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Original Article

A generalization of a half-discrete Hilbert's inequality

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Abstract Considering different parameters and by means of Hadamard's inequality, we obtain new and more general half-discrete Hilbert-type inequalities. Then we extract from our results some special cases that have been proved previously by other authors.

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1. Introduction

We study advanced variants of the following classical discrete Hilbert-type inequality [1]: if $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Inequality (1) has the following integral analogous:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dx\,dy \leq \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{\infty} g^q(x)dx\right)^{1/q}, \tag{2}$$

unless $f(x) \equiv 0$ or $g(x) \equiv 0$, where $p > 1$, $q = p/(p-1)$. The constant $\frac{\pi}{\sin(\pi/p)}$, in (1) and (2), is the best possible, see [1].

Inequalities (1) and (2), which have many generalizations see for example [2,3] and references therein, with their improvements have played fundamental roles in the development of many mathematical branches, see for instance [2,4,5] and references therein. A few results on the half-discrete Hilbert-type inequalities with non-homogeneous kernel can be found in [6]. Recently [7–10] gave some new half-discrete Hilbert-type inequalities. For example in [8] we find the following inequality

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with a non-homogeneous kernel: if $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \sum_{n=1}^\infty a_n^2 < \infty$, then

$$\sum_{n=1}^{\infty} a_n \int_0^\infty \frac{f(x)}{x+n} dx < \pi \left(\sum_{n=1}^{\infty} a_n^2 \int_0^\infty f^2(x) \right)^{1/2}, \tag{3}$$

where the constant π is the best possible. Then in [10], by using the way of weight coefficients and the idea of introducing parameters and by means of Hadamard's inequality, the authors gave the following more accurate inequality of (3):

$$\sum_{n=1}^{\infty} a_n \int_{-\frac{1}{2}}^\infty \frac{f(x)}{x+n} dx < \pi \left(\sum_{n=1}^{\infty} a_n^2 \int_{-\frac{1}{2}}^\infty f^2(x) \right)^{1/2}. \tag{4}$$

Inequalities (3) and (4) have many generalizations concerning the denominator of the left hand side, see for example [11–14].

Our main goal is to obtain a new generalization of the half-discrete Hilbert-type inequality (3). Before proving the main theorem of this paper, Theorem 2.1, let us state and prove the following lemma.

Lemma 1.1. For $0 < b < x < c$, $\alpha, r, \lambda_2\alpha \in (0, 1]$, with $\alpha > r$, $\lambda_1 \in (0, \infty)$, and $\lambda = \lambda_1 + \lambda_2$ with $\frac{\lambda_1}{\lambda_2} > p\left(\frac{\alpha}{r} - 1\right) \geq \frac{2}{\lambda_2 r} - 1$ define

$$w(n) := n^{\lambda_2\alpha} \int_b^c \frac{x^{\lambda_1\alpha - 1}}{(x^\alpha + n^r)^\lambda} dx, \tag{5}$$

and

$$\overline{w}(x) := x^{\lambda_1\alpha} \sum_{n=1}^{\infty} \frac{n^{p\lambda_2\alpha + (1-p)\lambda_2 r - 1}}{(x^\alpha + n^r)^\lambda}. \tag{6}$$

Then

$$w(n) = \frac{n^{\lambda_2(\alpha - r)}}{\alpha} (\beta(\lambda_1, \lambda_2) - \Psi(n)), \tag{7}$$

and

$$\overline{w}(x) < \frac{x^{p\lambda_2\alpha(\frac{\alpha}{r} - 1)}}{r} \beta(\xi, \zeta), \tag{8}$$

where $\Psi(n) = \int_0^{\frac{b^\alpha}{n^r}} \frac{u^{\lambda_1 - 1}}{(1+u)^\lambda} du + \int_0^{\frac{n^r}{c^\alpha}} \frac{u^{\lambda_2 - 1}}{(1+u)^\lambda} du$, and $\beta(\xi, \zeta)$ is the $\beta-$function *with* $\xi = \lambda_1 - p\lambda_2(\frac{\alpha}{r} - 1)$ *and* $\zeta = \lambda_2 + p\lambda_2(\frac{\alpha}{r} - 1)$.

Proof. Putting $u = \frac{x^\alpha}{n^r}$ in (5) gives

$$w(n) = \frac{n^{\lambda_2(\alpha - r)}}{\alpha} \int_{\frac{b^\alpha}{n^r}}^{\frac{c^\alpha}{n^r}} \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{1-\lambda_1} du$$
$$= \frac{n^{\lambda_2(\alpha - r)}}{\alpha} \left(\int_0^\infty \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{1-\lambda_1} du - \int_0^{\frac{b^\alpha}{n^r}} \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{1-\lambda_1} du - \int_{\frac{c^\alpha}{n^r}}^\infty \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{1-\lambda_1} du \right).$$

Use the definition of the Beta function $(\beta(\theta, \gamma) = \int_0^\infty \frac{z^{\theta - 1}}{(1+z)^{\theta+\gamma}} dz)$ in the first integral and the substitution $u = \frac{1}{v}$ in the third integral to have

$$w(n) = \frac{n^{\lambda_2(\alpha - r)}}{\alpha} \left(\beta(\lambda_1, \lambda_2) - \int_0^{\frac{b^\alpha}{n^r}} \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{1-\lambda_1} du - \int_0^{\frac{n^r}{c^\alpha}} \frac{1}{(1+v)^\lambda} \left(\frac{1}{v}\right)^{1-\lambda_2} dv \right),$$

as stated in (7).

In order to prove (8), for fixed $x \in (b, c)$, putting

$$f(t) = \frac{x^{\lambda_1\alpha} t^{p\lambda_2\alpha + (1-p)\lambda_2 r - 1}}{(x^\alpha + t^r)^\lambda}, \ t \in (0, \infty), \tag{9}$$

leads to

$$\frac{d}{dt} f(t) = x^{\lambda_1\alpha} \left(\frac{-r\lambda\, t^{p\lambda_2\alpha + (1-p)\lambda_2 r + r - 2}}{(x^\alpha + t^r)^{\lambda+1}} + \frac{(p\lambda_2\alpha + (1-p)\lambda_2 r - 1)\, t^{p\lambda_2\alpha + (1-p)\lambda_2 r - 2}}{(x^\alpha + t^r)^\lambda} \right) < 0,$$

while

$$\frac{d^2}{dt^2} f(t) = -\lambda r x^{\lambda_1\alpha} \left(\frac{-r(\lambda+1) t^{p\lambda_2\alpha + (1-p)\lambda_2 r + 2r - 3}}{(x^\alpha + t^r)^{\lambda+2}} + \frac{(p\lambda_2\alpha + (1-p)\lambda_2 r + r - 2) t^{p\lambda_2\alpha + (1-p)\lambda_2 r + r - 3}}{(x^\alpha + t^r)^{\lambda+1}} \right)$$
$$+ (p\lambda_2\alpha + (1-p)\lambda_2 r - 1) x^{\lambda_1\alpha} \left(\frac{-r\lambda t^{p\lambda_2\alpha + (1-p)\lambda_2 r + r - 3}}{(x^\alpha + t^r)^{\lambda+1}} + \frac{(p\lambda_2\alpha + (1-p)\lambda_2 r - 2) t^{p\lambda_2\alpha + (1-p)\lambda_2 r - 3}}{(x^\alpha + t^r)^\lambda} \right)$$
$$> 0.$$

Therefore, by Hadamard's inequality

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)dt, \ \ n \in \mathcal{N},$$

and (6) we obtain

$$\overline{w}(x) = \sum_{n=1}^{\infty} f(n) < \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)dt = \int_{\frac{1}{2}}^\infty f(t)dt$$
$$< \int_0^\infty f(t)dt = x^{\lambda_1\alpha} \int_0^\infty \frac{t^{p\lambda_2\alpha + (1-p)\lambda_2 r - 1}}{(x^\alpha + t^r)^\lambda} dt.$$

Letting $u = \frac{t^r}{x^\alpha}$ in the above inequality leads to

$$\overline{w}(x) < \frac{1}{r} x^{p\lambda_2\alpha(\frac{\alpha}{r} - 1)} \int_0^\infty \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{1 - (p\lambda_2\frac{\alpha}{r} + (1-p)\lambda_2)} du$$
$$= \frac{1}{r} x^{p\lambda_2\alpha(\frac{\alpha}{r} - 1)} \beta\Big(\lambda_1 - p\lambda_2(\frac{\alpha}{r} - 1), \lambda_2 + p\lambda_2\Big(\frac{\alpha}{r} - 1\Big)\Big).$$

This proves (8). □

In the following section we state the main result of this paper of which many special cases can be obtained.

2. Main results and discussion

In this section we state and discuss our main theorem together with its special cases. For three different parameters α, r, λ we have the following result.

Theorem 2.1. *Suppose that* $0 < b < c$, $0 < \alpha, 0 < r \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *(*$p \neq 0, 1$*)*, $\lambda_1 > 0$, $p\lambda_2\alpha + (1-p)\lambda_2 r \leq 1$, $\lambda = \lambda_1 + \lambda_2$, $a_n \geq 0$, *and* $f(x) \geq 0$ *is a real measurable function in* (b, c). *Then for* $p > 1$, *the following half-discrete Hilbert-type inequalities hold:*

$$J := \left(\sum_{n=1}^{\infty} n^{p\lambda_2\alpha - 1} \left[\int_b^c \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx \right]^p \right)^{\frac{1}{p}} \tag{10}$$

$$\leq \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\int_b^c f^p(x)\overline{w}(x)\Phi^{\frac{p}{q}}(\lambda_1, \lambda_2, n)\, x^{p(1-\lambda_1\alpha)-1} dx \right)^{\frac{1}{p}},$$

$$I := \sum_{n=1}^{\infty} a_n \int_b^c \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx \tag{11}$$

$$\leq \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\int_b^c f^p(x)\overline{w}(x)\Phi^{\frac{p}{q}}(\lambda_1, \lambda_2, n)\, x^{p(1-\lambda_1\alpha)-1} dx \right)^{\frac{1}{p}}$$

$$\times \left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2\alpha)-1} a_n^q \right)^{\frac{1}{q}},$$

where $\Phi(\lambda_1, \lambda_2, n) = \beta(\lambda_1, \lambda_2) - \Psi(n)$, $\beta(\lambda_1, \lambda_2)$ *is the* $\beta-$*function*, $\Psi(n)$ *and* $\overline{w}(x)$ *are as defined in Lemma 1.1.*

Proof. Using Hölder's inequality produces

$$\left[\int_b^c \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx \right]^p$$

$$= \left[\int_b^c \frac{1}{(x^\alpha + n^r)^\lambda} \left(\frac{x^{(1-\alpha\lambda_1)/q}}{n^{(1-\alpha\lambda_2)/p}} f(x) \right) \left(\frac{n^{(1-\alpha\lambda_2)/p}}{x^{(1-\alpha\lambda_1)/q}} \right) dx \right]^p$$

$$\leq \int_b^c \frac{1}{(x^\alpha + n^r)^\lambda} \frac{x^{(1-\alpha\lambda_1)(p-1)}}{n^{(1-\alpha\lambda_2)}} f^p(x) dx$$

$$\times \left[\int_b^c \frac{1}{(x^\alpha + n^r)^\lambda} \frac{n^{(1-\alpha\lambda_2)(q-1)}}{x^{(1-\alpha\lambda_1)}} dx \right]^{p-1}$$

$$= \int_b^c \frac{x^{(1-\alpha\lambda_1)(p-1)}}{n^{(1-\alpha\lambda_2)}(x^\alpha + n^r)^\lambda} f^p(x) dx \left[n^{q(1-\alpha\lambda_2)-1} w(n) \right]^{p-1}$$

$$= n^{1-p\lambda_2\alpha} w^{p-1}(n) \int_b^c \frac{f^p(x)}{(x^\alpha + n^r)^\lambda} \frac{x^{(1-\alpha\lambda_1)(p-1)}}{n^{(1-\alpha\lambda_2)}} dx. \tag{12}$$

Using Lebesgue term-by-term integration theorem (see [15]) and (12), then the left hand side of (10) can be written as follows:

$$J^p \leq \sum_{n=1}^{\infty} n^{p\lambda_2\alpha-1} n^{1-p\lambda_2\alpha} w^{p-1}(n) \int_b^c \frac{f^p(x)}{(x^\alpha + n^r)^\lambda} \frac{x^{(1-\alpha\lambda_1)(p-1)}}{n^{(1-\alpha\lambda_2)}} dx$$

$$= \int_b^c f^p(x) x^{\lambda_1\alpha} \sum_{n=1}^{\infty} \frac{n^{\lambda_2 - 1}}{(x^\alpha + n^r)^\lambda} w^{\frac{p}{q}}(n) x^{p(1-\lambda_1\alpha)-1} dx$$

$$= \left(\frac{1}{\alpha}\right)^{\frac{p}{q}} \int_b^c f^p(x) x^{\lambda_1\alpha} \sum_{n=1}^{\infty} \frac{n^{\lambda_2\alpha-1}}{(x^\alpha + n^r)^\lambda} n^{\lambda_2(\alpha-r)\frac{p}{q}}$$

$$\times (\beta(\lambda_1, \lambda_2) - \Psi(n))^{\frac{p}{q}} x^{p(1-\lambda_1\alpha)-1} dx,$$

which gives that

$$J \leq \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\int_b^c f^p(x)\overline{w}(x)\Phi^{\frac{p}{q}}(\lambda_1, \lambda_2, n)\, x^{p(1-\lambda_1\alpha)-1} dx \right)^{\frac{1}{p}}.$$

This completes the proof of (10). To prove (11), by Hölder's inequality and (10) we obtain

$$I := \sum_{n=1}^{\infty} a_n \int_b^c \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx$$

$$= \sum_{n=1}^{\infty} \left(n^{\frac{1}{p} - \lambda_2\alpha} a_n \right) \left(n^{\lambda_2\alpha - \frac{1}{p}} \int_b^c \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx \right)$$

$$\leq \left[\sum_{n=1}^{\infty} n^{\left(\lambda_2\alpha - \frac{1}{p}\right)p} \left(\int_b^c \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx \right)^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{\left(\frac{1}{p} - \lambda_2\alpha\right)q} a_n^q \right]^{\frac{1}{q}}$$

$$= J \left[\sum_{n=1}^{\infty} n^{\left(\frac{1}{p} - \lambda_2\alpha\right)q} a_n^q \right]^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\int_b^c f^p(x)\overline{w}(x)\Phi^{\frac{p}{q}}(\lambda_1, \lambda_2, n)\, x^{p(1-\lambda_1\alpha)-1} dx \right)^{\frac{1}{p}}$$

$$\times \left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2\alpha)-1} a_n^q \right)^{\frac{1}{q}}.$$

This completes the proof. □

As a special case of Theorem 2.1, focusing only on (11), when $c \to \infty$ and $b \to 0$ with $n < \infty$, which means that $\Psi(n) \equiv 0$, we have the following corollary.

Corollary 2.2. *Suppose that* $0 < \alpha, 0 < r \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *(*$p \neq 0, 1$*)*, $\lambda_1 > 0$, $p\lambda_2\alpha + (1-p)\lambda_2 r \leq 1$, $\lambda = \lambda_1 + \lambda_2$, $a_n \geq 0$, *and* $f(x) \geq 0$ *is a real measurable function in* $(0, \infty)$. *Then for* $p > 1$, *the following half-discrete Hilbert-type inequality holds:*

$$I := \sum_{n=1}^{\infty} a_n \int_0^\infty \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx \tag{13}$$

$$\leq \left(\frac{1}{\alpha} \beta(\lambda_1, \lambda_2) \right)^{\frac{1}{q}} \left(\frac{1}{r} \beta(\xi, \zeta) \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty x^{p\lambda_2\alpha(\frac{\alpha}{r}-1) + p(1-\lambda_1\alpha)-1} f^p(x) dx \right)^{\frac{1}{p}}$$

$$\times \left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2\alpha)-1} a_n^q \right)^{\frac{1}{q}},$$

where $\xi = \lambda_1 - p\lambda_2(\frac{\alpha}{r} - 1)$ *and* $\zeta = \lambda_2 + p\lambda_2(\frac{\alpha}{r} - 1)$.

Another special case is of Corollary 2.2 that is when $r = \alpha$, this leads to the following corollary (which has been proved in [10]).

Corollary 2.3. *Suppose that* $0 < \alpha \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *(*$p \neq 0, 1$*)*, $\lambda_1 > 0$, $\lambda_2\alpha \leq 1$, $\lambda = \lambda_1 + \lambda_2$, $a_n \geq 0$, *and* $f(x) \geq 0$ *is a real measurable function in* $(0, \infty)$. *Then for* $p > 1$, *the following half-discrete Hilbert-type inequality holds:*

$$I := \sum_{n=1}^{\infty} a_n \int_0^\infty \frac{f(x)}{(x^\alpha + n^\alpha)^\lambda} dx$$

$$\leq \frac{1}{\alpha} \beta(\lambda_1, \lambda_2) \left(\int_0^\infty x^{p(1-\lambda_1\alpha)-1} f^p(x) dx \right)^{\frac{1}{p}}$$

$$\times \left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2\alpha)-1} a_n^q \right)^{\frac{1}{q}}. \tag{14}$$

Remark 2.4. Putting $p = q = 2$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, and $\alpha = 1$ in (14) produces (3).

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