\mathcal{A}$. Note that since $\mathcal{A}$ is simple, the representation is necessarily injective.

We note that the class of completely representable cylindric algebras of dimension > 2 is not elementary. When we consider $< {}^{\omega}2$ many types then (*) becomes an instance of Martin's axiom restricted to countable Boolean algebras. Indeed, the notion of complete representations have been connected to Martin's axiom giving independent statements in set theory [7]. Our last result in this section unifies results on neat reducts and complete representations. Using the so called Rainbow construction for cylindric algebras the following is proved in [31]:

Theorem 3.7. *Let $n > 2$. Then any K such that $\mathfrak{Nr}_nCA_\omega \subseteq K \subseteq S_c\mathfrak{Nr}_nCA_{n+2}$ is not elementary.*

From which we readily get:

Corollary 3.8. *For $n > 2$ and $k \geqslant 2$, the class $\mathfrak{Nr}_nCA_{n+k}$ and the class of completely representable algebras of dimension n are not elementary.*

4. Neat embeddings and omitting types

First order logic (FOL) possesses some desirable properties, for example the completeness theorem, the omitting types theorem, the Craig interpolation theorem, and Beth's theorem. Daniele Mundici initiated the following type of investigations for FOL. Concerning various positive properties like Craig's interpolation Theorem, Mundici suggested to investigate how resource sensitive the positive result is. For example Craig's theorem says that to an implication $\phi \to \psi$ there exists an interpolant θ with $\phi \to \theta$ and $\theta \to \psi$. Now the question is, how much does θ depend on ϕ and ψ, or how complicated is θ relative to ϕ and ψ? Recent work measures expensiveness with the number of variables needed for θ. For example if both ψ and θ are built up of k variables, do we guarantee that the number of variables in the interpolant does not exceed k? Another example for such investigations is Monk's classical result that for any bound $k \in \omega$ there is a valid 3 variable formula which cannot be proved using only k variables. Note that ϕ can be proved using m variables for some finite $m > k$, for formulas contain finitely many variables and proofs are finite strings of formulas. This result was refined by Hirsch et al. showing that given any such k we can find a 3 variable formula as above, but also subject to the condition that it can be proved using $k + 1$ variables. Such results concerning proof theory for finite variable fragments of first order logic were first proved using algebraic logic. In this paper we apply this ''resource-oriented'' kind of the investigation to the classical Henkin–Orey omitting types theorem.

Let $\mathfrak{L}_n$ denote first order logic restricted to the first n variables. A systematic study of the fragments $\mathfrak{L}_n$ via cylindric algebras was initiated by Henkin via cylindric algebras of dimension n. The issue of ''resource-sensitivity'' is often addressed in the following form. We ask ourselves if certain distinguished properties of FOL are inherited by $\mathfrak{L}_n$. Examples of such distinguished properties studied in the literature for $\mathfrak{L}_n$ include interpolation, Beth definability [39], submodel preservation and completeness theorems [42]. A general first impression might be that, usually positive properties for FOL turn resource sensitive in such a strong way that a goal formulatable in $\mathfrak{L}_n$ cannot be soved in $\mathfrak{L}_n$. One might go further, by stipulating that a goal formulatable in $\mathfrak{L}_n$ cannot be solved, even in $\mathfrak{L}_{n+k}$ for every finite (fixed in advance) k. (Like Monk's result stated above). However, this is not true in such generality, some natural properties of substitutions in $\mathfrak{L}_n$ which are not provable in $\mathfrak{L}_n$ can be proved in $\mathfrak{L}_{n+2}$. A further counterexample is provided by the guarded fragment of FOL introduced by Andréka, Németi and van Benthem. Negative results (for finite variable fragments of first order logic) mentioned above do not occur for the guarded fragment of first order logic introduced in [41]. The guarded fragment (GF) was introduced as a fragment of first order logic which combines a great expressive power with nice modal behavior. It consists of relational first order formulas whose quantifiers are relativized by atoms in a certain way. *GF* has been established as a particularly well-behaved fragment of first order logic in many respects. The main point of the *GF* (and its variants e.g. the packed fragments) is that (inside the *GF*) we are safe of the above negative results for $\mathfrak{L}_n$, like essential incompleteness [55]. The omitting types theorem has not been investigated for the *GF*. However omitting types was investigated algebraically for other modifications of first order logic, the so called finitary logics of infinitary relations.

4.1. Omitting types fail in $\mathfrak{L}_n$

We work in usual *FOL*. For a formula ϕ and a first order structure $\mathfrak{M}$ in the language of ϕ we write ϕ^M to denote the set of all assignments that satisfy ϕ in M., i.e

$\phi^{\mathfrak{M}} = \{s \in {}^{\omega}M : \mathfrak{M} \models \phi[s]\}$.

For example if $\mathfrak{M} = (\mathfrak{N}, <)$ and ϕ is the formula $x_1 < x_2$ then a sequence $s \in {}^{\omega}\mathfrak{N}$ is in $\phi^{\mathfrak{M}}$ iff $s_1 < s_2$. Let Γ be a set of formulas (Γ may contain free variables). We say that Γ is realized in $\mathfrak{M}$ if $\bigcap_{\phi\in\Gamma}\phi^{\mathfrak{M}} \neq \emptyset$. Let ϕ be a formula and T be a theory. We say that ϕ ensures Γ in T if $T \vDash \phi \to \mu$ for all $\mu \in \Gamma$ and $T \models \exists\bar{x}\phi$. The classical Henkin–Orey omitting types theorem, *OTT* for short, states that if T is a consistent theory in a countable language $\mathfrak{L}$ and $\Gamma(x_1 \ldots x_n) \subseteq \mathfrak{L}$ is realized in every model of T, then there is a formula $\phi \in \mathfrak{L}$ such that ϕ ensures Γ in T. The formula ϕ is called a T-witness for Γ. Now the problem of resource sensitivity can be applied to *OTT* in the following sense. Can we always guarantee that the witness uses the same number of variables as T and Γ, or do we need extra variables? If we do need extra variables, is there perhaps an upper bound on the number of extra variables needed. In other words, let $\mathfrak{L}_n$ denote the set of formulas of $\mathfrak{L}$ which are built up using only n variables. The question is: If $T \cup \Gamma \subseteq \mathfrak{L}_n$, is there any guarantee that the witness stays in $\mathfrak{L}_n$, or do we occasionally have to step outside $\mathfrak{L}_n$?

Assume that $T \subseteq \mathfrak{L}_n$. We say that T is n complete iff for all sentences $\phi \in \mathfrak{L}_n$ we have either $T \vDash \phi$ or $T \models \neg\phi$. We say that T is n atomic iff for all $\phi \in \mathfrak{L}_n$, there is $\psi \in \mathfrak{L}_n$ such that $T \vDash \psi \to \phi$ and for all $\eta \in \mathfrak{L}_n$ either $T \vDash \psi \to \eta$ or $T \models \psi \to \neg\eta$.

Theorem 4.1. *Assume that $\mathfrak{L}$ is a countable first order language containing a binary relation symbol. For $n > 2$ and $k \geqslant 0$, there are a consistent n complete and n atomic theory T using only n variables, and a set $\Gamma(x_1)$ using only 3 variables (and only one free variable) such that Γ is realized in all models of T but each T-witness for T uses more that $n + k$ variables.*

Theorem 1 is proved using algebraic logic in [40]. For undefined terminology in the coming key Lemma the reader is referred to [40]:

Theorem 4.2. *Suppose that n is a finite ordinal with $n > 2$ and $k \geqslant 0$. There is a countable symmetric integral representable relation algebra $\mathbb{R}$ such that*

(i) *Its completion, i.e. the complex algebra of its atom structure is not representable, so $\mathbb{R}$ is representable but not completely representable.*
(ii) *$\mathbb{R}$ is generated by a single element.*
(iii) *The (countable) set $\mathcal{B}_n\mathbb{R}$ of all n by n basic matrices over $\mathbb{R}$ constitutes an n-dimensional cylindric basis. Thus $\mathcal{B}_n\mathbb{R}$ is a cylindric atom structure and the full complex algebra $\mathfrak{Cm}(\mathcal{B}_n\mathbb{R})$ with universe the power set of $\mathcal{B}_n\mathbb{R}$ is an n-dimensional cylindric algebra.*
(iv) *The term algebra over the atom structure $\mathcal{B}_n\mathbb{R}$, which is the countable subalgebra of $\mathfrak{Cm}(\mathcal{B}_n\mathbb{R})$ generated by the countable set of n by n basic matrices, $\mathfrak{Tm}(B_n\mathbb{R})$ for short, is a countable representable $\mathbf{CA}_n$, but $\mathfrak{Cm}(\mathcal{B}_n)$ is not representable.*
(v) *Hence $\mathcal{C}$ is a simple, atomic representable but not completely representable $\mathbf{CA}_n$.*
(vi) *$\mathcal{C}$ is generated by a single 2 dimensional element g, the relation algebraic reduct of $\mathcal{C}$ does not have a complete representation and is also generated by g as a relation algebra, and $\mathcal{C}$ is a sub-neat reduct of some simple representable $\mathcal{D} \in CA_{n+k}$ such that the relation algebraic reducts of $\mathcal{C}$ and $\mathcal{D}$ coincide.*

Sketch of Proof. We prove everything except that $\mathbb{R}$ can be generated by a single element, to which we refer to [40]. Let k be a cardinal. Let $\mathfrak{E}_k = \mathfrak{E}_k(2,3)$ denote the relation algebra which has k non-identity atoms, in which $a_i \leqslant a_j; a_l$ if $|\{i,j,l\}| \in \{2,3\}$ for all non-identity atoms a_i, a_j, a_k. (This means that all triangles are allowed except the monochromatic ones.) These algebras were defined by Maddux. Let k be finite, let I be the set of non-identity atoms of $\mathfrak{E}_k(2,3)$ and let $P_0, P_1 \ldots P_{k-1}$ be an enumeration of the elements of I. Let $l \in \omega$, $l \geqslant 2$ and let J_l denote the set of all subsets of I of cardinality l. Define the symmetric ternary relation on ω by $E(i,j,k)$ if and only if i, j, k are evenly distributed, that is

$$(\exists p,q,r)\{p,q,r\} = \{i,j,k\}, r - q = q - p.$$

Now assume that $n > 2$, $l \geqslant 2n - 1$, $k \geqslant (2n-1)l$, $k \in \omega$. Let $\mathfrak{M} = \mathfrak{E}_k(2,3)$. Then $\mathfrak{M}$ is a simple, symmetric finite atomic relation algebra. Also,

$$(\forall V_2 \ldots, V_n, W_2 \ldots W_n \in J_l)(\exists T \in J_l)(\forall 2 \leqslant i \leqslant n)$$
$$(\forall a \in V_i)\forall b \in W_i)(\forall c \in T_i)(a \leqslant b; c).$$

That is $(J4)_n$ formulated in [40] p. 72 is satisfied. Therefore, as proved in [40] p. 77, B_n the set of all n by n basic matrices is a cylindric basis of dimension n. But we also have

$$(\forall P_2, \ldots, P_n, Q_2 \ldots Q_n \in I)(\forall W \in J_l)(W \cap P_2; Q_2 \cap \ldots \cap P_n : Q_n \neq 0).$$

That is $(J5)_n$ formulated on p. 79 of [40] holds. According to Definition 3.1 (ii) (J,E) is an n blur for $\mathfrak{M}$, and clearly E is definable in $(\omega, <)$. Let $\mathcal{C}$ be as defined in Lemma 4.3 in [40]. Then, by Lemma 4.3, $\mathcal{C}$ is a subalgebra of $\mathfrak{Cm}\mathcal{B}_n$, hence it contains the term algebra $\mathfrak{Tm}\mathcal{B}_n$. Denote $\mathcal{C}$ by $\mathfrak{Bb}_n(\mathfrak{M}, J, E)$. Then by Theorem 4.6 in [40] $\mathcal{C}$ is representable, and by Theorem 4.4 in [40] for $m < n$ $\mathfrak{Bb}_m(\mathfrak{M}, J, E) = \mathfrak{Nr}_m\mathfrak{Bb}_n(\mathfrak{M}, J, E)$. However $\mathfrak{Cm}\mathcal{B}_n$ is not representable. In [40] $\mathbb{R} = \mathfrak{Bb}(\mathfrak{M}, \mathbb{J}, \mathbb{E})$ is proved to be generated by a single element. □

Now we give a proof of Theorem 4.1 modulo Theorem 4.2.

Proof of Theorem 4.1. Let $g, \mathcal{C}$ and $\mathcal{D}$ be as in Theorem 2 (vi). Then g generates $\mathcal{C}$ and g is 2 dimensional in $\mathcal{C}$. We can write up a theory $T \subseteq \mathfrak{L}_n$ such that for any model $\mathfrak{M}$ we have

$\mathfrak{M} = (M, G) \models T$ iff $\mathcal{C}_n(\mathfrak{M})$

$\cong \mathcal{C}$ and G corresponds to g via this isomorphism.

Now $T \subseteq L_n$, T is consistent and n complete and n atomic because C is simple and atomic. We now specify $\Gamma(x, y)$. For $a \in At$, let τ_a be a relation algebraic term such that $\tau_a(g) = a$ in R, the relation algebra reduct of $\mathcal{C}$. For each τ_a there is a formula $\mu_a(x,y)$ such that $\tau_a(g) = \mu_a^{\mathfrak{M}}$. Define $\Gamma(x, y) = \{\neg\mu_a : a \in At\}$. We will show that Γ is as required. First we show that Γ is realized in every model of T. Let $\mathfrak{M} \models T$. Then $\mathcal{C}_n(\mathfrak{M}) \cong \mathcal{C}$, hence $\mathfrak{M}$ gives a representation of $\mathbb{R}$ because $\mathbb{R}$ is the relation algebraic reduct of $\mathcal{C}_n(\mathfrak{M})$. But $\mathbb{R}$ has no complete representation, which means that $X = \bigcup\{\mu_a^{\mathfrak{M}} : a \in At\} \subset M \times M$, i.e. proper subset, so let $(u,v) \in M \times M \sim X$. This means that Γ is realized by (u,v) in $\mathfrak{M}$. We have seen that Γ is realized in each model of T. Assume that $\phi \in \mathfrak{L}_{n+k}$ such that $T \models \exists\bar{x}\phi$. We may assume that ϕ has only two free variables, say x, y. Take the representable $\mathcal{D} \in CA_{n+k}$ from Theorem 2 (iv). Recall that $g \in C \subseteq D$ and $\mathcal{D}$ is simple. Let $\mathfrak{M} = (M, g)$ where M is the base set of $\mathcal{D}$. Then $\mathfrak{M} \models T$ because $\mathcal{C}$ is a subreduct of $\mathcal{D}$ generated by g. By $T \models \exists\bar{x}\phi$, we have $\phi^{\mathfrak{M}} \neq \emptyset$. Also $\phi^{\mathfrak{M}} \in \mathcal{D}$ and is 2 dimensional, hence $\phi^{\mathfrak{M}} \in R$, since $\mathbb{R}$ is the relation algebraic reduct of $\mathcal{D}$, as well. But $\mathbb{R}$ is atomic hence $\phi^{\mathfrak{M}} \cap \mu_a \neq \emptyset$ for some $a \in At$. This shows that it is not the case that $\mathfrak{M} \models \phi \to \neg\mu_a$ where $\neg\mu_a \in \Gamma$, thus ϕ is not a T-witness for Γ. Now we modify T, Γ so that Γ uses only one free variable. We use the technique of so-called partial pairing functions. Let $g, \mathcal{C}, \mathcal{D}$ be as in Theorem 2 (iv) with $\mathcal{D} \in CA_{2n+2k}$. We may assume that g is disjoint from the identity $1'$ because $1'$ is an atom in the relation algebraic reduct of $\mathcal{C}$. Let U be the base set of $\mathcal{C}$. We may assume that U and $U \times U$ are disjoint. Let $M = U \cup (U \times U)$, let$G = g \cup \{(u,(u,v)) : u, v \in U\} \cup \{((u,v), v) : u, v \in U\} \cup \{((u,v),(u,v)) : u, v \in U\}$ and let $\mathcal{M} = (M, G)$. From G we can define $U \times U$ as $\{x : G(x,x)\}$ and from $U \times U$ and G we can define the projection functions between $U \times U$ and U, and g. All these definitions use only 3 variables. Thus for all $t \geqslant 3$ for all $\phi(x,y) \in \mathfrak{L}_t$ there is a $\psi(x) \in \mathfrak{L}_t$ such that $\psi^{\mathfrak{M}} = \{(u,v) \in U \times U : \phi^{(U,g)}(u,v)\}$. For any $a \in At$ let $\psi_a(x)$ be the formula corresponding to $\mu_a(x,y)$ this way. Conversely for any $\psi \in \mathfrak{L}_t$ there is a $\phi \in \mathfrak{L}_{2t}$ such that the projection of $\psi^{\mathfrak{M}}$ to U is $\phi^{(U,g)}$. Now define T as the $\mathfrak{L}_n$ theory of $\mathfrak{M}$, and set $\Gamma(x) = \{\neg\psi_a(x) : a \in At\}$. Then it can be easily checked that Γ and T are as required. □

Let us call an atom structure $\mathfrak{M}$ strongly representable if $\mathfrak{Cm}\mathfrak{M}$ is representable, and weakly representable if $\mathfrak{Tm}\mathfrak{M}$ is representable. As shown above, the construction of weakly representable atom structures that are not strongly representable [6,40] lead to atomic algebras with no complete representations and proves that $\mathbf{RCA}_n$ is not closed under completions i.e. is not atom-canonical, for $\mathfrak{Cm}At\mathcal{A}$ is the completion of $\mathfrak{Tm}At\mathcal{C}$. Such algebras were first constructed by Hirsch and Hodkinson [50]. Several variations on such constructions can be found in [36–38]. It also shows that OTT fails in L_n the first order logic restricted to the first n variables as long as $n > 2$. For $n = 2$ the Omitting types theorem fails for L_n in a more subtle way. A moment's reflection reveals that what we actually showed above is that Vaught's famous theorem on existence of atomic models for atomic theories fails for L_n when $n > 2$. For L_2 the analogue of Vaught's theorem holds [64], but the omitting types fails (an unpublished result of Andreka and Nemeti, cf. concluding remarks [40] p. 87.) In passing we note that for usual FOL Vaught's theorem follows from the omitting types theorem, so this method cannot be used for L_2. Hirsch and Hodkinson show that the class of strongly representable atom structures of relation algebras (and cylindric algebras) is not elementary. The construction makes use of the probabilistic method of Erdös to show that there are finite graphs with arbitrarily large chromatic number and girth. In his pioneering paper of 1959, Erdos took a radically new approach to construct such graphs: for each n he defined a probability space on the set of graphs with n vertices, and showed that, for some carefully chosen probability measures, the probability that an n vertex graph has these properties is positive for all large enough n. This approach, now called the *probabilistic method* has since unfolded into a sophisticated and versatile proof technique, in graph theory and in other branches of discrete mathematics. This method was used first in algebraic logic by Hirsch and Hodkinson to show that the class of strongly representable atom structures of cylindric and relation algebras is not elementary and that varieties of representable relation algebras are barely canonical. But yet again using these methods of Erdös in [58] it is shown that there exist continuum-many canonical equational classes of Boolean algebras with operators that are not generated by the complex algebras of any first-order definable class of relational structures. Using a variant of this construction the authors resolve the long-standing question of Fine, by exhibiting a bimodal logic that is valid in its canonical frames, but is not sound and complete for any first-order definable class of Kripke frames.

We know that every $\mathcal{A} \in \mathfrak{Nr}_n CA_\omega$ is representable. While $\mathbf{RCA}_n$ is a variety, the class $\mathfrak{Nr}_n CA_\omega$ is a pseudo elementary class, that is not elementary; furthermore; its elementary closure, $UpUr\mathfrak{Nr}_n CA_\omega$ is not finitely axiomatizable. The class of neat reducts is treated at length in [54,55]. In [28] the following question is investigated. When does $\mathcal{A} \in \mathfrak{Nr}_n CA_\omega$ posses a cylindric representation preserving a given set of (infinite) meets carrying them to set theoretic intersection? Then if $\mathcal{A}$ has a representation preserving arbitrary meets, then $\mathcal{A}$ is atomic. Conversely, when $\mathcal{A}$ is countable and atomic then $\mathcal{A}$ has such a representation. The example used above to violate OTT for $\mathfrak{L}_n$ together with an unpublished example of the author, can be used to show that countability is essential and we cannot replace $\mathfrak{Nr}_n CA_\omega$ by $\mathfrak{Nr}_n CA_{n+k} \cap RCA_n$ for any finite k. We also investigate the question of when representations preserve a given (possibly infinite) set of meets. More concretely, if $\mathcal{A} \in \mathfrak{Nr}_n CA_\omega$ is countable, κ is a cardinal and $(X_i : i < \kappa)$ is a family of subsets of $\mathcal{A}$ such that $\prod X_i = 0$, when does there exist a (generalized) set algebra $\mathcal{B}$ and isomorphism $f : \mathcal{A} \to \mathcal{B}$ such that for all $i \in \kappa$, $\bigcap_{x \in X_i} f(x) = \emptyset$. (This is an algebraic version of omitting κ many types in n variable logics.) Let $2^{\aleph_0}$ denote the power of the continuum. We show that when the meets are ultrafilters then preservation of $< {}^2\aleph_0$ many meets is possible (in ZFC), while if they are not then we are led to a statement that is independent of ZFC. The consistency of such a statement is proved by showing that is a consequence of a combinatorial consequence of Martin's axiom, namely P_0 stated before. The independence is proved using iterated forcing. Let us be even more explicit and formulate and state the results of [28].

Definition 4.3.

(i) Let κ be a cardinal. Let $OTT(\kappa)$ be the following statement. $\mathcal{A} \in \mathfrak{Nr}_n CA_\omega$ is countable and for $i \in \kappa$, $X_i \subseteq A$ are such that $\prod X_i = 0$, then for all $a \neq 0$, there exists a set algebra $\mathcal{C}$ with countable base, $f : \mathcal{A} \to \mathcal{C}$ such that $f(a) \neq 0$ and for all $i \in \kappa$, $\bigcap_{x \in X_i} f(x) = 0$.

(ii) Let OTT be the statement that

$$(\forall\kappa < 2^{\aleph_0}) OTT(\kappa).$$

(iii) Let $OTT_m(\kappa)$ be the statement obtained from $OTT(\kappa)$ by replacing X_i with "nonprincipal ultrafilter F_i" and OTT_m be the statement

$(\forall\kappa < 2^{\aleph_0})OTT_m(\kappa)$.

The proofs of the following theorems can be found in [28].

Theorem 4.4.

(i) *OTT is independent from $ZFC + \neg CH$. In fact for any regular cardinal $\kappa > \omega_1$, there is a model of ZFC in which $\kappa = 2^{\aleph_0}$ and OTT holds. Conversely, there is a model of ZFC in which $\omega_3 = 2^{\aleph_0}$ and $OTT(\omega_2)$ is false.*
(ii) *OTT_m is provable in ZFC.*

Using Shelah's techniques from stability theory, we also investigate preservation of $< 2^{\lambda}$ many (maximal) meets, where λ is a regular uncountable cardinal, for uncountable algebras in $\mathfrak{Nr}_n CA_\omega$. In more detail:

Theorem 4.5. *Let $\mathcal{A} \in \mathfrak{Nr}_n CA_\omega$ be infinite such that $|A| = \lambda$, λ is a regular cardinal. Let $\kappa < 2^{\lambda}$. Let $(X_i : i \in \kappa)$ be a family of non-principal ultrafilters of $\mathcal{A}$. Then there exists a representation $f : \mathcal{A} \to \wp({}^nX)$ such that $\bigcap_{x\in X_i} f(x) = \emptyset$ for all $i \in \kappa$.*

The last theorem is a new omitting types theorem addressing the uncountable case for $\mathfrak{L}_n$. Before we give a logical counterpart of the above theorems we review briefly the notion of quantifier elimination. Quantifier elimination is a concept that occurs in mathematical logic model theory, and theoretical computer science. One way of classifying formulas is by the amount of quantification. Formulae with less depth of quantifier alternation are thought of as simpler and the quantifier free formulae as the simplest. *A theory has quantifier elimination if for every formula α there exists a formula α_{QF} without quantifiers which is equivalent to it (modulo the theory).* Quantifier elimination is particularly useful in proving that a given theory is decidable. Examples of theories that have been shown decidable using quantifier elimination are Presburger arithmetic, real closed fields, atomless Boolean algebras, term algebras, dense linear orders, and random graphs.

Now one metalogical reading of the last two theorems is:

Theorem 4.6. *Let T be an $\mathfrak{L}_n$ consistent theory that admits elimination of quantifiers. Assume that $|T| = \lambda$ is a regular cardinal. Let $\kappa < 2^{\lambda}$. Let $(\Gamma_i : i \in \kappa)$ be a set of non-principal maximal types in T. Then there is a model $\mathcal{M}$ of T that omits all the Γ_i's.*

Proof. If $\mathcal{A} = \mathfrak{Fm}_T$ denotes the cylindric algebra corresponding to T, then since T admits elimination of quantifiers, then $\mathcal{A} \in \mathfrak{Nr}_n CA_\omega$. This follows from the following reasoning. Let $\mathcal{B} = \mathfrak{Fm}_{T_\omega}$ be the locally finite cylindric algebra based on T but now allowing ω many variables. Consider the map $\phi/T \mapsto \phi/T_\omega$. Then this map is from $\mathcal{A}$ into $\mathfrak{Nr}_n\mathcal{B}$. But since T admits elimination of quantifiers the map is onto. The Theorem now follows.

We feel that some clarification is in order. We mentioned above that we have an uncountable atomic simple algebra $\mathcal{A} \in \mathfrak{Nr}_n CA_\omega$ which is not completely representable. First impression might be that this is incompatible with Theorem 4.4. To construct a complete representation of this $\mathcal{A}$ one has to construct a representation that preserves $X = \{-a : a \in At\mathcal{A}\}$. But this is not an ultrafilter so the last theorem does not apply. Even more the atoms of algebra $\mathcal{A}$ are mutually disjoint and uncountable, so even Martin's axiom cannot offer solace in this context, for the algebra in question does not satisfy the countable chain condition.

5. Some further final comments

Hirsch proved the analogous result of Theorem 3.8 for relation algebras (**RA**) [49]. Let **RRA** denote the class of representable **RA**'s. For $\mathcal{C} \in CA_n$, $n \geqslant 4$, $\mathfrak{Ra}\mathcal{C}$, the relation algebra reduct of $\mathcal{C}$, is defined as in [55] 5.3.7. For **RA**'s we do have a NET to the effect that $RRA = S\mathfrak{Ra}CA_\omega = S\mathfrak{Ra}RCA_\omega$. If a representable relation algebra $\mathcal{A}$ generates at most one $\mathbf{RCA}_\omega$ then $\mathcal{A} \in APbase(RRA)$. This is another way of saying that an **RA** has the $UNEP$. In particular, $\mathbf{QRA} \subseteq APbase(\mathbf{RRA})$. **QRA** defined in e.g [74] p. 242 is the class of relation algebras with quasi-projections. In fact, we have $\mathbf{QRA} \subseteq SUPAP$-$base(\mathbf{RRA})$ [29]. A recent reference dealing with representability of **QRA**'s via a NET for **CA**'s is [73]. So for **RA**'s, **QRA** is a "natural" class such that each of its members has NS and $UNEP$. The **CA** analogue of this class is the class of directed cylindric algebras invented by Németi, and studied by András Simon and Gábor Sági [67,70]. These algebras are strongly related to higher order logics and they provide a solution to the finitizability problem (FP) in non-well founded set theories. The representability of such algebras, providing a solution to the finite dimensional version of the (FP), can be also proved using a NET. Positive solution exists in non-well founded set theories, because one can generate infinitely many extra dimensions, forcing a neat embedding theorem, by digging "downwards" with nothing to stop him! This view comes across very much in the case of Németi's directed cylindric algebras. Furthermore for such algebras neat reducts commute with forming subalgebras, hence this class has $SUPAP$ [30]. A solution to the infinite dimensional version of the FP is provided by Sain [71] (in usual set theory) using also a NET. These algebras are obtained by expanding the language of quasipolyadic algebras by finitely many infinitary substitutions and adding *finitely* many new axioms in the bigger language that enforces a NET. For those algebras neat reducts also commute with forming subalgebras, and so they have $SUPAP$ [5]. The real technical difficulty that comes up here is that when we expand our languages and add axioms to code extra dimensions somehow, in the hope of obtaining a NET, then usually we succeed in representing the already existing operations; the difficult problem is that the new operations turn out representable as well! Sain succeeded to overcome this difficulty for first order logic *without* equality. Adding equality proves problematic so far. A sophisticated categorial formulation of the FP, is to look at *inverses* of the *Neat reduct* functor going from one category to another in ω extra dimensions, and try to reflect those in an adjoint situation. A solution to the finitizability problem is thereby presented as an equivalence of two categories.

In [45] the NET of Henkin is likened to his completeness proof (the extra dimensions play the role of added witnesses to existential formulas); therefore it is not a coincidence that interpolation results and omitting types for variants of first order logic can be proved algebraically by using appropriate variations on the *NET* [9,34]. We find it timely to make the following observation. There are algebras for which the *NET* does not hold, that is, neat embeddability in algebras in ω extra dimensions does not enforce representability. Surprisingly this occurs at the "end points". The *NET* fails for the class of diagonal free cylindric algebras ($\mathfrak{D}f$) and polyadic equality algebras of infinite dimension (**PEA**). In between, there is a whole stratum of proper reducts of **PEA**'s that are proper expansions of $\mathfrak{D}f$'s (like **CA**'s, Sain's algebras introduced in [71] and **PA**'s) for which the *NET* holds. Finally (*) above after Theorem 3.6, which is basically a variation on a *NET*, is related to many statements from Lattice theory and topology in [14].

6. Open problems

We end this paper with the following two questions:

(i) Let $n > 2$ and $k \geqslant 2$. Is the class $S\mathfrak{N}r_nCA_{n+k}$ closed under completions?
(ii) Does the class of completely representable polyadic algebras of infinite dimension coincide with the atomic algebras?

Problem (i), attributed to the present author, appears in [52] problem 12 p. 627. For a partial result to (i), the reader is referred to [37].

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