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ORIGINAL ARTICLE

Characterizations of mixture of two-component exponentiated family of distributions based on generalized order statistics

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Abstract In this paper, we shall characterize mixture of two components of exponentiated family of distributions based on recurrence relations for moment and conditional moment generating functions of generalized order statistics. Results for ordinary order statistics and upper kth record values are obtained as special cases.

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1. Introduction

The concept of the generalized order statistics (gOSs) is a mathematical generalization of all ordered data such as, ordinary order statistics (oOSs), ordinary record values (oRVs), sequential order statistics and ordering via truncated distributions can be discussed as they are special cases of the gOSs. This concept had been introduced by Kamps [1,2]. Keseling [3] characterized some continuous distributions based on conditional distributions of gOSs. Characterization of the exponential distribution based on independence of functions of gOSs and estimation of its parameters have been introduced by Ahsanullah [4]. Recurrence relations for moments of gOSs within a class of doubly truncated distributions have been derived by Ahmad and Fawzy [5]. AL-Hussaini et al. [6] obtained recurrence relations for moment and conditional moment generating functions of gOSs based on mixed distribution. Mahmoud and Ghazal [7,8] derived recurrence relations for moments, conditional moment generating functions and product moments of gOSs based on exponentiated family of distributions and doubly truncated exponentiated family of distributions. Many authors Abdel-Hamid and AL-Hussaini [9] and AL-Hussaini [10] studied specified exponentiated distributions from other points of view.

A mixture cumulative distribution function (cdf) of two components $F_1(x)$ and $F_2(x)$, is given, for $0 \leqslant p \leqslant 1$, by

$$F(x) = p\ F_1(x) + (1-p)\ F_2(x). \tag{1.1}$$

If for $i = 1, 2$, $F_i(x)$ belongs to exponentiated family of distributions, that is,

$$F_i(x) = (1 - e^{-\lambda(x)})^{\theta_i}, \quad x \geqslant 0, \tag{1.2}$$

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where $\lambda(x)$ is a non-negative, continuous, monotone increasing, differentiable function of x such that $\lambda(x) \to 0$ as $x \to 0^+$, $\lambda(x) \to \infty$ as $x \to \infty$ and the parameters $\theta_i > 0$.

The finite mixture probability density function (pdf) is given by

$$f(x) = p\, f_1(x) + (1-p)\, f_2(x), \tag{1.3}$$

where

$$f_i(x) = \theta_i \lambda'(x)\, e^{-\lambda(x)}\, [1 - e^{-\lambda(x)}]^{\theta_i - 1}, \quad i = 1, 2.$$

Using (1.1) and (1.2), then a mixture cdf $F(x)$ of such two components may be written as

$$F(x) = p\, (1 - e^{-\lambda(x)})^{\theta_1} + (1-p)\, (1 - e^{-\lambda(x)})^{\theta_2}. \tag{1.4}$$

We need to prove the following lemma.

Lemma 1. *Let X be a mixed random variable which has a cdf F(x) with F(x) being twice differentiable at x > 0 and 0 < F(x) < 1. Then the random variable X has the cdf (1.4) iff for distinct positive constants θ_1 and θ_2, the differential equation*

$$\left[\frac{(1 - e^{\lambda(x)})\, \overline{F}'(x)}{\lambda'(x)}\right]' \frac{(1 - e^{\lambda(x)})}{\lambda'(x)} + (\theta_1 + \theta_2) \frac{(1 - e^{\lambda(x)})\, \overline{F}'(x)}{\lambda'(x)} + \theta_1\theta_2\, \overline{F}(x) - \theta_1\theta_2 = 0, \tag{1.5}$$

is satisfied, where $\overline{F}(x) = 1 - F(x)$.

Proof. The necessary part is obvious. For sufficiency, we need to solve the differential Eq. (1.5). Eq. (1.5) can be rewritten as

$$\frac{(1 - e^{\lambda(x)})}{\lambda'(x)} \left[\frac{(1 - e^{\lambda(x)})\, F'(x)}{\lambda'(x)}\right]' + (\theta_1 + \theta_2) \frac{(1 - e^{\lambda(x)})\, F'(x)}{\lambda'(x)} + \theta_1\theta_2\, F(x) = 0. \tag{1.6}$$

Put $z = 1 - e^{-\lambda(x)}$, then (1.6) reduces to

$$z^2 \frac{d^2 F[\lambda^{-1}(ln(\frac{1}{1-z}))]}{dz^2} + [1 - (\theta_1 + \theta_2)]\, z \frac{dF[\lambda^{-1}(ln(\frac{1}{1-z}))]}{dz} + \theta_1\theta_2\, F\left[\lambda^{-1}\left(ln\left(\frac{1}{1-z}\right)\right)\right] = 0. \tag{1.7}$$

Setting $z = e^t$, then (1.7) reduces to

$$\frac{d^2 F[\lambda^{-1}(ln(\frac{1}{1-e^t}))]}{dt^2} - (\theta_1 + \theta_2) \frac{dF[\lambda^{-1}(ln(\frac{1}{1-e^t}))]}{dt} + \theta_1\theta_2\, F\left[\lambda^{-1}\left(ln\left(\frac{1}{1-e^t}\right)\right)\right] = 0. \tag{1.8}$$

The general solution of (1.8) is

$$F\left[\lambda^{-1}\left(ln\left(\frac{1}{1-e^t}\right)\right)\right] = p_1\, e^{\theta_1\, t} + p_2\, e^{\theta_2\, t},$$

or equivalently,

$$F(x) = p_1 (1 - e^{-\lambda(x)})^{\theta_1} + p_2 (1 - e^{-\lambda(x)})^{\theta_2},$$

where $p_1 + p_2 = 1$. □

2. Characterizations of finite mixtures based on recurrence relations of moment generating functions of gOSs

Let $X_{1;n;m;k}, X_{2;n;m;k}, \ldots, X_{n;n;m;k}$ be n gOSs from the pdf (1.3), (m and k are real numbers, $n > 1$ and $k \geqslant 1$). The pdf of $X_{r;n;m;k}$, $1 \leqslant r \leqslant n$, is given by Kamps [1] as follows:

$$f_{X_{r,n,m,k}}(x) = \frac{C_{r-1}}{(r-1)!} [\overline{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \quad x \in \chi, \tag{2.1}$$

where χ is the domain on which $f_{X_{r,n,m,k}}(x)$ is positive,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

and for $z \in (0, 1)$,

$$g_m(z) = \begin{cases} \frac{[1-(1-z)^{m+1}]}{m+1}, & \text{m} \neq -1, \\ -\ln(1-z), & \text{m} = -1. \end{cases}$$

Theorem 2.1. *Let X be a random variable, $r \geqslant 1$, m and k be real numbers such that $m \geqslant -1$, $k > 1$. Then for integers a such that $a \geqslant 1$, the following recurrence relation is satisfied iff X has the cdf (1.4)*

$$\begin{aligned} M^{(a)}_{r;n;m;k}(t) - M^{(a)}_{r-1;n;m;k}(t) &= \frac{a\, t(\theta_1 + \theta_2)}{\theta_1\theta_2\gamma_r} E\left[\frac{X^{a-1}_{r;n;m;k}\; e^{t\; X^a_{r;n;m;k}} (1 - e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right] \\ &+ \frac{a\, t(\gamma_r - 1)}{\theta_1\theta_2\gamma_r} E\left[\frac{X^{a-1}_{r;n;m;k}\; H(X_{r;n;m;k})\; e^{t\; X^a_{r;n;m;k}} (1 - e^{\lambda(X_{r;n;m;k})})^2}{\lambda'^{\,2}(X_{r;n;m;k})}\right] \\ &- \frac{a\, t}{\theta_1\theta_2\gamma_r} E\left[\left[\frac{X^{a-1}_{r;n;m;k}\; e^{t\; X^a_{r;n;m;k}} (1 - e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right]' \frac{(1 - e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right] \\ &- \frac{a\, t}{\theta_1\theta_2} E\left[\frac{X^{a-1}_{r-1;n;m;k}\; H(X_{r-1;n;m;k}) e^{t\; X^a_{r-1;n;m;k}} (1 - e^{\lambda(X_{r-1;n;m;k})})^2}{\lambda'^{\,2}(X_{r-1;n;m;k})}\right] \\ &+ \frac{a\; t\; C_{r-1}}{\gamma_r\; C^*_{r-1}}\; E\left[\frac{X^{a-1}_{r;n-1;m;k+m}\; e^{t\, X^a_{r;n-1;m;k+m}}}{H(X_{r;n-1;m;k+m})}\right], \end{aligned} \tag{2.2}$$

where $C^*_{r-1} = \prod_{i=1}^{r} \gamma^*_i, \gamma^*_i = \gamma_i - 1 = (k+m) + (n-1-i)(m+1), ' = \frac{d}{d\,x}$ and $H(x) = f(x)/\overline{F}(x)$.

Proof. If X has the cdf (1.4), from (2.1) we have

$$M^{(a)}_{r;n;m;k}(t) = E[e^{t\; X^{(a)}_{r;n;m;k}}] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty e^{t\; x^a} [\overline{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1} \times (F(x))\; dx = \frac{C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty e^{t\; x^a}\; g_m^{r-1}(F(x))\; d[-[\overline{F}(x)]^{\gamma_r}]. \tag{2.3}$$

Integrating by parts, we obtain

$$M^{(a)}_{r;n;m;k}(t) = \frac{at\; C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^{a-1}\; e^{t\; x^a}\; [\overline{F}(x)]^{\gamma_r}\; g_m^{r-1}(F(x))\; dx + \frac{C_{r-2}}{(r-2)!} \int_0^\infty e^{t\; x^a} [\overline{F}(x)]^{\gamma_{r-1} - 1}\; f(x)\; g_m^{r-2}(F(x))\; dx. \tag{2.4}$$

The second term in the right hand side is $M^{(a)}_{r-1;n;m;k}(t)$, so we obtain

$$M^{(a)}_{r;n;m;k}(t) - M^{(a)}_{r-1;n;m;k}(t) = \frac{a\ t\ C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^{a-1}\ e^{t\ x^a} [\overline{F}(x)]^{\gamma_r} \times g_m^{r-1}(F(x))\ dx, \tag{2.5}$$

which can be written as

$$M^{(a)}_{r;n;m;k}(t) - M^{(a)}_{r-1;n;m;k}(t) = \frac{a\ t\ C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty \frac{x^{a-1}\ e^{t\ x^a}}{\lambda'(x)} \times [\overline{F}(x)]^{\gamma_r - 1}\ g_m^{r-1}(F(x)) \{\lambda'(x)\overline{F}(x)\}\ dx. \tag{2.6}$$

Making use of (1.5) in (2.6) yields

$$M^{(a)}_{r;n;m;k}(t) - M^{(a)}_{r-1;n;m;k}(t) = \frac{a\, t\, C_{r-1}}{\theta_1\theta_2\gamma_r(r-1)!} \int_0^\infty \frac{x^{a-1}\, e^{t\, x^a}}{\lambda'(x)} [\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x)) \times \left\{ (\theta_1+\theta_2)(1-e^{\lambda(x)}) f(x) + \theta_1\theta_2\lambda'(x) + \left[\frac{(1-e^{\lambda(x)})f(x)}{\lambda'(x)}\right]' (1-e^{\lambda(x)}) \right\} dx.$$

Then

$$M^{(a)}_{r;n;m;k}(t) - M^{(a)}_{r-1;n;m;k}(t) = I_1 + I_2 + I, \tag{2.7}$$

where

$$I_1 = \frac{a\ t(\theta_1+\theta_2)\ C_{r-1}}{\theta_1\ \theta_2\ \gamma_r (r-1)!} \int_0^\infty \times \frac{x^{a-1}\ e^{t\ x^a}(1-e^{\lambda(x)})}{\lambda'(x)} [\overline{F}(x)]^{\gamma_r-1} f(x)\ g_m^{r-1}(F(x))\ dx = \frac{a\ t(\theta_1+\theta_2)}{\theta_1\theta_2\ \gamma_r} E\left[\frac{X^{a-1}_{r;n;m;k}\ e^{t\ X^a_{r;n;m;k}}(1-e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right].$$

$$I_2 = \frac{a\ t\ C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty \frac{x^{a-1}\ e^{t\ x^a}}{H(x)} [\overline{F}(x)]^{\gamma_r^*-1} f(x)\ g_m^{r-1}(F(x))\ dx = \frac{a\ t\ C_{r-1}}{\gamma_r\ C^*_{r-1}} E\left[\frac{X^{a-1}_{r;n-1;m;k+m}\ e^{t\ X^a_{r;n-1;m;k+m}}}{H(X_{r;n-1;m;k+m})}\right].$$

where $C^*_{r-1} = \prod_{i=1}^r \gamma_i^*, \gamma_i^* = \gamma_i - 1 = (k+m) + (n-1-i)\ (m+1)$.

$$I = \frac{a\, t\, C_{r-1}}{\theta_1\theta_2\gamma_r(r-1)!} \int_0^\infty \times \frac{x^{a-1}\, e^{t\, x^a}(1-e^{\lambda(x)})}{\lambda'(x)} [\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x))\, d\left[\frac{(1-e^{\lambda(x)})f(x)}{\lambda'(x)}\right].$$

Integrating by parts, gives

$$I = I_3 - I_4 - I_5, \tag{2.8}$$

where

$$I_3 = \frac{a\ t(\gamma_r-1)\ C_{r-1}}{\theta_1\theta_2\ \gamma_r (r-1)!} \int_0^\infty \times \frac{x^{a-1}\ H(x)\ e^{t\ x^a}(1-e^{\lambda(x)})^2}{\lambda'^{\,2}(x)} [\overline{F}(x)]^{\gamma_r-1} f(x)\ g_m^{r-1}(F(x))\ dx = \frac{a\ t(\gamma_r-1)}{\theta_1\theta_2\ \gamma_r} E\left[\frac{X^{a-1}_{r;n;m;k}\ H(X_{r;n;m;k})\ e^{t\ X^a_{r;n;m;k}}\ (1-e^{\lambda(X_{r;n;m;k})})^2}{\lambda'^{\,2}(X_{r;n;m;k})}\right].$$

$$I_4 = \frac{a\ t\ C_{r-1}}{\theta_1\theta_2\gamma_r (r-1)!} \int_0^\infty \left[\frac{x^{a-1}\ e^{t\ x^a}(1-e^{\lambda(x)})}{\lambda'(x)}\right]' \frac{(1-e^{\lambda(x)})}{\lambda'(x)} \times [\overline{F}(x)]^{\gamma_r-1} f(x)\ g_m^{r-1}(F(x))\ dx = \frac{a\ t}{\theta_1\theta_2\gamma_r} E\left[\left[\frac{X^{a-1}_{r;n;m;k}\ e^{t\ X^a_{r;n;m;k}}(1-e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right]' \frac{(1-e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right].$$

$$I_5 = \frac{a\ t(r-1)\ C_{r-1}}{\theta_1\theta_2\gamma_r (r-1)!} \int_0^\infty \frac{x^{a-1}\ H(x)\ e^{t\ x^a}(1-e^{\lambda(x)})^2}{\lambda'^{\,2}(x)} \times [\overline{F}(x)]^{\gamma_{r-1}-1} f(x)\ g_m^{r-2}(F(x))\ dx = \frac{a\ t}{\theta_1\theta_2} E\left[\frac{X^{a-1}_{r-1;n;m;k}\ H(X_{r-1;n;m;k})\ e^{t\ X^a_{r-1;n;m;k}}(1-e^{\lambda(X_{r-1;n;m;k})})^2}{\lambda'^{\,2}(X_{r-1;n;m;k})}\right],$$

where $H(x) = f(x)/\overline{F}(x)$. Substituting I_1, I_2 and I in (2.7), we obtain (2.2).

Conversely, if the characterizing Condition (2.2) holds, then from (2.7) and (2.8), we have

$$M^{(a)}_{r;n;m;k}(t) - M^{(a)}_{r-1;n;m;k}(t) = I_1 + I_2 + I_3 - I_4 - I_5. \tag{2.9}$$

I_3 can be written as

$$I_3 = \frac{a\ t\ C_{r-1}}{\theta_1\theta_2\gamma_r (r-1)!} \int_0^\infty \frac{x^{a-1}\ e^{t\ x^a}(1-e^{\lambda(x)})}{\lambda'(x)}\ g_m^{r-1}(F(x)) \times \left[\frac{(1-e^{\lambda(x)})\ f(x)}{\lambda'(x)}\right] d[-[\overline{F}(x)]^{\gamma_r-1}].$$

Upon integrating by parts, we obtain

$$I_3 = I_4 + I_5 + I_6, \tag{2.10}$$

where

$$I_6 = \frac{a\ t\ C_{r-1}}{\theta_1\theta_2\gamma_r (r-1)!} \int_0^\infty \frac{x^{a-1}\ e^{t\ x^a}(1-e^{\lambda(x)})}{\lambda'(x)} g_m^{r-1}(F(x)) \times \left[\frac{(1-e^{\lambda(x)})\ f(x)}{\lambda'(x)}\right]' \overline{F}(x)]^{\gamma_r-1}\ dx.$$

Substituting (2.10) in (2.8), we obtain

$$M^{(a)}_{r;n;m;k}(t) - M^{(a)}_{r-1;n;m;k}(t) = I_1 + I_2 + I_6. \tag{2.11}$$

Eq. (2.11) can be written as follows

$$\frac{a\ t\ C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^{a-1}\ e^{t\ x^a} [\overline{F}(x)]^{\gamma_r-1}\ g_m^{r-1}(F(x))\ dx \times \left\{ \overline{F}(x) - \frac{1}{\theta_1\theta_2}\left[\frac{(1-e^{\lambda(x)})\ f(x)}{\lambda'(x)}\right]' \frac{(1-e^{\lambda(x)})}{\lambda'(x)} - \frac{\theta_1+\theta_2}{\theta_1\theta_2} \frac{(1-e^{\lambda(x)})\ f(x)}{\lambda'(x)} - 1 \right\} dx = 0.$$

Applying the extension of Müntz–Szàsz theorem [11], we obtain the differential equation

$$\left[\frac{(1-e^{\lambda(x)})\ \overline{F}'(x)}{\lambda'(x)}\right]' \frac{(1-e^{\lambda(x)})}{\lambda'(x)} + (\theta_1+\theta_2) \frac{(1-e^{\lambda(x)})\ \overline{F}'(x)}{\lambda'(x)} + \theta_1\theta_2\ \overline{F}(x) - \theta_1\theta_2 = 0.$$

This is condition (1.5) in Lemma 1 which has the solution

$F(x) = p(1-e^{-\lambda(x)})^{\theta_1} + (1-p)(1-e^{-\lambda(x)})^{\theta_2}$. □

Remark 2.1. By differentiating both sides of (2.2) with respect to t and then setting $t = 0$, we obtain

$$\mu^{(a)}_{r;n;m;k} - \mu^{(a)}_{r-1;n;m;k} = \frac{a(\theta_1+\theta_2)}{\theta_1\theta_2\gamma_r} E\left[\frac{X^{a-1}_{r;n;m;k}(1-e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right] + \frac{a(\gamma_r-1)}{\theta_1\theta_2\gamma_r} E\left[\frac{X^{a-1}_{r;n;m;k}\ H(X_{r;n;m;k})\ (1-e^{\lambda(X_{r;n;m;k})})^2}{\lambda'^{\,2}(X_{r;n;m;k})}\right] - \frac{a}{\theta_1\theta_2\gamma_r} E\left[\left[\frac{X^{a-1}_{r;n;m;k}(1-e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right]' \frac{(1-e^{\lambda(X_{r;n;m;k})})}{\lambda'(X_{r;n;m;k})}\right] - \frac{a}{\theta_1\theta_2} E\left[\frac{X^{a-1}_{r-1;n;m;k}\ H(X_{r-1;n;m;k})(1-e^{\lambda(X_{r-1;n;m;k})})^2}{\lambda'^{\,2}(X_{r-1;n;m;k})}\right] + \frac{a\ C_{r-1}}{\gamma_r\ C^*_{r-1}} E\left[\frac{X^{a-1}_{r;n-1;m;k+m}}{H(X_{r;n-1;m;k+m})}\right], \tag{2.12}$$

where $\mu^{(a)}_{r;n;m;k} = E[X^a_{r;n;m;k}]$.

Remark 2.2. If we put $m = 0$ and $k = 1$ in (2.2) and (2.12), we obtain the recurrence relations of oOSs, in the form

$$M^{(a)}_{r;n}(t) - M^{(a)}_{r-1;n}(t) = \frac{a\ t(\theta_1+\theta_2)}{\theta_1\theta_2(n-r+1)} E\left[\frac{X^{a-1}_{r;n}\ e^{t\ X^a_{r;n}}(1-e^{\lambda(X_{r;n})})}{\lambda'(X_{r;n})}\right] + \frac{a\ t(n-r)}{\theta_1\theta_2(n-r+1)} E\left[\frac{X^{a-1}_{r;n}\ H(X_{r;n})\ e^{t\ X^a_{r;n}}\ (1-e^{\lambda(X_{r;n})})^2}{\lambda'^{\,2}(X_{r;n})}\right] - \frac{a\ t}{\theta_1\theta_2(n-r+1)} E\left[\left[\frac{X^{a-1}_{r;n}\ e^{t\ X^a_{r;n}}(1-e^{\lambda(X_{r;n})})}{\lambda'(X_{r;n})}\right]' \frac{(1-e^{\lambda(X_{r;n})})}{\lambda'(X_{r;n})}\right] - \frac{a\ t}{\theta_1\theta_2} E\left[\frac{X^{a-1}_{r-1;n}\ H(X_{r-1;n})\ e^{t\ X^a_{r-1;n}}(1-e^{\lambda(X_{r-1;n})})^2}{\lambda'^{\,2}(X_{r-1;n})}\right] + \frac{a\ t\ n}{(n-r)(n-r+1)} E\left[\frac{X^{a-1}_{r;n-1}\ e^{t\ X^a_{r;n-1}}}{H(X_{r;n-1})}\right]. \tag{2.13}$$

$$\mu^{(a)}_{r;n} - \mu^{(a)}_{r-1;n} = \frac{a(\theta_1+\theta_2)}{\theta_1\theta_2(n-r+1)} E\left[\frac{X^{a-1}_{r;n}\ (1-e^{\lambda(X_{r;n})})}{\lambda'(X_{r;n})}\right] + \frac{a(n-r)}{\theta_1\theta_2(n-r+1)} E\left[\frac{X^{a-1}_{r;n}\ H(X_{r;n})\ (1-e^{\lambda(X_{r;n})})^2}{\lambda'^{\,2}(X_{r;n})}\right] - \frac{a}{\theta_1\theta_2(n-r+1)} E\left[\left[\frac{X^{a-1}_{r;n}(1-e^{\lambda(X_{r;n})})}{\lambda'(X_{r;n})}\right]' \frac{(1-e^{\lambda(X_{r;n})})}{\lambda'(X_{r;n})}\right] - \frac{a}{\theta_1\theta_2} E\left[\frac{X^{a-1}_{r-1;n}\ H(X_{r-1;n})(1-e^{\lambda(X_{r-1;n})})^2}{\lambda'^{\,2}(X_{r-1;n})}\right] + \frac{a\,n}{(n-r)(n-r+1)} E\left[\frac{X^{a-1}_{r;n-1}}{H(X_{r;n-1})}\right]. \tag{2.14}$$

Remark 2.3. If we put $m = -1$ in (2.2) and (2.12), upper kth Rvs, we have

$$M^{(a)}_{U(r),\,k}(t) - M^{(a)}_{U(r),\,k}(t) = \frac{a\,t(\theta_1+\theta_2)}{k\ \theta_1\theta_2} \times E\left[\frac{X^{a-1}_{U(r),\,k}\ e^{t\ X^a_{U(r),\,k}}(1-e^{\lambda(X_{U(r),\,k})})}{\lambda'(X_{U(r),\,k})}\right] + \frac{a\,t(k-1)}{k\ \theta_1\theta_2} \times E\left[\frac{X^{a-1}_{U(r),\,k}\ H(X_{U(r),\,k})\ e^{t\ X^a_{U(r),\,k}}(1-e^{\lambda(X_{U(r),\,k})})^2}{\lambda'^{\,2}(X_{U(r),\,k})}\right] - \frac{a\ t}{k\ \theta_1\theta_2} E\left[\left[\frac{X^{a-1}_{U(r),\,k}\ e^{t\ X^a_{U(r),\,k}}(1-e^{\lambda(X_{U(r),\,k})})}{\lambda'(X_{U(r),\,k})}\right]' \frac{(1-e^{\lambda(X_{U(r),\,k})})}{\lambda'(X_{U(r),\,k})}\right] - \frac{a\ t}{\theta_1\theta_2} E\left[\frac{X^{a-1}_{U(r-1),\,k}\ H(X_{U(r-1),\,k})\ e^{t\ X^a_{U(r-1),\,k}}(1-e^{\lambda(X_{U(r-1),\,k})})^2}{\lambda'^{\,2}(X_{U(r-1),\,k})}\right] + \frac{a\ t}{k(1-k^{-1})^r} E\left[\frac{X^{a-1}_{U(r),\,k-1}\ e^{t\ X^a_{U(r),\,k-1}}}{H(X_{U(r),\,k-1})}\right]. \tag{2.15}$$

$$\mu^{(a)}_{U(r),\ k} - \mu^{(a)}_{U(r),\ k} = \frac{a(\theta_1+\theta_2)}{k\ \theta_1\theta_2} E\left[\frac{X^{a-1}_{U(r),\ k}(1-e^{\lambda(X_{U(r),\ k})})}{\lambda'(X_{U(r),\ k})}\right] + \frac{a(k-1)}{k\ \theta_1\theta_2} E\left[\frac{X^{a-1}_{U(r),\ k}\ H(X_{U(r),\ k})(1-e^{\lambda(X_{U(r),\ k})})^2}{\lambda'^{\,2}(X_{U(r),\ k})}\right] - \frac{a}{k\ \theta_1\theta_2} E\left[\left[\frac{X^{a-1}_{U(r),\ k}(1-e^{\lambda(X_{U(r),\ k})})}{\lambda'(X_{U(r),\ k})}\right]' \frac{(1-e^{\lambda(X_{U(r),\ k})})}{\lambda'(X_{U(r),\ k})}\right] - \frac{a}{\theta_1\theta_2} E\left[\frac{X^{a-1}_{U(r-1),\ k}\ H(X_{U(r-1),\ k})(1-e^{\lambda(X_{U(r-1),\ k})})^2}{\lambda'^{\,2}(X_{U(r-1),\ k})}\right] + \frac{a}{k(1-k^{-1})^r} E\left[\frac{X^{a-1}_{U(r),\ k-1}}{H(X_{U(r),\ k-1})}\right], \tag{2.16}$$

where $X_{U(r),\ k}$ is the upper kth record values.

3. Characterizations of finite mixtures based on recurrence relations for conditional moment generating functions of gOSs

Theorem 3.1. *Let X be a random variable, r, s be two integers such that $1 \leqslant r \leqslant s \leqslant n$, m and k be real numbers such that $m \geqslant -1$, $k > 1$. Then for integers a such that $a \geqslant 1$, the following recurrence relation is satisfied iff X has the cdf (1.4)*

$$M_{X^a_{s;n;m;k}|X_{r;n;m;k}}(t|y) - M_{X^a_{s-1;n;m;k}|X_{r;n;m;k}}(t|y) = \frac{a\,t(\theta_1+\theta_2)}{\theta_1\theta_2\gamma_s} \times E\left[\frac{x^{a-1}_{s;n;m;k}\ e^{t\ x^a_{s;n;m;k}}(1-e^{\lambda(x_{s;n;m;k})})}{\lambda'(x_{s;n;m;k})}|X_{r;n;m;k}=y\right] + \frac{a\,t(\gamma_s-1)}{\theta_1\theta_2\gamma_s} \times E\left[\frac{x^{a-1}_{s;n;m;k}\ H(x_{s;n;m;k})\ e^{t\ x^a_{s;n;m;k}}\ (1-e^{\lambda(x_{s;n;m;k})})^2}{\lambda'^{\,2}(x_{s;n;m;k})}|X_{r;n;m;k}=y\right] - \frac{a\,t}{\theta_1\theta_2\gamma_s} E\left[\left[\frac{x^{a-1}_{s;n;m;k}\ e^{t\ x^a_{s;n;m;k}}(1-e^{\lambda(x_{s;n;m;k})})}{\lambda'(x_{s;n;m;k})}\right]' \frac{(1-e^{\lambda(x_{s;n;m;k})})}{\lambda'(x_{s;n;m;k})}|X_{r;n;m;k}=y\right] - \frac{a\,t}{\theta_1\theta_2} E\left[\frac{x^{a-1}_{s-1;n;m;k}\ H(x_{s-1;n;m;k})\ e^{t\ x^a_{s-1;n;m;k}}(1-e^{\lambda(x_{s-1;n;m;k})})^2}{\lambda'^{\,2}(x_{s-1;n;m;k})}|\ X_{r;n;m;k}=y\right] + \frac{a\,t\,C_{s-1}\ C^*_{r-1}[\overline{F}_d(x)]^{\gamma^*_{r+1}-\gamma_{r+1}}}{\gamma_s\ C_{r-1}\ C^*_{s-1}} E\left[\frac{x^{a-1}_{s;n-1;m;k+m}\ e^{t\ x^a_{s;n-1;m;k+m}}}{H(x_{s;n-1;m;k+m})}|X_{r;n;m;k}=y\right]. \tag{3.1}$$

It easy to prove this theorem in the same manner of Theorem 2.1

Remark 3.1. By differentiating both sides of (3.1) with respect to t and then setting $t = 0$, we obtain the following recurrence relation for moments of gOSs

$$E\left[X^a_{s;n;m;k}|X_{r;n;m;k}=y\right]-E\left[X^a_{s-1;n;m;k}|X_{r;n;m;k}=y\right]=\frac{a(\theta_1+\theta_2)}{\theta_1\theta_2\gamma_s}$$
$$\times E\left[\frac{X^{a-1}_{s;n;m;k}(1-e^{\lambda(X_{s;n;m;k})})}{\lambda'(X_{s;n;m;k})}|X_{r;n;m;k}=y\right]+\frac{a(\gamma_s-1)}{\theta_1\theta_2\gamma_s}$$
$$\times E\left[\frac{X^{a-1}_{s;n;m;k}\,H(X_{s;n;m;k})\,(1-e^{\lambda(X_{s;n;m;k})})^2}{\lambda'^2(X_{s;n;m;k})}|X_{r;n;m;k}=y\right]$$
$$-\frac{a}{\theta_1\theta_2\gamma_s}\,E\left[\left[\frac{X^{a-1}_{s;n;m;k}(1-e^{\lambda(X_{s;n;m;k})})}{\lambda'(X_{s;n;m;k})}\right]'\frac{(1-e^{\lambda(X_{s;n;m;k})})}{\lambda'(X_{s;n;m;k})}|X_{r;n;m;k}=y\right]$$
$$+\frac{a}{\theta_1\theta_2}\,E\left[\frac{X^{a-1}_{s-1;n;m;k}\,H(X_{s-1;n;m;k})(1-e^{\lambda(X_{s-1;n;m;k})})^2}{\lambda'^2(X_{s-1;n;m;k})}|X_{r;n;m;k}=y\right]$$
$$-\frac{a\,C_{s-1}\,C^*_{r-1}[\overline{F}_d(x)]^{\gamma^*_{r+1}-\gamma_{r+1}}}{\gamma_s\,C_{r-1}\,C^*_{s-1}}\,E\left[\frac{X^{a-1}_{s;n-1;m;k+m}}{H(X_{s;n-1;m;k+m})}|X_{r;n;m;k}=y\right]. \tag{3.2}$$

Remark 4.2. If we put $m = 0$ and $k = 1$ in (3.1) and (3.2), we obtain

$$M_{X^a_{s;n}|X_{r;n}}(t|y)-M_{X^a_{s-1;n}|X_{r;n}}(t|y)=\frac{a\,t(\theta_1+\theta_2)}{\theta_1\theta_2(n-s+1)}$$
$$\times E\left[\frac{X^{a-1}_{s;n}\,e^{t\,X^a_{s;n}}(1-e^{\lambda(X_{s;n})})}{\lambda'(X_{s;n})}|X_{r;n}=y\right]+\frac{a\,t(n-s)}{\theta_1\theta_2(n-s+1)}$$
$$\times E\left[\frac{X^{a-1}_{s;n}\,H(X_{s;n})\,e^{t\,X^a_{s;n}}\,(1-e^{\lambda(X_{s;n})})^2}{\lambda'^2(X_{s;n})}|X_{r;n}=y\right]$$
$$-\frac{a\,t}{\theta_1\theta_2(n-s+1)}\,E\left[\left[\frac{X^{a-1}_{s;n}\,e^{t\,X^a_{s;n}}(1-e^{\lambda(X_{s;n})})}{\lambda'(X_{s;n})}\right]'\frac{(1-e^{\lambda(X_{s;n})})}{\lambda'(X_{s;n})}|X_{r;n}=y\right]$$
$$-\frac{a\,t}{\theta_1\theta_2}E\left[\frac{X^{a-1}_{s-1;n}\,H(X_{s-1;n})\,e^{t\,X^a_{s-1;n}}(1-e^{\lambda(X_{s-1;n})})^2}{\lambda'^2(X_{s-1;n})}|X_{r;n}=y\right]$$
$$+\frac{a\,t(n-r)}{(n-s)(n-s+1)\overline{F}_d(x)}E\left[\frac{X^{a-1}_{s;n-1}\,e^{t\,X^a_{s;n-1}}}{H(X_{s;n-1})}|X_{r;n}=y\right], \tag{3.3}$$

$$E\left[X^a_{s;n}|X_{r;n}=y\right]-E\left[X^a_{s-1;n}|X_{r;n}=y\right]=\frac{a(\theta_1+\theta_2)}{\theta_1\theta_2(n-s+1)}$$
$$\times E\left[\frac{X^{a-1}_{s;n}(1-e^{\lambda(X_{s;n})})}{\lambda'(X_{s;n})}|X_{r;n}=y\right]+\frac{a(n-s)}{\theta_1\theta_2(n-s+1)}$$
$$\times E\left[\frac{X^{a-1}_{s;n}\,H(X_{s;n})\,(1-e^{\lambda(X_{s;n})})^2}{\lambda'^2(X_{s;n})}|X_{r;n}=y\right]-\frac{a}{\theta_1\theta_2(n-s+1)}$$
$$\times E\left[\left[\frac{X^{a-1}_{s;n}(1-e^{\lambda(X_{s;n})})}{\lambda'(X_{s;n})}\right]'\frac{(1-e^{\lambda(X_{s;n})})}{\lambda'(X_{s;n})}|X_{r;n}=y\right]$$
$$+\frac{a}{\theta_1\theta_2}E\left[\frac{X^{a-1}_{s-1;n}\,H(X_{s-1;n})(1-e^{\lambda(X_{s-1;n})})^2}{\lambda'^2(X_{s-1;n})}|X_{r;n}=y\right]$$
$$+\frac{a(n-r)}{(n-s)(n-s+1)\overline{F}_d(x)}\,E\left[\frac{X^{a-1}_{s;n-1}}{H(X_{s;n-1})}|X_{r;n}=y\right]. \tag{3.4}$$

Remark 4.3. If we put $m = -1$ in (3.1) and (3.2), can be deduced

$$M_{X^a_{U(s),\,k}|X_{U(r),\,k}}(t|y)-M_{X^a_{U(s-1),\,k}|X_{U(r),\,k}}(t|y)=\frac{a\,t(\theta_1+\theta_2)}{k\,\theta_1\theta_2}$$
$$\times E\left[\frac{X^{a-1}_{U(s),\,k}\,e^{t\,X^a_{U(s),\,k}}(1-e^{\lambda(X_{U(s),\,k})})}{\lambda'(X_{U(s),\,k})}|X_{U(r),\,k}=y\right]+\frac{a\,t(k-1)}{k\,\theta_1\theta_2}$$
$$\times\,E\left[\frac{X^{a-1}_{U(s),\,k}\,H(X_{U(s),\,k})\,e^{t\,X^a_{U(s),\,k}}(1-e^{\lambda(X_{U(s),\,k})})^2}{\lambda'^2(X_{U(s),\,k})}|X_{U(r),\,k}=y\right]$$
$$-\frac{a\,t}{k\,\theta_1\theta_2}\,E\left[\left[\frac{X^{a-1}_{U(s),\,k}\,e^{t\,X^a_{U(s),\,k}}(1-e^{\lambda(X_{U(s),\,k})})}{\lambda'(X_{U(s),\,k})}\right]'\frac{(1-e^{\lambda(X_{U(s),\,k})})}{\lambda'(X_{U(s),\,k})}|X_{U(r),\,k}=y\right]$$
$$-\frac{a\,t}{\theta_1\theta_2}E\left[\frac{X^{a-1}_{U(s-1),\,k}\,H(X_{U(s-1),\,k})e^{t\,X^a_{U(s-1),\,k}}(1-e^{\lambda(X_{U(s-1),\,k})})^2}{\lambda'^2(X_{U(s-1),\,k})}|X_{U(r),\,k}=y\right]$$
$$+\frac{a\,t(1-k^{-1})^r}{k(1-k^{-1})^s\overline{F}_d(x)}E\left[\frac{X^{a-1}_{U(s),\,k-1}\,e^{t\,X^a_{U(s),\,k-1}}}{H(X_{U(s),\,k-1})}|X_{U(r),\,k}=y\right]. \tag{3.5}$$

Table 1 Examples of (1.4) distributions.

Distribution	$F(x)$
Mixture of two exponentiated Weibull	$p(1-e^{-\alpha x^\beta})^{\theta_1}+(1-p)(1-e^{-\alpha x^\beta})^{\theta_2}$ $\lambda(x) = \alpha x^\beta,\quad x,\,\alpha,\,\beta > 0$
Mixture of two exponentiated exponential	$p(1-e^{-\alpha x})^{\theta_1}+(1-p)(1-e^{-\alpha x})^{\theta_2}$ $\lambda(x) = \alpha x,\quad x,\,\alpha > 0$
Mixture of two exponentiated Rayleigh	$p(1-e^{-\alpha x^2})^{\theta_1}+(1-p)(1-e^{-\alpha x^2})^{\theta_2}$ $\lambda(x) = \alpha x^2,\quad x,\,\alpha > 0$
Mixture of two exponentiated Pareto	$p(1-(1+x)^{-\alpha})^{\theta_1}+(1-p)(1-(1+x)^{-\alpha})^{\theta_2}$ $\lambda(x) = \alpha\ln(1+x),\quad x,\,\alpha > 0$
Mixture of two exponentiated Gamma	$p(1-e^{-x}(1+x))^{\theta_1}+(1-p)(1-e^{-x}(1+x))^{\theta_2}$ $\lambda(x) = x-\ln(1+x),\quad x > 0$
Mixture of two exponentiated Beta	$p\left(1-(1-x)^{\frac{1}{\alpha}}\right)^{\theta_1}+(1-p)\left(1-(1-x)^{\frac{1}{\alpha}}\right)^{\theta_2}$ $\lambda(x) = \frac{1}{\alpha}\,\ln\left(\frac{1}{1-x}\right),\quad 0<x<1,\alpha>0$
Mixture of two exponentiated Power function	$p(1-(1-x)^{\alpha})^{\theta_1}+(1-p)(1-(1-x)^{\alpha})^{\theta_2}$ $\lambda(x) = -\alpha\ln(1-x),\quad 0>x>1,\,\alpha > 0$

$$E\left[X^a_{U(s),\,k}|X_{U(r),\,k}=y\right]-E\left[X^a_{U(s-1),\,k}|X_{U(r),\,k}=y\right]$$
$$=\frac{a(\theta_1+\theta_2)}{k\,\theta_1\theta_2}\,E\left[\frac{X^{a-1}_{U(s),\,k}(1-e^{\lambda(X_{U(s),\,k})})}{\lambda'(X_{U(s),\,k})}|X_{U(r),\,k}=y\right]$$
$$+\frac{a(k-1)}{k\,\theta_1\theta_2}E\left[\frac{X^{a-1}_{U(s),\,k}\,H(X_{U(s),\,k})\,(1-e^{\lambda(X_{U(s),\,k})})^2}{\lambda'^{\,2}(X_{U(s),\,k})}|X_{U(r),\,k}=y\right]$$
$$-\frac{a}{k\,\theta_1\theta_2}\,E\left[\left[\frac{X^{a-1}_{U(s),\,k}(1-e^{\lambda(X_{U(s),\,k})})}{\lambda'(X_{U(s),\,k})}\right]'\frac{(1-e^{\lambda(X_{U(s),\,k})})}{\lambda'(X_{U(s),\,k})}|X_{U(r),\,k}=y\right]$$
$$-\frac{a}{\theta_1\theta_2}\,E\left[\frac{X^{a-1}_{U(s-1),\,k}\,H(X_{U(s-1),\,k})(1-e^{\lambda(X_{U(s-1),\,k})})^2}{\lambda'^{\,2}(X_{U(s-1),\,k})}|X_{U(r),\,k}=y\right]$$
$$+\frac{a(1-k^{-1})^r}{k(1-k^{-1})^s\overline{F_d}(x)}E\left[\frac{X^{a-1}_{U(s),\,k-1}}{H(X_{U(s),\,k-1})}|X_{U(r),\,k}=y\right]. \tag{3.6}$$

The previous table gives some distributions with proper choice of $\lambda(x)$ as examples on Theorems 2.1, 3.1 (see Table 1).

4. Conclusions

In this paper we succeeded to characterize a finite mixtures of two components of exponentiated family of distributions based on recurrence relations for moment and conditional moment generating functions of generalized order statistics. Results for ordinary order statistics and kth record values are obtained as special cases.

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