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ORIGINAL ARTICLE

Mean square convergent three points finite difference scheme for random partial differential equations

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Abstract In this paper, the random finite difference method with three points is used in solving random partial differential equations problems mainly: random parabolic, elliptic and hyperbolic partial differential equations. The conditions of the mean square convergence of the numerical solutions are studied. The numerical solutions are computed through some numerical case studies.
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1. Introduction

Random partial differential equations (RPDEs) are defined as partial differential equations involving random inputs which may be a random variable or a stochastic process. In recent years, some of the main numerical methods for solving stochastic partial differential equations (SPDEs), like finite difference and finite element schemes, have been considered [1–8]. Various numerical methods and approximation schemes for RPDEs and ordinary differential equations have also been developed, analyzed, and tested (see [9–19]). This paper solves some RPDEs in mean square sense using random finite difference scheme. Mean square consistency of the random difference scheme for (RPDE) is established. Sufficient conditions for the mean square stability of the proposed numerical solution are given.

This paper is organized as follows. In Section 2, some important preliminaries are discussed. In Section 3, the random parabolic partial differential equation is discussed. Section 4 discusses the random elliptic partial differential equation. In Section 5, the random hyperbolic partial differential equation is discussed. Section 6 presents some new results. Section 7 presents the solution of some numerical examples in random partial differential equations using random finite difference method. The general conclusions are presented in the end section.

2. Preliminaries

2.1. Mean square calculus

Definition 1 [16]. Let us consider the properties of a class of real r.v.'s

$X_{11}, X_{12}, \ldots, X_{21}, X_{22}, \ldots, X_{nk}, \ldots$ Whose second moments, $E\{X_{11}^2\}, E\{X_{12}^2\}, \ldots, E\{X_{nk}^2\}, \ldots$ are finite. In this case, they are called "second order random variables", (2.r.v's).

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Definition 2 [16]. The linear vector space of second order random variables with inner product, norm and distance, is called an L_2-space. A s.p. $\{X(t),\ t \in T\}$ is called a **"second order stochastic process"** (2.s.p) if for $t_1, t_2, \ldots, t_n$, the r.v's $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ are elements of L_2-space. A second order s.p. $\{X(t),\ t \in T\}$ is characterized by

$$\|X(t)\|^2 = E\{X^2(t)\} < \infty, \quad t \in T$$

2.1.1. The convergence in mean square [16]

A sequence of r.v's $\{X_{nk},\ n,\ k > o\}$ converges in mean square (m.s) to a random variable X if:

$$l.i.m_{n,k\to\infty}\|X_{nk} - X\| = 0 \text{ i.e. } X_{nk} \xrightarrow{m.s} X \text{ or } L.i.m_{n,k\to\infty} X_{nk} = X$$

where l.i.m is the limit in mean square sense.

3. Random parabolic partial differential equation

This section of the paper is interested in studying the following random parabolic differential problem of the form:

$$\begin{aligned} &u_t(x,t) = \beta u_{xx}(x,t), \quad t \in [0,T], \quad x \in [0,X] \\ &u(x,0) = u_0(x), \quad x \in [0,X] \\ &u(0,t) = u(X,t) = 0 \end{aligned} \tag{3.1}$$

Randomness may exist in the initial condition, in the differential equation it self or in the boundary conditions. The random finite difference method is used to obtain an approximate solution for problem (3.1).

3.1. Random Finite Difference Scheme (RFDS)

In this section, we extend one kind of the finite difference methods to random case in order to approximate random parabolic differential equations of the form:

$$u_t(x,t) = \beta u_{xx}(x,t), \ \beta(\text{random variable}), \ t \in [0,T], \ x \in [0,X] \tag{3.2}$$

$$u(x,0) = u_0(x), \ x \in [0,X] \tag{3.3}$$

$$u(0,t) = u(X,t) = 0 \tag{3.4}$$

For difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis. Let u_k^n approximates $u(x,t)$ at point $(k\Delta x, n\Delta t)$. Hence $u_k^0 = u_0(k\Delta x)$. On this mesh, we have:

$$u_t(x,t) \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \quad \text{Then} \quad u_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} - u_k^n}{\Delta t}$$

Similarly:

$$u_{xx}(x,t) \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \quad \text{Then} \quad u_{xx}(k\Delta x, n\Delta t) \approx \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2}$$

Hence for (3.1):

$$u_k^{n+1} = u_k^n + r\beta\left(u_{k+1}^n - 2u_k^n + u_{k-1}^n\right) \tag{3.5}$$

$$u_k^0 = u_0(x) \tag{3.6}$$

$$u_0^n = u_X^n = 0 \tag{3.7}$$

where $r = \frac{\Delta t}{\Delta x^2}$.

The above scheme is a random version of (3.5)–(3.7). For a RPDE, say: $Lv = G$, where L is a differential operator and $G \in L^2(R)$. In the other hand, we represent finite difference scheme at the point $(k\Delta x, n\Delta t)$ by $L_k^n u_k^n = G_k^n$.

The above random difference scheme (3.5) can be written in the form:

$$u_k^{n+1} = (1 - 2r\beta)u_k^n + r\beta\left(u_{k+1}^n + u_{k-1}^n\right).$$

3.2. Consistency of RFDS

Definition 3 (17–19). A random difference scheme $L_k^n u_k^n = G_k^n$ approximating RPDE $Lv = G$ is **consistent** in mean square at time $t = (n+1)\Delta t$, if fo r any continuously differentiable function $\Phi = \Phi(x,t)$, we have in mean square:

$$E\left|(L\Phi - G)_k^n - \left(L_k^n\Phi(k\Delta x, n\Delta t) - G_k^n\right)\right|^2 \to 0$$
$$\text{As} \quad \Delta x \to 0, \quad \Delta t \to 0 \quad \text{and} \quad (k\Delta x, n\Delta t) \to (x,t)$$

Theorem 3.1. *The random difference scheme* (3.5)–(3.7) *is* **consistent** *in mean square sense*

$$\text{as} \quad \Delta x \to 0, \quad \Delta t \to 0, \quad E(\beta) \to 0 \quad \text{and} \quad (k\Delta x, n\Delta t) \to (x,t)$$

Proof. Assume that $\Phi(x,t)$ is a smooth function then:

$$L(\Phi)|_k^n = \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \beta\int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)ds$$

and

$$\begin{aligned} L_k^n\Phi = {} & \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - r(\Phi((k+1)\Delta x, n\Delta t) \\ & - 2\Phi(k\Delta x, n\Delta t) + \Phi((k-1)\Delta x, n\Delta t)). \end{aligned}$$

Then we have

$$\begin{aligned} E\left|(L\Phi)_k^n - L_k^n\Phi\right|^2 &= E\left[-\beta\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds + \beta\frac{\Delta t}{\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t) - 2\Phi(k\Delta x,n\Delta t) + \Phi((k-1)\Delta x,n\Delta t))\right]^2 \\ &= E\left[-\beta\left[\int_{n\Delta t}^{(n+1)\Delta t}\Phi_x(k\Delta x,s)ds - \frac{\Delta t}{\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t) - 2\Phi(k\Delta x,n\Delta t) + \Phi((k-1)\Delta x,n\Delta t))\right]\right]^2 \\ &= E\left[\beta^2\left[\int_{n\Delta t}^{(n+1)\Delta t}\Phi_x(k\Delta x,s)ds - \frac{\Delta t}{\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t) - 2\Phi(k\Delta x,n\Delta t) + \Phi((k-1)\Delta x,n\Delta t))\right]^2\right] \\ &= E\left[\beta^2\left[\int_{n\Delta t}^{(n+1)\Delta t}\Phi_x(k\Delta x,s)ds\right]^2 - 2\beta^2\left[\left(\int_{n\Delta t}^{(n+1)\Delta t}\Phi_x(k\Delta x,s)ds\right)\left(\frac{k^2\Delta t}{k^2\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t) - 2\Phi(k\Delta x,n\Delta t)\right.\right.\right. \\ &\quad \left.\left. + \Phi((k-1)\Delta x,n\Delta t)))\right] + \beta^2\left[\frac{k^2\Delta t}{k^2\Delta x^2}(\Phi(k+1)\Delta x,n\Delta t) - 2\Phi(k\Delta x,n\Delta t) + \Phi((k-1)\Delta x,n\Delta t)\right]^2\right] \end{aligned}$$

If $(k\Delta x,\ n\Delta t) \to (x,t)$ and $\Delta x,\ \Delta t,\ E(\beta) \to 0$ then $E\left|(L\Phi)_k^n - \left(L_k^n\Phi(k\Delta x, n\Delta t)\right)\right|^2 \to 0$

Hence the random difference scheme (3.5)–(3.7) is **consistent** in mean square sense. □

3.3. Stability of RFDS

Definition 4 (17–19). A random difference scheme is **stable** in mean square if there exist some positive constants ε,δ and constants k, b such that:

$$E\left|u_k^{n+1}\right|^2 \leqslant k e^{bt} E|u^0|^2$$

For all $0 \leqslant t = (n+1)\Delta t 0 \leqslant \Delta x \leqslant \varepsilon$ and $0 \leqslant \Delta t \leqslant \delta$.

Theorem 3.2. *The random difference scheme* (3.5)–(3.7) *is* **stable** *in mean square sense.*

Proof. Since $u_k^{n+1} = (1-2\beta r)u_k^n + r\beta(u_{k+1}^n + u_{k-1}^n)$ then

$$\begin{aligned} E\left|u_k^{n+1}\right|^2 &= E\left|(1-2r\beta)u_k^n + r\beta\left(u_{k+1}^n + u_{k-1}^n\right)\right|^2 \\ &= E\Big[(1-2r\beta)^2\left|u_k^n\right|^2 + 2|r\beta||1-2r\beta|\left|u_k^n\left(u_{k+1}^n + u_{k-1}^n\right)\right| \\ &\quad + \left(r\beta\right)^2\left|u_{k+1}^n + u_{k-1}^n\right|^2\Big] \\ &= E(1-2r\beta)^2E\left|u_k^n\right|^2 + 2E|r\beta||1-2r\beta|E\left|u_k^n\left(u_{k+1}^n + u_{k-1}^n\right)\right| \\ &\quad + E(r\beta)^2E\left|u_{k+1}^n + u_{k-1}^n\right|^2 \\ &= E(1-2r\beta)^2E\left|u_k^n\right|^2 + 2E|r\beta||1-2r\beta|E\left|u_k^n\left(u_{k+1}^n\right)\right. \\ &\quad \left. + u_k^n u_{k-1}^n\right| + E(r\beta)^2E\left|u_{k+1}^n + u_{k-1}^n\right|^2. \end{aligned}$$

Since:

$$E|X+Y|^S \leqslant k(E|X|^s + E|Y|^s), \quad k = \begin{cases} 1 & s \leqslant 1 \\ 2^{s-1} & s \geqslant 1 \end{cases}$$

then we have

$$\begin{aligned} &= E(1-2r\beta)^2E\left|u_k^n\right|^2 + 2E|r\beta||1-2r\beta|E\left|u_k^n(u_{k+1}^n) + u_k^n u_{k-1}^n\right| \\ &\quad + E(r\beta)^2E\left|u_{k+1}^n + u_{k-1}^n\right|^2 \\ &\leqslant E(1-2r\beta)^2E\left|u_k^n\right|^2 + 2E|r\beta||1 \\ &\quad - 2r\beta|\left(E\left|u_k^n\left(u_{k+1}^n\right)\right| + E\left|u_k^n u_{k-1}^n\right|\right) \\ &\quad + 2E(r\beta)^2\left(E\left|u_{k+1}^n\right|^2 + E\left|u_{k-1}^n\right|^2\right) \\ &\leqslant E(1-2r\beta)^2\sup_k E\left|u_k^n\right|^2 + 4E|r\beta||1-2r\beta|\sup_k E\left|u_k^n\right|^2 \\ &\quad + 4E(r\beta)^2\sup_k E\left|u_k^n\right|^2 \end{aligned}$$

since: $|r\beta||1-2r\beta| = |r\beta(2r\beta-1)|$ then

$$\begin{aligned} &= E|1-2r\beta|^2\sup_k E\left|u_k^n\right|^2 + 4E|r\beta(2r\beta-1)|\sup_k E\left|u_k^n\right|^2 \\ &\quad + 4E|r\beta|^2\sup_k E\left|u_k^n\right|^2 \end{aligned}$$

From: "Jensen's inequality": $|E(X)|^P \leqslant E(|X|^P),\ p \geqslant 1$ we have

$$\begin{aligned} &= E|1-2r\beta|^2\sup_k E\left|u_k^n\right|^2 + 4E|r\beta(2r\beta-1)|\sup_k E\left|u_k^n\right|^2 \\ &\quad + 4E|r\beta|^2\sup_k E\left|u_k^n\right|^2 \\ &\quad = |E(1-2r\beta)|^2\sup_k E\left|u_k^n\right|^2 + 4|E[r\beta(2r\beta-1)]|\sup_k E\left|u_k^n\right|^2 \\ &\quad\quad + 4|E(r\beta)|^2\sup_k E\left|u_k^n\right|^2 \\ &\quad = [|E(1-2r\beta)| + 2|E(r\beta)|]^2\sup_k E\left|u_k^n\right|^2 \\ &\quad = [|1-2E(r\beta)| + 2|E(r\beta)|]^2\sup_k E\left|u_k^n\right|^2 \end{aligned}$$

Now, with: $0 \leqslant E(r\beta) = rE(\beta) = \frac{\Delta t}{\Delta x^2}E(\beta) \leqslant \frac{1}{2}$ then

$|1-2E(r\beta)| = |1| - |2E(r\beta)| = 1 - 2|E(r\beta)|$.

Finally, we get

$$E\left|u_k^{n+1}\right|^2 \leqslant \sup_k E(u_k^n)^2$$

This holds for all k, and so, we have:

$$\sup_k E\left|u_k^{n+1}\right|^2 \leqslant \sup_k E\left|u_k^n\right|^2 \leqslant \sup_k E\left|u_k^{n-1}\right|^2 \leqslant \cdots \leqslant \sup_k E\left|u_k^0\right|^2$$

Then:

$$E|u^{n+1}|^2 \leqslant e^{0(t)}E|u^0|^2$$

where $k = 1$ and $b = 0$, then the random difference scheme (3.5)–(3.7) is **stable** in mean square sense with the condition $0 \leqslant E(r\beta) = rE(\beta) = \frac{\Delta t}{\Delta x^2}E(\beta) \leqslant \frac{1}{2}$. □

3.4. Convergence of RFDS

Definition 5 (17–19). A random difference scheme $L_k^n u_k^n = G_k^n$ approximating RPDE $Lv = G$ is **convergent** in mean square at time $t = (n+1)\Delta t$, if:

$$E\left|u_k^n - u\right|^2 \to 0$$

As $\Delta x \to 0$, $\Delta t \to 0$ and $(k\Delta x, n\Delta t) \to (x,t)$

3.4.1. A stochastic version of Lax–Richtmyer theorem [19]

A random difference scheme $L_k^n u_k^n = G_k^n$ approximating SPDE $Lv = G$ is **convergent** in mean square at time $t = (n+1)\Delta t$, if it is **consistent** and **stable**.

Theorem 3.3. *The random difference scheme* (3.5)–(3.7) *is* **convergent** *in mean square sense*

$$\text{as}\quad \Delta x \to 0,\quad \Delta t \to 0,\quad E(\beta) \to 0\quad \text{and}\quad (k\Delta x, n\Delta t) \to (x,t)$$

Proof.

$$E\left|u_k^n - u\right|^2 = E\left|(L_k^n)^{-1}\left(L_k^n u_k^n - L_k^n u\right)\right|^2$$

Since the scheme is **consistent** then we have

$$L_k^n u_k^n \overset{m.s}{\to} L_k^n u.$$

Then we obtain $E\left|L_k^n u_k^n - L_k^n u\right|^2 \to 0$ as $\Delta x \to 0$, $\Delta t \to 0$, $E(\beta) \to 0$ and $(k\Delta x,\ n\Delta t) \to (x,t)$.

Since the scheme is **stable** then $(L_k^n)^{-1}$ is bounded.

Hence

$$E\left|u_k^n - u\right|^2 \to 0 \quad \text{As} \quad \Delta x \to 0, \quad \Delta t \to 0, \quad E(\beta) \to 0.$$

Then the random difference scheme (3.5)–(3.7) is **convergent** in mean square sense. □

4. Random elliptic partial differential equation

This section is interested in studying the following random elliptic differential problem of the form:

$$\begin{aligned} &u_{yy}(x,y) + \beta u_{xx}(x,y) = 0, \quad y \in [0,m], \quad x \in [0,l], \quad \beta > 0 \\ &u(x,0) = u_0(x), \quad u(x,m) = u_m(x) \\ &u(0,y) = u_0(y), \quad u(l,y) = u_l(y) \end{aligned} \tag{4.1}$$

Randomness may exist in the differential equation (it self), in the initial conditions or in the boundary conditions. The random finite difference method is used to obtain an approximate solution for problem (4.1).

4.1. Random Finite Difference Scheme (RFDS)

In this section, we extend one kind of the finite difference methods to random case in order to approximate random elliptic differential equations of the form:

$$u_{yy}(x,y) + \beta u_{xx}(x,y) = 0, \quad y \in [0,m], \quad x \in [0,l], \quad \beta(\text{random variable}), \quad \beta > 0 \tag{4.2}$$

$$u(x,0) = u_0(x), \quad u(x,m) = u_m(x) \tag{4.3}$$

$$u(0,y) = u_0(y), \quad u(l,y) = u_l(y) \tag{4.4}$$

For the difference method, consider a uniform mesh with step size Δx and Δy on x and y axes. u_k^n will approximate $u(x,y)$ at point $(k\Delta x, n\Delta y)$. Hence

$$u_k^0 = u_0(k\Delta x), \quad u_k^m = u_m(k\Delta x), \quad u_0^n = u_0(n\Delta y), \quad u_l^n = u_l(n\Delta y).$$

On this mesh, we have:

$$u_{xx}(x,y) \approx \frac{u(x+\Delta x,y) - 2u(x,y) + u(x-\Delta x,y)}{\Delta x^2}$$

$$\text{Then} \quad u_{xx}(k\Delta x, n\Delta y) \approx \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2}$$

Similarly:

$$u_{yy}(x,y) \approx \frac{u(x,y+\Delta y) - 2u(x,y) + u(x,y-\Delta y)}{\Delta y^2}$$

$$\text{Then} \quad u_{yy}(k\Delta x, n\Delta y) \approx \frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{\Delta y^2}$$

Hence for (4.2)

$$u_k^{n+1} = (2+2r\beta)u_k^n - u_k^{n-1} - r\beta\left(u_{k+1}^n + u_{k-1}^n\right), \quad y \in [0,m], \quad x \in [0,l] \tag{4.5}$$

$$u(x,0) = u_0(x), \quad u(x,m) = u_m(x) \tag{4.6}$$

$$u(0,y) = u_0(y), \quad u(l,y) = u_l(y) \tag{4.7}$$

where $r = \frac{\Delta y^2}{\Delta x^2}$.

The above scheme is a random version of (4.5)–(4.7).

It is possible to prove the **consistency**, **stability** and the **convergence** of RFDS (4.5)–(4.7) according to Definitions 3–5 as follows.

4.2. Consistency of RFDS (4.5)–(4.7)

Theorem 4.1. *The random difference scheme* (4.5)–(4.7) *is* **consistent** *in mean square sense*

$$\text{as} \quad \Delta x \to 0, \quad \Delta y \to 0, \quad E(\beta) \to 0 \quad \text{and} \quad (k\Delta x, n\Delta y) \to (x,y)$$

Proof. Assume that $\Phi(x,y)$ be a smooth function then:

$$L(\Phi)|_k^n = \int_{k\Delta x}^{(k+1)\Delta x} \Phi_{yy}(z, n\Delta y)dz + \beta \int_{n\Delta y}^{(n+1)\Delta y} \Phi_{xx}(k\Delta x, s)ds$$

and

$$\begin{aligned} L_k^n\Phi = {} & \Phi(k\Delta x,(n+1)\Delta y) - 2\Phi(k\Delta x,n\Delta y) + \Phi(k\Delta x,(n-1)\Delta y) \\ & + r(\Phi((k+1)\Delta x,n\Delta y) - 2\Phi(k\Delta x,n\Delta y) \\ & + \Phi((k-1)\Delta x,n\Delta y)). \end{aligned}$$

Then we have:

$$\begin{aligned} E\left|(L\Phi)_k^n - L_k^n\Phi\right|^2 = {} & E\Big[\int_{k\Delta x}^{(k+1)\Delta x} \Phi_{yy}(z,n\Delta y)dz \\ & +\beta\int_{n\Delta y}^{(n+1)\Delta y} \Phi_{xx}(k\Delta x,s)ds - \Phi(k\Delta x,(n+1)\Delta y) \\ & +2\Phi(k\Delta x,n\Delta y) - \Phi(k\Delta x,(n-1)\Delta y) \\ & -\beta\frac{\Delta y^2}{\Delta x^2}(\Phi((k+1)\Delta x,n\Delta y) - 2\Phi(k\Delta x,n\Delta y) \\ & +\Phi((k-1)\Delta x,n\Delta y))\Big]^2 \\ = {} & E\Big[\Big[\int_{k\Delta x}^{(k+1)\Delta x} \Phi_{tt}(z,n\Delta y)dz \\ & +\beta\int_{n\Delta y}^{(n+1)\Delta y} \Phi_{xx}(k\Delta x,s)ds\Big]^2 \\ & -2\Big[\int_{k\Delta x}^{(k+1)\Delta x} \Phi_{yy}(z,n\Delta y)dz \\ & +\beta\int_{n\Delta y}^{(n+1)\Delta y} \Phi_{xx}(k\Delta x,s)ds\Big][\Phi(k\Delta x,(n+1)\Delta y) \\ & -2\Phi(k\Delta x,n\Delta y) + \Phi(k\Delta x,(n-1)\Delta y) \\ & +\beta\frac{k^2\Delta y^2}{k^2\Delta x^2}(\Phi((k+1)\Delta x,n\Delta y) - 2\Phi(k\Delta x,n\Delta y) \\ & +\Phi((k-1)\Delta x,n\Delta y)) \\ & +[\Phi(k\Delta x,(n+1)\Delta y) - 2\Phi(k\Delta x,n\Delta y) \\ & +\Phi(k\Delta x,(n-1)\Delta y) \\ & +\beta\frac{k^2\Delta y^2}{k^2\Delta x^2}(\Phi((k+1)\Delta x,n\Delta y) - 2\Phi(k\Delta x,n\Delta y) \\ & +\Phi((k-1)\Delta x,n\Delta y))]^2\Big] \end{aligned}$$

If $(k\Delta x, \; n\Delta y) \to (x,y)$ and $\Delta x, \; \Delta y, \; E(\beta) \to 0$ then $E\left|(L\Phi)_k^n - (L_k^n\Phi(k\Delta x, n\Delta y))\right|^2 \to 0$.

Hence the random difference scheme (4.5)–(4.7) is **consistent** in mean square sense. □

4.3. Stability of RFDS (4.5)–(4.7)

Theorem 4.2. *The random difference scheme (4.5)–(4.7) is* **stable** *in mean square sense.*

Proof. Since $u_k^{n+1} = (2+2\beta r)u_k^n - u_k^{n-1} - r\beta(u_{k+1}^n + u_{k-1}^n)$ **then**

$$E|u_k^{n+1}|^2 = E|(2+2r\beta)u_k^n - u_k^{n-1} - r\beta(u_{k+1}^n + u_{k-1}^n)|^2$$
$$= E\Big[(2+2r\beta)^2|u_k^n|^2 + |u_k^{n-1}|^2 - 2|2+2r\beta||u_k^n u_k^{n-1}|$$
$$-2[|2+2r\beta||u_k^n| - |u_k^{n-1}|][|r\beta||u_{k+1}^n + u_{k-1}^n|]$$
$$+\Big[(r\beta)^2|u_{k+1}^n + u_{k-1}^n|^2\Big]$$
$$= E(2+2r\beta)^2 E|u_k^n|^2 + E|u_k^{n-1}|^2$$
$$-2E|2+2r\beta|E|u_k^n u_k^{n-1}| - 2[E|2+2r\beta|E|u_k^n|$$
$$-E|u_k^{n-1}|][E|r\beta|E|u_{k+1}^n + u_{k-1}^n|] + E(r\beta)^2 E|u_{k+1}^n$$
$$+u_{k-1}^n|^2 = E(2+2r\beta)^2 E|u_k^n|^2 + E|u_k^{n-1}|^2$$
$$-2E|2+2r\beta|E|u_k^n u_k^{n-1}| - 2E|2+2r\beta|E|u_k^n|E|r\beta|E|u_{k+1}^n$$
$$+u_{k-1}^n| + 2E|u_k^{n-1}|E|r\beta|E|u_{k+1}^n + u_{k-1}^n|$$
$$+E(r\beta)^2 E|u_{k+1}^n + u_{k-1}^n|^2$$
$$= E(2+2r\beta)^2 E|u_k^n|^2 + E|u_k^{n-1}|^2 - 2E|2+2r\beta|E|u_k^n u_k^{n-1}|$$
$$-2E|2+2r\beta||r\beta|E|u_k^n(u_{k+1}^n + u_{k-1}^n)|$$
$$+2E|r\beta|E|u_k^{n-1}(u_{k+1}^n + u_{k-1}^n)| + E(r\beta)^2 E|u_{k+1}^n + u_{k-1}^n|^2$$
$$= E(2+2r\beta)^2 E|u_k^n|^2 + E|u_k^{n-1}|^2 - 2E|2$$
$$+2r\beta|E|u_k^n u_k^{n-1}| - 2E|2+2r\beta||r\beta|E|u_k^n u_{k+1}^n + u_k^n u_{k-1}^n|$$
$$+2E|r\beta|E|u_k^{n-1}u_{k+1}^n + u_k^{n-1}u_{k-1}^n| + E(r\beta)^2 E|u_{k+1}^n + u_{k-1}^n|^2$$

since

$$E|X+Y|^S \leqslant k(E|X|^s + E|Y|^s), \quad k = \begin{cases} 1 & s \leqslant 1 \\ 2^{s-1} & s \geqslant 1 \end{cases}$$

then we have

$$= E(2+2r\beta)^2 E|u_k^n|^2 + E|u_k^{n-1}|^2 - 2E|2+2r\beta|E|u_k^n u_k^{n-1}|$$
$$-2E|2+2r\beta||r\beta|[E|u_k^n u_{k+1}^n| + E|u_k^n u_{k-1}^n|]$$
$$+2E|r\beta|[E|u_k^{n-1}u_{k+1}^n| + E|u_k^{n-1}u_{k-1}^n|] + 2E(r\beta)^2\Big[E|u_{k+1}^n|^2 + E|u_{k-1}^n|^2\Big]$$
$$\leqslant E(2+2r\beta)^2 \sup_{k,n} E|u_k^n|^2 + \sup_{k,n} E|u_k^n|^2$$
$$-2E|2+2r\beta|\sup_{k,n} E|u_k^n|^2 - 4E|2+2r\beta||r\beta|\sup_{k,n} E|u_k^n|^2$$
$$+4E|r\beta|\sup_{k,n} E|u_k^n|^2 + 4E(r\beta)^2 \sup_{k,n} E|u_k^n|^2$$

since:

$$|r\beta||1-2r\beta| = |r\beta(2r\beta-1)|$$

then

$$= E|2+2r\beta|^2 \sup_{k,n} E|u_k^n|^2 + \sup_{k,n} E|u_k^n|^2 - 2E|2$$
$$+2r\beta|\sup_{k,n} E|u_k^n|^2 - 4E|r\beta(2+2r\beta)|\sup_{k,n} E|u_k^n|^2$$
$$+4E|r\beta|\sup_{k,n} E|u_k^n|^2 + 4E|r\beta|^2 \sup_{k,n} E|u_k^n|^2$$

From: "Jensen's inequality": $|E(X)|^P \leqslant E(|X|^P)$, $p \geqslant 1$ we have

$$= |E(2+2r\beta)|^2 \sup_{k,n} E|u_k^n|^2 + \sup_{k,n} E|u_k^n|^2$$
$$-2|E(2+2r\beta)|\sup_{k,n} E|u_k^n|^2 - 4|E[r\beta(2+2r\beta)]|\sup_{k,n} E|u_k^n|^2$$
$$+4|E(r\beta)|\sup_{k,n} E|u_k^n|^2 + 4|E(r\beta)|^2 \sup_{k,n} E|u_k^n|^2$$
$$= [|E(2+2r\beta)| - 2|E(r\beta)|]^2 + 1 - 2|E[(2+2r\beta)]|$$
$$+4|E(r\beta)|]\sup_{k,n} E|u_k^n|^2$$
$$= [|2+2E(r\beta)| - 2|E(r\beta)|]^2 + 1 - 2|2+2E(r\beta)|$$
$$+4|E(r\beta)|]\sup_{k,n} E|u_k^n|^2 = [2|1+E(r\beta)| - 2|E(r\beta)|]^2$$
$$+1 - 4|1+E(r\beta)| + 4|E(r\beta)|]\sup_{k,n} E|u_k^n|^2$$

Now, with: $0 \leqslant E(r\beta) = rE(\beta) = \frac{\Delta y^2}{\Delta x^2} E(\beta) \leqslant \infty$ then $|1 + E(r\beta)| = 1 + |E(r\beta)|$.

Finally, we get:

$$E|u_k^{n+1}|^2 \leqslant \sup_k E(u_k^n)^2.$$

Hence

$$\sup_k E|u_k^{n+1}|^2 \leqslant \sup_k E|u_k^n|^2 \leqslant \sup_k E|u_k^{n-1}|^2 \leqslant \cdots \leqslant \sup_k E|u_k^0|^2$$

Then

$$E|u^{n+1}|^2 \leqslant \sup_k E|u^0|^2,$$

where $k = 1$ and $b = 0$ then the random difference scheme (4.5)–(4.7) is **stable** in mean square sense with the condition $0 \leqslant E(r\beta) = rE(\beta) = \frac{\Delta y^2}{\Delta x^2} E(\beta) \leqslant 1$. □

4.4. Convergence of RFDS (4.5)–(4.7)

4.4.1. A stochastic version of Lax–Richtmyer theorem [19]
A random difference scheme $L_k^n u_k^n = G_k^n$ approximating SPDE $Lv = G$ is **convergent** in mean square at time$t = (n+1)\Delta t$, if it is **consistent** and **stable**.

Theorem 4.3. *The random difference scheme (4.5)–(4.7) is* **convergent** *in mean square sense*

As $\Delta x \to 0$, $\Delta y \to 0$, $E(\beta) \to 0$ and $(k\Delta x, n\Delta y) \to (x, y)$

Proof.

$$E|u_k^n - u|^2 = E\Big|(L_k^n)^{-1}(L_k^n u_k^n - L_k^n u)\Big|^2$$

Since the scheme (4.5)–(4.7) is **consistent** then we have:

$$L_k^n u_k^n \overset{m.s}{\to} L_k^n u.$$

Then we obtain $E|L_k^n u_k^n - L_k^n u|^2 \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$, $E(\beta) \to 0$ and $(k\Delta x, n\Delta y) \to (x, y)$.

Since the scheme is **stable** then $(L_k^n)^{-1}$ is bounded.

Hence

$$E|u_k^n - u|^2 \to 0 \quad \text{As} \quad \Delta x \to 0, \quad \Delta y \to 0, \quad E(\beta) \to 0.$$

Then the random difference scheme (4.5)–(4.7) is **convergent** in mean square sense. □

5. Random hyperbolic partial differential equation

This section is interested in studying the following random hyperbolic differential problem of the form:

$$\begin{aligned} &u_{tt}(x,t)=\beta^2 u_{xx}(x,t), \quad t\in[0,m], \quad x\in[0,l]\\ &u(x,0)=u_0(x)\\ &u(0,t)=0, \quad u(l,t)=0\\ &\frac{\partial u(x,0)}{\partial t}=g(x)\end{aligned} \tag{5.1}$$

Randomness may exist in the differential equation (it self), in the initial conditions or in the boundary conditions. The random finite difference method is used to obtain an approximate solution for problem (5.1).

5.1. Random Finite Difference Scheme (RFDS)

In this section, we extend one kind of the finite difference methods to random case in order to approximate random hyperbolic differential equations of the form:

$$u_{tt}(x,t)=\beta^2 u_{xx}(x,t), \quad t\in[0,m], \quad \beta(\text{random variable}), \quad x\in[0,l] \tag{5.2}$$

$$u(x,0)=u_0(x) \tag{5.3}$$

$$u(0,t)=0, \quad u(l,t)=0 \tag{5.4}$$

$$\frac{\partial u(x,0)}{\partial t}=g(x) \tag{5.5}$$

For difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis. u_k^n will approximate $u(x,t)$ at point ($k\Delta x$, $n\Delta t$).

On this mesh, we have:

$$u_{xx}(x,t)\approx\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2}$$

$$\text{Then} \quad u_{xx}(k\Delta x,n\Delta t)\approx\frac{u_{k+1}^n-2u_k^n+u_{k-1}^n}{\Delta x^2}.$$

Similarly

$$u_{tt}(x,t)\approx\frac{u(x,t+\Delta t)-2u(x,t)+u(x,t-\Delta t)}{\Delta t^2}$$

$$\text{Then} \quad u_{tt}(k\Delta x,n\Delta t)\approx\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{\Delta t^2}.$$

Hence for (5.2)–(5.5):

$$u_k^{n+1}=(2-2r\beta^2)u_k^n-u_k^{n-1}+r\beta^2(u_{k+1}^n+u_{k-1}^n), \quad t\in[0,m], \quad x\in[0,l] \tag{5.6}$$

$$u(x,0)=u_0(x) \tag{5.7}$$

$$u(0,t)=0, \quad u(l,t)=0 \tag{5.8}$$

$$\frac{\partial u(x,0)}{\partial t}=g(x) \tag{5.9}$$

where $r=\frac{\Delta t^2}{\Delta x^2}$.

The above scheme is a random version of (5.6)–(5.9).

It is possible to prove the **consistency**, **stability** and the **convergence** of RFDS (5.6)–(5.9) according to Definitions 3–5 as follows:

5.2. Consistency of RFDS (5.6)–(5.9)

Theorem 5.1. *The random difference scheme* (5.6)–(5.9) *is* **consistent** *in mean square sense* as $\Delta x\to 0$, $\Delta t\to 0$, $E(\beta)\to 0$ and $(k\Delta x,n\Delta t)\to(x,t)$.

Proof. Assume that $\Phi(x,t)$ be a smooth function then:

$$L(\Phi)|_k^n=\int_{k\Delta x}^{(k+1)\Delta x}\Phi_{tt}(z,n\Delta t)dz-\beta^2\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds$$

and

$$\begin{aligned} L_k^n\Phi&=\Phi(k\Delta x,(n+1)\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi(k\Delta x,(n-1)\Delta t)\\ &\quad-r(\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t))\end{aligned}$$

Then we have:

$$\begin{aligned} E\left|(L\Phi)_k^n-L_k^n\Phi\right|^2&=E\Bigg[\Bigg[\int_{k\Delta x}^{(k+1)\Delta x}\Phi_{tt}(z,n\Delta t)dz-\beta^2\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds\Bigg]\\ &\quad-[\Phi(k\Delta x,(n+1)\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi(k\Delta x,(n-1)\Delta t)\\ &\quad-\beta^2\frac{\Delta t^2}{\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t))\Bigg]\Bigg]^2\\ &=E\Bigg[\Bigg[\int_{k\Delta x}^{(k+1)\Delta x}\Phi_{tt}(z,n\Delta t)dz-\beta^2\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds\Bigg]^2\\ &\quad-2\Bigg[\int_{k\Delta x}^{(k+1)\Delta x}\Phi_{tt}(z,n\Delta t)dz-\beta^2\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds\Bigg]\\ &\quad\times[\Phi(k\Delta x,(n+1)\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi(k\Delta x,(n-1)\Delta t)\\ &\quad-\beta^2\frac{k^2\Delta t^2}{k^2\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t))\Bigg]\\ &\quad+[\Phi(k\Delta x,(n+1)\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi(k\Delta x,(n-1)\Delta t)\\ &\quad-\beta^2\frac{k^2\Delta t^2}{k^2\Delta x^2}(\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t))]^2\Bigg]\end{aligned}$$

If $(k\Delta x,\ n\Delta t)\to(x,t)$ and Δx, Δt, $E(\beta)\to 0$ then $E\left|(L\Phi)_k^n-\left(L_k^n\Phi(k\Delta x,n\Delta t)\right)\right|^2\to 0$

Hence the random difference scheme (5.6)–(5.9) is **consistent** in mean square sense. □

5.3. Stability of RFDS (5.6)–(5.9)

Theorem 5.2. *The random difference scheme* (5.6)–(5.9) *is* **stable** *in mean square sense.*

Proof. Since $u_k^{n+1}=(2-2\beta^2 r)u_k^n-u_k^{n-1}+r\beta^2(u_{k+1}^n+u_{k-1}^n)$ then:

$$\begin{aligned} E\left|u_k^{n+1}\right|^2&=E\left|(2-2r\beta^2)u_k^n-u_k^{n-1}+r\beta^2(u_{k+1}^n+u_{k-1}^n)\right|^2\\ &=E\Big[(2-2r\beta^2)^2\left|u_k^n\right|^2+\left|u_k^{n-1}\right|^2-2|2-2r\beta^2|\left|u_k^nu_k^{n-1}\right|\\ &\quad+2\left[|2-2r\beta^2|\left|u_k^n\right|-\left|u_k^{n-1}\right|\right]\left[|r\beta^2|\left|u_{k+1}^n+u_{k-1}^n\right|\right]\\ &\quad+\left[(r\beta^2)^2\left|u_{k+1}^n+u_{k-1}^n\right|^2\right]\Big]\\ &=E(2-2r\beta^2)^2E\left|u_k^n\right|^2\\ &\quad+E\left|u_k^{n-1}\right|^2-2E|2-2r\beta^2|E\left|u_k^nu_k^{n-1}\right|\\ &\quad+2\left[E|2-2r\beta^2|E\left|u_k^n\right|-E\left|u_k^{n-1}\right|\right]\left[E|r\beta^2|E\left|u_{k+1}^n+u_{k-1}^n\right|\right]\\ &\quad+E(r\beta^2)^2E\left|u_{k+1}^n+u_{k-1}^n\right|^2\end{aligned}$$

$$= E(2-2r\beta^2)^2E|u_k^n|^2 + E|u_k^{n-1}|^2 - 2E|2-2r\beta^2|E|u_k^nu_k^{n-1}| + 2E|2-2r\beta^2|E|u_k^n|E|r\beta^2|E|u_{k+1}^n + u_{k-1}^n| - 2E|u_k^{n-1}|E|r\beta^2|E|u_{k+1}^n + u_{k-1}^n| + E(r\beta^2)^2E|u_{k+1}^n + u_{k-1}^n|^2$$
$$= E(2-2r\beta^2)^2E|u_k^n|^2 + E|u_k^{n-1}|^2 - 2E|2-2r\beta^2|E|u_k^nu_k^{n-1}| + 2E|2-2r\beta^2||r\beta^2|E|u_k^n(u_{k+1}^n + u_{k-1}^n)| - 2E|r\beta^2|E|u_k^{n-1}(u_{k+1}^n + u_{k-1}^n)| + E(r\beta^2)^2E|u_{k+1}^n + u_{k-1}^n|^2$$
$$= E(2-2r\beta^2)^2E|u_k^n|^2 + E|u_k^{n-1}|^2 - 2E|2-2r\beta^2|E|u_k^nu_k^{n-1}| + 2E|2-2r\beta^2||r\beta^2|E|u_k^nu_{k+1}^n + u_k^nu_{k-1}^n| - 2E|r\beta^2|E|u_k^{n-1}u_{k+1}^n + u_k^{n-1}u_{k-1}^n| + E(r\beta^2)^2E|u_{k+1}^n + u_{k-1}^n|^2$$

Since:

$$E|X+Y|^S \leqslant k(E|X|^s + E|Y|^s), \quad k = \begin{cases} 1 & s \leqslant 1 \\ 2^{s-1} & s \geqslant 1 \end{cases}$$

then we have:

$$= E(2-2r\beta^2)^2E|u_k^n|^2 + E|u_k^{n-1}|^2 - 2E|2-2r\beta^2|E|u_k^nu_k^{n-1}| + 2E|2-2r\beta^2||r\beta^2|[E|u_k^nu_{k+1}^n| + E|u_k^nu_{k-1}^n|] - 2E|r\beta^2|[E|u_k^{n-1}u_{k+1}^n| + E|u_k^{n-1}u_{k-1}^n|] + 2E(r\beta^2)^2\left[E|u_{k+1}^n|^2 + E|u_{k-1}^n|^2\right]$$
$$\leqslant E(2-2r\beta^2)^2\sup_{k,n}E|u_k^n|^2 + \sup_{k,n}E|u_k^n|^2 - 2E|2-2r\beta^2|\sup_{k,n}E|u_k^n|^2 + 4E|2-2r\beta^2||r\beta^2|\sup_{k,n}E|u_k^n|^2 - 4E|r\beta^2|\sup_{k,n}E|u_k^n|^2 + 4E(r\beta^2)^2\sup_{k,n}E|u_k^n|^2$$

Since:

$$|r\beta^2||2-2r\beta^2| = |r\beta^2(2r\beta^2-2)|$$

then:

$$= E|2-2r\beta^2|^2\sup_{k,n}E|u_k^n|^2 + \sup_{k,n}E|u_k^n|^2 - 2E|2 - 2r\beta^2|\sup_{k,n}E|u_k^n|^2 + 4E|r\beta^2(2-2r\beta^2)|\sup_{k,n}E|u_k^n|^2 - 4E|r\beta^2|\sup_{k,n}E|u_k^n|^2 + 4E|r\beta^2|^2\sup_{k,n}E|u_k^n|^2$$

From: "Jensen's inequality": $|E(X)|^P \leqslant E(|X|^P)$, $p \geqslant 1$ then we have:

$$= |E(2-2r\beta^2)|^2\sup_{k,n}E|u_k^n|^2 + \sup_{k,n}E|u_k^n|^2 - 2|E(2-2r\beta^2)| \times \sup_{k,n}E|u_k^n|^2 + 4|E[r\beta^2(2-2r\beta^2)]|\sup_{k,n}E|u_k^n|^2 - 4|E(r\beta^2)| \times \sup_{k,n}E|u_k^n|^2 + 4|E(r\beta^2)|^2\sup_{k,n}E|u_k^n|^2$$
$$= \left[\left|E(2-2r\beta^2)\right| + 2\left|E(r\beta^2)\right|\right]^2 + 1 - 2|E[(2-2r\beta^2)]| - 4\left|E(r\beta^2)\right|\Big] \times \sup_{k,n}E|u_k^n|^2 = [|2-2E(r\beta^2)| + 2|E(r\beta^2)|]^2 + 1 - 2|2-2E(r\beta^2)| - 4|E(r\beta^2)|]\sup_{k,n}E|u_k^n|^2 = [2|1-E(r\beta^2)| + 2|E(r\beta^2)|]^2 + 1 - 4|1-E(r\beta^2)| - 4|E(r\beta^2)|]\sup_{k,n}E|u_k^n|^2$$

Now, with: $0 \leqslant E(r\beta^2) = rE(\beta^2) = \frac{\Delta t^2}{\Delta x^2}E(\beta^2) \leqslant 1$ then $|1 - E(r\beta^2)| = 1 - |E(r\beta^2)|$ then we get:

$$E|u_k^{n+1}|^2 \leqslant \sup_k E(u_k^n)^2.$$

Hence

$$\sup_k E|u_k^{n+1}|^2 \leqslant \sup_k E|u_k^n|^2 \leqslant \sup_k E|u_k^{n-1}|^2 \leqslant \cdots \leqslant \sup_k E|u_k^0|^2,$$

then

$$E|u^{n+1}|^2 \leqslant \sup_k E|u^0|^2,$$

where $k = 1$ and $b = 0$ then the random difference scheme (5.6)–(5.9) is **stable** in mean square sense with the condition $0 \leqslant E(r\beta^2) = rE(\beta^2) = \frac{\Delta t^2}{\Delta x^2}E(\beta^2) \leqslant 1$. □

5.4. Convergence of RFDS (5.6)–(5.9)

5.4.1. A stochastic version of Lax–Richtmyer theorem [19]

A random difference scheme $L_k^nu_k^n = G_k^n$ approximating SPDE $Lv = G$ is **convergent** in mean square at time $t = (n+1)\Delta t$, if it is **consistent** and **stable**.

Theorem 5.3. *The random difference scheme* (5.6)–(5.9) *is* **convergent** *in mean square sense*

$$\text{as} \quad \Delta x \to 0, \quad \Delta t \to 0, \quad E(\beta) \to 0 \quad \text{and} \quad (k\Delta x, n\Delta t) \to (x,t).$$

Proof.

$$E|u_k^n - u|^2 = E\left|\left(L_k^n\right)^{-1}\left(L_k^nu_k^n - L_k^nu\right)\right|^2$$

Since the scheme (5.6)–(5.9) is **consistent** then we have

$$L_k^nu_k^n \overset{m.s}{\to} L_k^nu,$$

then we obtain $E|L_k^nu_k^n - L_k^nu|^2 \to 0$ as $\Delta x \to 0$, $\Delta t \to 0$, $E(\beta) \to 0$ and $(k\Delta x, n\Delta t) \to (x,t)$.

Since the scheme is **stable** then $(L_k^n)^{-1}$ is bounded.

Hence

$$E|u_k^n - u|^2 \to 0 \quad \text{As} \quad \Delta x \to 0, \quad \Delta t \to 0, \quad E(\beta) \to 0.$$

The random difference scheme (5.6)–(5.9) is **convergent** in mean square sense. □

6. Some results

Theorem 6.1. *Let* $\{X_{nk}, n, k > o\}$, $\{Y_{nk}, n, k > o\}$ *be sequences of 2-r.v's over the same probability space and suppose that:*

$$\underset{n,k\to 0}{L.i.m}\, X_{nk} = X, \quad \underset{n,k\to\infty}{L.i.m}\, Y_{nk} = Y$$

Then:

(i) $\lim_{n\to\infty} E\{X_{nk}\} = E(X)$

(ii) $\lim_{\substack{k\to\infty \\ n\to\infty}} E\{X_{nk}^2\} = E(X^2)$

(iii) $\lim_{\substack{k\to\infty \\ n\to\infty}} Var\{X_{nk}\} = Var(X)$

(v) $\lim_{\substack{k\to\infty \\ n\to\infty}} PDF(X_{nk}) = PDF(X)$

$k \to \infty$

Proof.

(i) From Schwarz inequality: $E|XY| \leqslant (E(X^2))^{\frac{1}{2}}E(E(Y^2))^{\frac{1}{2}}$ we have:

$$|E\{X_{nk}.1\}| \leqslant 1.(E(X_{nk}^2))^{\frac{1}{2}} = \|X_{nk}\|$$

Then

$$|E\{X_{nk}\}| \leqslant E|X_{nk}| \leqslant \|X_{nk}\| < \infty \quad (6.1.1)$$

In (6.1.1) put $X_{nk} - X$ instead of X_{nk} then we have:

$$|E\{X_{nk} - X\}| = |E\{X_{nk}\} - E\{X\}| \leqslant E|X_{nk} - X| \leqslant \|X_{nk} - X\|$$

As $n, k \to \infty$ then $|E\{X_{nk}\} - E\{X\}| = 0$ i.e.,

$$\lim_{\substack{n \to \infty \\ k \to \infty}} E\{X_{nk}\} = E(X)$$

(ii) From successive applications of the triangle and Schwarz inequalities:

$$\begin{aligned}|E(XY) - E(X_{nk}Y_{nk})| &= |E(XY) - E(X_{nk}Y_{nk}) \pm E(XY_{nk}) \\ &\pm E(XY) \pm E(X_{nk}Y)| = |E(Y - Y_{nk})X \\ &+ E(X - X_{nk})Y - E((X - X_{nk})(Y - Y_{nk}))| \\ &\leqslant |E(Y - Y_{nk})X| + |E(X - X_{nk})Y| \\ &+ |E(X - X_{nk})(Y - Y_{nk})| \leqslant |X|\|Y_{nk} - Y\| \\ &+ |Y|\|X_{nk} - X\| + \|X_{nk} - X\|\|Y_{nk} - Y\|\end{aligned}$$

But each of the terms on the right-hand side tends to zero by hypothesis as n, $k \to \infty$.Then

$$E(XY) = \lim_{n,k\to\infty} E\{X_{nk}Y_{nk}\}$$

As $X_{nk} = Y_{nk}$ then we obtain $\lim_{n,k\to\infty} E\{X_{nk}^2\} = E\{X^2\}$

(iii) Since

$$\lim_{\substack{n \to \infty \\ k \to \infty}} E\{X_{nk}\} = E(X) \quad \text{and} \quad \lim_{n,k\to\infty} E\{X_{nk}^2\} = E\{X^2\}$$

And

$$Var\{X_{nk}\} = E\{X_{nk}^2\} - (E\{X_{nk}\})^2$$

Then

$$\begin{aligned}\lim_{n,k\to\infty} Var\{X_{nk}\} &= \lim_{n,k\to\infty} (E\{X_{nk}^2\} - (E\{X_{nk}\})^2) \\ &= \lim_{n,k\to\infty} E\{X_{nk}^2\} - \lim_{n,k\to\infty} (E\{X_{nk}\})^2 \\ &= E\{X^2\} - \{E(X)\}^2 = Var(X)\end{aligned}$$

Then we obtain

$$\lim_{n,k\to\infty} Var\{X_n\} = Var\{X\} \qquad \square$$

Definition 6 [16]. "The convergence in probability"A sequence of r.v's $\{X_{nk}\}$ converges in probability to a random variable X

as $\mathrm{n}, \mathrm{k} \to \infty$ if $\lim_{n,k\to\infty} p\{|X_{nk} - X| > \varepsilon\} = 0 \quad \forall \varepsilon > 0$

Definition 7 [16]. "The convergence in distribution"

A sequence of r.v's $\{X_{nk}\}$ converge in distribution to a random variable X

As $\mathrm{n}, \mathrm{k} \to \infty$ if $\lim_{n,k\to\infty} F_{x_{nk}}(x) = F_x(x)$

Lemma 6.1.1 [16]. *The convergence in m.s implies convergence in probability*

Lemma 6.1.2 [16]. *The convergence in probability implies convergence in distribution*

Theorem 6.2. *If* $X_{nk} \overset{m.s}{\to} X$ *then PDF of* $\{X_{nk}\} \overset{m.s}{\to}$ *PDF of* $\{X\}$

i.e. $\lim_{n,k\to\infty} f_{x_{nk}}(x) = f_X(x)$

The proof

Since we have shown that If $X_{nk} \overset{m.s}{\to} X$ then $X_{nk} \overset{d}{\to} X$

i.e. if $X_{nk} \overset{m.s}{\to} X$ then $\lim_{n,k\to\infty} F_{x_{nk}}(x) = F_X(x)$

Then $\lim_{n,k\to\infty} \frac{d}{dx} F_{X_{nk}}(x) = \frac{d}{dx} F_X(x)$ hence $\lim_{n,k\to\infty} f_{x_{nk}}(x) = f_X(x)$

7. Numerical examples

Example 1. Solve the random parabolic partial differential equation:

$$u_t(x,t) = \beta u_{xx}(x,t), \quad t \in [0,T], \quad x \in [0,X] \quad (7.1)$$

$$u(x,0) = \sin \pi x, \quad x \in [0,X] \quad \beta \text{ (random variable)} \quad (7.2)$$

$$u(0,t) = u(X,t) = 0 \quad (7.3)$$

The exact solution:

$$u(x,t) = e^{-\beta\pi^2 t} \sin \pi x$$

The numerical solution:

For the difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis.

Where $\Delta x = \frac{X}{M}$, $\Delta t = \frac{T}{N}$ and $M, N > 0$

Notational, u_k^n will be approximate of $u(x,t)$ at point $(k\Delta x, n\Delta t)$, $u_k^0 = u_0(k\Delta x)$. On this mesh, the difference scheme for this problem is:

$$u_k^{n+1} = (1 - 2r)u_k^n + r\left(u_{k+1}^n + u_{k-1}^n\right) \quad (7.4)$$

$$u_k^0 = \sin \pi x_k \quad (7.5)$$

$$u_0^n = u_X^n = 0 \quad (7.6)$$

where $r = \frac{\beta\Delta t}{\Delta x^2}$, $x_k = k\Delta x$, $t_n = n\Delta t$

First from the initial condition we have:

$$\begin{aligned}u(0,0) &= u_{00} = 0 \\ u(1,0) &= u_{10} = \sin(\Delta x\pi) \\ u(2,0) &= u_{20} = \sin(2\Delta x\pi)\end{aligned}$$

Then:

$$u(k,0) = u_{k0} = \sin(k\Delta x\pi) \quad (7.7)$$

From (7.4):

$$\begin{aligned}u(1,1) &= u_{11} = (1-2r)u_{10} + r(u_{20} + u_{00}) \\ &= (1-2r)u_{10} + ru_{20} \\ &= (1-2r)[\sin(\Delta x\pi)] + r[\sin(2\Delta x\pi)] \\ u(2,1) &= u_{21} = (1-2r)u_{20} + r(u_{30} + u_{10}) \\ &= (1-2r)[\sin(2\Delta x\pi)] + r[\sin(3\Delta x\pi) + \sin(\Delta x\pi)]\end{aligned}$$

Finally:

$$u(k,1)=u_{k1}=(1-2r)[\sin(k\Delta x\pi)]+r[\sin(k+1)\Delta x\pi+\sin(k-1)\Delta x\pi]$$
$$=(1-2r)[\sin(k\Delta x\pi)]+2r[\sin(k\Delta x\pi)\cos(\Delta x\pi)]$$
$$=\sin(k\Delta x\pi)[1-2r+2r\cos(\Delta x\pi)]=\sin(k\Delta x\pi)[1-2r(1-\cos(\Delta x\pi))]$$
$$=\sin(k\Delta x\pi)\left[1-2r\left(2\sin^2\left(\frac{\Delta x\pi}{2}\right)\right)\right]$$
$$=\sin(k\Delta x\pi)\left[1-4r\sin^2\left(\frac{\Delta x\pi}{2}\right)\right]=\sin(k\Delta x\pi)\left[1-4r\left(\frac{e^{\frac{i\Delta x\pi}{2}}-e^{\frac{-i\Delta x\pi}{2}}}{2i}\right)^2\right]$$
$$=\sin(k\Delta x\pi)\left[1+r\left(e^{\frac{i\Delta x\pi}{2}}-e^{\frac{-i\Delta x\pi}{2}}\right)^2\right]$$
$$=\sin(k\Delta x\pi)[1+r(e^{i\Delta x\pi}-2+e^{-i\Delta x\pi})]$$
$$=\sin(k\Delta x\pi)\left[1+r\left(-2+1+i\Delta x\pi-\frac{(\Delta x\pi)^2}{2!}-\frac{i(\Delta x\pi)^3}{3!}+\frac{(\Delta x\pi)^4}{4!}+\cdots\right.\right.$$
$$\left.\left.+1-i\Delta x\pi-\frac{(\Delta x\pi)^2}{2!}+\frac{i(\Delta x\pi)^3}{3!}+\frac{(\Delta x\pi)^4}{4!}-\cdots\right)\right]$$
$$=\sin(k\Delta x\pi)\left[1+r\left(-\frac{2(\Delta x\pi)^2}{2!}+\frac{2(\Delta x\pi)^4}{4!}-\frac{2(\Delta x\pi)^2}{6!}+\cdots\right)\right]$$
$$=\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right)]$$

Then:

$$u_{k1}=\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \tag{7.8}$$

$$u(1,2)=u_{12}=(1-2r)u_{11}+r(u_{21}+u_{01})$$
$$=(1-2r)\sin(\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]+r$$
$$\times\ \sin 2\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$u(2,2)=u_{22}=(1-2r)u_{21}+r(u_{31}+u_{11})$$
$$=(1-2r)\sin(2\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]+r$$
$$\times\ \sin 3\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]+r$$
$$\times\ \sin \Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$u(3,2)=u_{32}=(1-2r)u_{31}+r(u_{41}+u_{21})$$
$$=(1-2r)\sin(3\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]+r$$
$$\times\ \sin 4\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]+r$$
$$\times\ \sin 2\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

Finally:

$$u_{k2}=(1-2r)\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$+\ r[\sin(k+1)\Delta x\pi]\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$+\ r[\sin(k-1)\Delta x\pi]\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$u(k,2)=u_{k2}=(1-2r)\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$+r\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right][\sin(k+1)\Delta x\pi+\sin(k-1)\Delta x\pi]$$

$$u(k,2)=u_{k2}=(1-2r)\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$+2r\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right][\sin(k\Delta x\pi)\cos(\Delta x\pi)]$$
$$=\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right][1-2r+2r\cos(\Delta x\pi)]$$
$$=\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2$$

Then:

$$u_{k2}=\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2 \tag{7.9}$$

$$u(1,3)=u_{13}=(1-2r)u_{12}+r(u_{22}+u_{02})$$
$$=(1-2r)\sin(\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2+r$$
$$\times\ \sin 2\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2$$

$$u(2,3)=u_{23}=(1-2r)u_{22}+r(u_{32}+u_{12})$$
$$=(1-2r)\sin(2\Delta x\pi)[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^2$$
$$+\ r\sin 3\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2$$
$$+\ r\sin \Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2$$

$$
\begin{aligned}
u(3,3) = u_{33} &= (1-2r)u_{32} + r(u_{42}+u_{22}) \\
&= (1-2r)\sin(3\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2 + r \\
&\quad \times \sin 4\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2 + r \\
&\quad \times \sin 2\Delta x\pi\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2
\end{aligned}
$$

Finally:

$$
\begin{aligned}
u(k,3) &= u_{k3} \\
&= (1-2r)\sin(k\Delta x\pi)[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^2 + r[\sin(k \\
&\quad +1)\Delta x\pi]\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2 + r[\sin(k \\
&\quad -1)\Delta x\pi]\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2 \\
&= (1-2r)\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2 \\
&\quad + r\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2[\sin(k+1)\Delta x\pi + \sin(k \\
&\quad -1)\Delta x\pi] \\
&= (1-2r)\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2 \\
&\quad + 2r\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2[\sin(k\Delta x\pi)\cos(\Delta x\pi)] \\
&= \sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2[1-2r+2r \\
&\quad \times \cos(\Delta x\pi)] \\
&= \sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^3
\end{aligned}
$$

Then:

$$
u_{k3} = \sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^3 \tag{7.10}
$$

Finally, the numerical solution for this problem is:

$$
\begin{aligned}
u_{kn} &= \sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n \\
&= \sin(k\Delta x\pi)\left[1+2\frac{\beta\Delta t}{\Delta x^2}\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n
\end{aligned} \tag{7.11}
$$

We can prove that:

(i)$u_k^n \overset{m.s}{\rightarrow} u$

Proof. Since:

$$
\underset{\substack{n\to\infty \\ k\to\infty}}{l.i.m}\; u_k^n = \underset{\substack{\Delta x,\Delta t\to 0 \\ k\Delta x, n\Delta t \to x,t}}{l.i.m}\; u_k^n = u
$$

(if and only if)

$$
\lim_{\substack{n\to\infty \\ k\to\infty}} E\left|u_k^n - u\right|^2 = 0
$$

or

$$
\lim_{\substack{\Delta x,\Delta t\to 0 \\ k\Delta x, n\Delta t \to x,t}} E\left|u_k^n - u\right|^2 = 0
$$

$$
\begin{aligned}
u_k^n - u &= \sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n - e^{-\beta\pi^2 t} \\
&\quad \times \sin\pi x\left|u_k^n - u\right|^2 \\
&= \left|\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n - e^{-\beta\pi^2 t}\sin\pi x\right|^2 \\
&= \left[\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n\right]^2 \\
&\quad -2\left[\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n\right][e^{-\beta\pi^2 t} \\
&\quad \times \sin\pi x] + [e^{-\beta\pi^2 t}\sin\pi x]^2 \\
&= [\sin^2(k\Delta x\pi)]\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2n} \\
&\quad -2\left[\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n\right]\quad [e^{-\beta\pi^2 t} \\
&\quad \times \sin\pi x] + [e^{-2\beta\pi^2 t}\sin^2\pi x]^2
\end{aligned}
$$

Then:

$$
\begin{aligned}
E\left|u_k^n - u\right|^2 &= E\left[[\sin^2(k\Delta x\pi)]\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2n}\right. \\
&\quad -2\left[\sin(k\Delta x\pi)\left[1+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n\right] \\
&\quad \left.\times\left[e^{-\beta\pi^2 t}\sin\pi x] + [e^{-2\beta\pi^2 t}\sin^2\pi x]^2\right]\right]
\end{aligned}
$$

At time $t = (n+1)\Delta t$ then:

$$
\begin{aligned}
E\left|u_k^n - u\right|^2 &= E\left[[\sin^2(k\Delta x\pi)]\left[1+2\frac{\beta\Delta t}{\Delta x^2}\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2n}\right] \\
&\quad -2E\left[\left[\sin(k\Delta x\pi)\left[1+2\frac{\beta\Delta t}{\Delta x^2}\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n\right][e^{-\beta\pi^2(n+1)\Delta t}\sin\pi x]\right] \\
&\quad + E[[e^{-2\beta\pi^2(n+1)\Delta t}\sin^2\pi x]] \\
&= E[[\sin^2(k\Delta x\pi)][1+2\beta\Delta t\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}]^{2n}] \\
&\quad -2E\left[\left[\sin(k\Delta x\pi)[1+2\frac{\Delta t}{\Delta x^2}\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^n\right][e^{-\beta\pi^2(n+1)\Delta t}\sin\pi x]\right] \\
&\quad + E[[e^{-2\beta\pi^2(n+1)\Delta t}\sin^2\pi x]]
\end{aligned}
$$

By taking the limit as $\Delta x \to 0$, $\Delta t \to 0$, $E(\beta) \to 0$, $(k\Delta x, n\Delta t) \to (x,t)$ and under condition $0 \leqslant \frac{\Delta t}{\Delta x^2} E(\beta) \leqslant \frac{1}{2}$

Then we obtain:

$$E\left|u_k^n - u\right|^2 \to 0 \quad \text{Then} \quad u_k^n \xrightarrow{m.s} u \quad \square$$

Example 2. Solve the random elliptic partial differential equation:

$$u_{yy}(x,y) + u_{xx}(x,y) = 0, \quad y > 0, \quad x \in [0,1] \tag{7.12}$$

$$u(x,0) = \beta \sin \pi x, \quad \beta \sim \text{random variable} \tag{7.13}$$

$$u(0,y) = u_0(y) = u(1,y) = 0 \tag{7.14}$$

The exact solution:

$$u(x,t) = \beta e^{-\pi y} \sin \pi x$$

The numerical solution:

For the difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis.

Where $\Delta x = \frac{1}{M}$, $\Delta y = \frac{Y}{N}$ and $M, N > 0$

Notational, u_k^n will be approximate of $u(x,t)$ at point $(k\Delta x, n\Delta t)$, the difference scheme for this problem is:

$$u_k^{n+1} = (2+2r)u_k^n - u_k^{n-1} - r\left(u_{k+1}^n + u_{k-1}^n\right), \quad y \in [0, Y], \quad x \in [0,1] \tag{7.15}$$

$$u(x_k, 0) = \beta \sin \pi x_k \tag{7.16}$$

$$u(0,y) = 0, \quad u(1,y) = 0 \tag{7.17}$$

Where $r = \frac{\Delta y^2}{\Delta x^2}$, $x_k = k\Delta x$, $y_n = n\Delta y$

First from the initial condition we have:

$$u(0,0) = u_{00} = 0$$
$$u(1,0) = u_{10} = \beta \sin(\Delta x\pi)$$
$$u(2,0) = u_{20} = \beta \sin(2\Delta x\pi)$$

Then:

$$u(k,0) = u_{k0} = \beta \sin(k\Delta x\pi) \tag{7.18}$$

From (7.15):

$$\begin{aligned}
u(1,1) &= u_{11} = (2+2r)u_{10} - r(u_{20} + u_{00}) \\
&= (2+2r)u_{10} - ru_{20} \\
&= \beta(2+2r)[\sin(\Delta x\pi)] - r\beta[\sin(2\Delta x\pi)] \\
u(2,1) &= u_{21} = (2+2r)u_{20} - r(u_{30} + u_{10}) \\
&= \beta(2+2r)[\sin(2\Delta x\pi)] - \beta r[\sin(3\Delta x\pi) + \sin(\Delta x\pi)]
\end{aligned}$$

Finally:

$$\begin{aligned}
u(k,1) = u_{k1} &= \beta(2+2r)[\sin(k\Delta x\pi)] - r\beta[\sin(k+1)\Delta x\pi + \sin(k-1)\Delta x\pi] \\
&= \beta(2+2r)[\sin(k\Delta x\pi)] - 2r\beta[\sin(k\Delta x\pi)\cos(\Delta x\pi)] \\
&= \beta\sin(k\Delta x\pi)[2+2r-2r\cos(\Delta x\pi)] \\
&= \beta\sin(k\Delta x\pi)[2+2r(1-\cos(\Delta x\pi))] \\
&= \beta\sin(k\Delta x\pi)\left[2+2r(2\sin^2\left(\frac{\Delta x\pi}{2}\right)\right] \\
&= \beta\sin(k\Delta x\pi)\left[2+4r\sin^2\left(\frac{\Delta x\pi}{2}\right)\right] \\
&= \beta\sin(k\Delta x\pi)\left[2+4r\left(\frac{e^{\frac{i\Delta x\pi}{2}} - e^{\frac{-i\Delta x\pi}{2}}}{2i}\right)^2\right] \\
&= \beta\sin(k\Delta x\pi)\left[2-r\left(e^{\frac{i\Delta x\pi}{2}} - e^{\frac{-i\Delta x\pi}{2}}\right)^2\right] \\
&= \beta\sin(k\Delta x\pi)[2-r(e^{i\Delta x\pi} - 2 + e^{-i\Delta x\pi})] \\
&= \beta\sin(k\Delta x\pi)\left[2-r(-2+1+i\Delta x\pi - \frac{(\Delta x\pi)^2}{2!} - \frac{i(\Delta x\pi)^3}{3!}\right. \\
&\quad \left. + \frac{(\Delta x\pi)^4}{4!} + \cdots + 1 - i\Delta x\pi - \frac{(\Delta x\pi)^2}{2!} + \frac{i(\Delta x\pi)^3}{3!} + \frac{(\Delta x\pi)^4}{4!} - \cdots)\right] \\
&= \beta\sin(k\Delta x\pi)\left[2-r\left(-\frac{2(\Delta x\pi)^2}{2!} + \frac{2(\Delta x\pi)^4}{4!} - \frac{2(\Delta x\pi)^2}{6!} + \cdots\right)\right] \\
&= \beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!})\right]
\end{aligned}$$

Then:

$$u_{k1} = \beta\sin(k\Delta x\pi)\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \tag{7.19}$$

$$\begin{aligned}
u(1,2) = u_{12} &= (2+2r)u_{11} - u_{10} - r(u_{21} + u_{01}) \\
&= (2+2r)\beta\sin(\Delta x\pi)\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad - \beta\sin\Delta x\pi - r\beta\sin 2\Delta x\pi\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]
\end{aligned}$$

$$\begin{aligned}
u(2,2) = u_{22} &= (2+2r)u_{21} - u_{20} - r(u_{31} + u_{11}) \\
&= (2+2r)\beta\sin(2\Delta x\pi)\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad - \beta\sin 2\Delta x\pi - r\beta\sin 3\Delta x\pi\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad - r\beta\sin\Delta x\pi\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]
\end{aligned}$$

$$\begin{aligned}
u(3,2) = u_{32} &= (2+2r)u_{31} - u_{30} - r(u_{41} + u_{21}) \\
&= (2+2r)\beta\sin(3\Delta x\pi)\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad - \beta\sin 3\Delta x\pi - r\beta\sin 4\Delta x\pi\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad - r\beta\sin 2\Delta x\pi\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]
\end{aligned}$$

Finally:

$$\begin{aligned}
u(k,2) &= u_{k2} \\
&= (2+2r)\beta\sin(k\Delta x\pi)\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad - \beta\sin k\Delta x\pi - r\beta[\sin(k \\
&\quad + 1)\Delta x\pi]\left[2 - 2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] - r\beta[\sin(k \\
&\quad - 1)\Delta x\pi]\left[2 - 2r\sum(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]
\end{aligned}$$

$$u(k,2) = u_{k2}$$
$$= (2+2r)\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$-\beta\sin k\Delta x\pi - r\beta\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right][\sin(k$$
$$+1)\Delta x\pi+\sin(k-1)\Delta x\pi]$$

$$u(k,2) = u_{k2}$$
$$= (2+2r)\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$-\beta\sin k\Delta x\pi$$
$$-2r\beta\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right][\sin(k\Delta x\pi)\cos(\Delta x\pi)]$$
$$=\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right][2+2r-2r$$
$$\times\cos(\Delta x\pi)]-\beta\sin k\Delta x\pi$$
$$=\beta\sin(k\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$

Then:

$$u_{k2}=\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}$$
$$-\beta\sin(k\Delta x\pi) \tag{7.20}$$

$$u(1,3)=u_{13}=(2+2r)u_{12}-u_{11}-r(u_{22}+u_{02})$$
$$=(2+2r)\beta\sin(\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-r\beta\sin 2\Delta x\pi\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-\beta\sin(\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$u(2,3)=u_{23}=(2+2r)u_{22}-u_{21}-r(u_{32}+u_{12})$$
$$=(2+2r)\beta\sin(2\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-r\beta\sin 3\Delta x\pi\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-r\beta\sin\Delta x\pi\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-\beta\sin(2\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$u(3,3)=u_{33}=(2+2r)u_{32}-u_{31}-r(u_{42}+u_{22})$$
$$=(2+2r)\beta\sin(3\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-r\beta\sin 4\Delta x\pi\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-r\beta\sin 2\Delta x\pi\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-\beta\sin(3\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

Finally:

$$u(k,3)=u_{k3}=(2+2r)\beta\sin(k\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)}\right]-r\beta[\sin(k+1)\Delta x\pi]\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$-r\beta[\sin(k-1)\Delta x\pi]\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$=(2+2r)\beta\sin(k\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]-r\beta\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$[\sin(k+1)\Delta x\pi+\sin(k-1)\Delta x\pi]-\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$=(2+2r)\beta\sin(k\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]-2r\beta\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$[\sin(k\Delta x\pi)\cos(\Delta x\pi)]-\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$
$$=\beta\sin(k\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right][2+2r-2r\cos(\Delta x\pi)]$$
$$-\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]=\beta\sin(k\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]$$
$$\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]-\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

Then:

$$u_{k3}=\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{2}-1\right]-1\right]$$

$$u_{k3}=\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{3}-2\beta\sin(k\Delta x\pi)\left[\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]\right]$$

Finally, the numerical solution for this problem is:

$$u_{k}^{n}=\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{n}-(n$$
$$-1)\beta\sin(k\Delta x)\left[2-2r\sum_{i=1}^{\infty}(-1)^{i}\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{(n-2)}$$

We can prove that:

(i)$u_{k}^{n}\overset{m.s}{\to}u$

Proof. Since:

$$\underset{\substack{n\to\infty\\ k\to\infty}}{l.i.m}\, u_k^n = \underset{\substack{\Delta x,\Delta t\to 0\\ k\Delta x, n\Delta t\to x,t}}{l.i.m}\, u_k^n = u \text{ (if and only if)} \lim_{\substack{n\to\infty\\ k\to\infty}} E\left|u_k^n - u\right|^2 = 0 \text{ or}$$

$$\lim_{\substack{\Delta x,\Delta t\to 0\\ k\Delta x, n\Delta t\to x,t}} E\left|u_k^n - u\right|^2 = 0$$

$$u_k^n - u = \beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n - (n-1)\beta\sin(k\Delta x)\left[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{(n-2)} - \beta e^{-\pi y}\sin\pi x\left|u_k^n-u\right|^2$$

$$= \left|\beta\sin(k\Delta x\pi)[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^n - (n-1)\beta\sin(k\Delta x)[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^{(n-2)} - \beta e^{-\pi y}\sin\pi x\right|^2 = \left[\beta\sin(k\Delta x\pi)[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^n - (n-1)\beta\sin(k\Delta x)[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^{(n-2)}\right]^2$$

$$-2\left[\beta\sin(k\Delta x\pi)\left[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n - (n-1)\beta\sin(k\Delta x)\left[2-2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{(n-2)}\right]\left[\beta e^{-\pi y}\sin\pi x\right] + [\beta e^{-\pi y}\sin\pi x]^2$$

Then:

$$E\left|u_k^n-u\right|^2 = E\left[\beta\sin(k\Delta x\pi)\left[2-2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}\right]^n - (n-1)\beta\sin(k\Delta x)\left[2-2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}\right]^{(n-2)}\right]^2$$

$$-2E\left[\beta\sin(k\Delta x\pi)\left[2-2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}\right]^n - (n-1)\beta\sin(k\Delta x)[2-2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}]^{(n-2)}\right][\beta e^{-\pi^2 t}\sin\pi x] + E[\beta^2 e^{-2\pi y}\sin^2\pi x]$$

At time $t = (n + 1)\Delta t$ and by taking the limit as $\Delta x\to 0$, $\Delta t\to 0$, $E(\beta)\to 0$ and $(k\Delta x, n\Delta t)\to(x,t)$

We obtain: $E\left|u_k^n-u\right|^2\to 0$ Then: $u_k^n \overset{m.s}{\to} u$ □

Example 3. Solve the random hyperbolic partial differential equation:

$$u_{tt}(x,t) - 4u_{xx}(x,t) = 0, \quad t>0, \quad x\in[0,1] \tag{7.21}$$

$$u(x,0) = \beta\sin\pi x, \quad \beta\sim \text{random variable} \tag{7.22}$$

$$u(0,t) = u(1,t) = 0, \tag{7.23}$$

$$\frac{\partial u(x,0)}{\partial t} = 0 \tag{7.24}$$

The exact solution:

$$u(x,t) = \beta\sin\pi x\cos(2\pi x)$$

The numerical solution:

For the difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis.

Where $\Delta x = \frac{1}{M}$, $\Delta t = \frac{T}{N}$ and $M, N > 0$

Notational, u_k^n will be approximate of $u(x,t)$ at point $(k\Delta x, n\Delta t)$, the difference scheme for this problem is:

$$\begin{aligned} &u_k^{n+1} = (2-2r)u_k^n - u_k^{n-1} + r\left(u_{k+1}^n + u_{k-1}^n\right)\\ &u(x_k,0) = \beta\sin\pi x_k\\ &u(0,t) = u(1,t) = 0\\ &u_t(x,0) = 0\end{aligned} \tag{7.25}$$

where $r = 4\frac{\Delta t^2}{\Delta x^2}$, $x_k = k\Delta x$, $t_n = n\Delta t$

First from the initial condition we have:

$$\begin{aligned} u(0,0) &= u_{00} = 0\\ u(1,0) &= u_{10} = \beta\sin(\Delta x\pi)\\ u(2,0) &= u_{20} = \beta\sin(2\Delta x\pi)\end{aligned}$$

Then:

$$u(k,0) = u_{k0} = \beta\sin(k\Delta x\pi) \tag{7.26}$$

From (7.25):

$$u(1,1) = u_{11} = (2-2r)u_{10} - u_{1,-1} + r(u_{20}+u_{00})$$

But from the boundary condition we have:

$\left(\frac{\partial u}{\partial}\right)_{k,0} = \frac{u_{k1}-u_{k,-1}}{2k} = g(x_k)$ And science here $g(x_k) = 0$ then $u_{k,-1} = u_{k1}$

Then:

$$u(1,1) = u_{11} = (2-2r)u_{10} - u_{1,-1} + r(u_{20}+u_{00})$$

$$\begin{aligned} u(1,1) = u_{11} &= (2-2r)u_{10} - u_{11} + r(u_{20}+u_{00})\\ &= \frac{1}{2}[(2-2r)u_{10} + ru_{20}]\\ &= \frac{1}{2}[\beta(2-2r)[\sin(\Delta x\pi)] + r\beta[\sin(2\Delta x\pi)]]\end{aligned}$$

$$u(2,1) = u_{21} = (2-2r)u_{20} - u_{2,-1} + r(u_{30}+u_{10})$$

$$\begin{aligned} u(2,1) = u_{21} &= (2-2r)u_{20} - u_{21} + r(u_{30}+u_{10})\\ &= \frac{1}{2}[\beta(2-2r)[\sin(2\Delta x\pi)] + \beta r[\sin(3\Delta x\pi) + \sin(\Delta x\pi)]]\end{aligned}$$

Finally:

$$
\begin{aligned}
u(k,1) &= u_{k1} \\
&= \frac{1}{2}[\beta(2-2r)[\sin(k\Delta x\pi)] + r\beta[\sin(k+1)\Delta x\pi + \sin(k-1)\Delta x\pi]] \\
&= \frac{1}{2}[\beta(2-2r)[\sin(k\Delta x\pi)] + 2r\beta[\sin(k\Delta x\pi)\cos(\Delta x\pi)]] \\
&= \frac{1}{2}[\beta\sin(k\Delta x\pi)[2-2r+2r\cos(\Delta x\pi)]] \\
&= \frac{1}{2}[\beta\sin(k\Delta x\pi)[2-2r(1-\cos(\Delta x\pi))]] \\
&= \frac{1}{2}[\beta\sin(k\Delta x\pi)[2-2r(2\sin^2(\frac{\Delta x\pi}{2})]] \\
&= \frac{1}{2}[\beta\sin(k\Delta x\pi)[2-4r\sin^2(\frac{\Delta x\pi}{2})]] \\
&= \frac{1}{2}\left[\beta\sin(k\Delta x\pi)\left[2-4r\left(\frac{e^{\frac{i\Delta x\pi}{2}}-e^{\frac{-i\Delta x\pi}{2}}}{2i}\right)^2\right]\right] \\
&= \frac{1}{2}\left[\beta\sin(k\Delta x\pi)\left[2+r\left(e^{\frac{i\Delta x\pi}{2}}-e^{\frac{-i\Delta x\pi}{2}}\right)^2\right]\right] \\
&= \frac{1}{2}[\beta\sin(k\Delta x\pi)[2+r(e^{i\Delta x\pi}-2+e^{-i\Delta x\pi})]] \\
&= \frac{1}{2}\Bigg[\beta\sin(k\Delta x\pi)\Bigg[2+r(-2+1+i\Delta x\pi-\frac{(\Delta x\pi)^2}{2!} \\
&\quad -\frac{i(\Delta x\pi)^3}{3!}+\frac{(\Delta x\pi)^4}{4!}+\ldots\ldots\ldots+1-i\Delta x\pi \\
&\quad -\frac{(\Delta x\pi)^2}{2!}+\frac{i(\Delta x\pi)^3}{3!}+\frac{(\Delta x\pi)^4}{4!}-\ldots\ldots\ldots)\Bigg]\Bigg] \\
&= \frac{1}{2}\Bigg[\beta\sin(k\Delta x\pi)\Bigg[2 \\
&\quad +r\left(-\frac{2(\Delta x\pi)^2}{2!}+\frac{2(\Delta x\pi)^4}{4!}-\frac{2(\Delta x\pi)^2}{6!}+\ldots\ldots.\right)\Bigg]\Bigg] \\
&= \frac{1}{2}\left[\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!})\right]\right]
\end{aligned}
$$

Then:

$$
u_{k1} = \frac{1}{2}\left[\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]\right] \tag{7.27}
$$

$$
\begin{aligned}
u(1,2) = u_{12} &= (2-2r)u_{11} - u_{10} + r(u_{21}+u_{01}) \\
&= (2-2r)\frac{1}{2}\beta\sin(\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad -\beta\sin\Delta x\pi + r\frac{1}{2}\beta\sin 2\Delta x\pi\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]
\end{aligned}
$$

$$
\begin{aligned}
u(2,2) = u_{22} &= (2-2r)u_{21} - u_{20} + r(u_{31}+u_{11}) \\
&= (2-2r)\frac{1}{2}\beta\sin(2\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad -\beta\sin 2\Delta x\pi + \frac{1}{2}r\beta\sin 3\Delta x\pi\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad +\frac{1}{2}r\beta\sin\Delta x\pi\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]
\end{aligned}
$$

$$
\begin{aligned}
u(3,2) = u_{32} &= (2-2r)u_{31} - u_{30} + r(u_{41}+u_{21}) \\
&= (2-2r)\frac{1}{2}\beta\sin(3\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad -\beta\sin 3\Delta x\pi \\
&\quad +\frac{1}{2}r\beta\sin 4\Delta x\pi\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad +\frac{1}{2}r\beta\sin 2\Delta x\pi\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)}{(2i)!}\right]
\end{aligned}
$$

Finally:

$$
\begin{aligned}
u(k,2) = u_{k2} &= \frac{1}{2}(2-2r)\beta\sin(k\Delta x\pi)[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}] \\
&\quad -\beta\sin k\Delta x\pi + \frac{1}{2}r\beta[\sin(k+1)\Delta x\pi]\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad +\frac{1}{2}r\beta[\sin(k-1)\Delta x\pi]\left[2+2r\sum(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]
\end{aligned}
$$

$$
\begin{aligned}
u(k,2) = u_{k2} &= \frac{1}{2}(2-2r)\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad -\beta\sin k\Delta x\pi - \frac{1}{2}r\beta\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right][\sin(k+1) \\
&\quad \times\Delta x\pi + \sin(k-1)\Delta x\pi]
\end{aligned}
$$

$$
\begin{aligned}
u(k,2) &= u_{k2} \\
&= \frac{1}{2}(2-2r)\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad -\beta\sin k\Delta x\pi + r\beta\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad \times[\sin(k\Delta x\pi)\cos(\Delta x\pi)] \\
&\quad = \frac{1}{2}\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right] \\
&\quad \times[2-2r+2r\cos(\Delta x\pi)] - \beta\sin k\Delta x\pi \\
&\quad = \frac{1}{2}\beta\sin(k\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]
\end{aligned}
$$

Then:

$$
u_{k2} = \frac{1}{2}\left[\beta\sin(k\Delta x\pi)[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^2 - 2\beta\sin(k\Delta x\pi)\right] \tag{7.28}
$$

$$u(1,3)=u_{13}$$

$$=(2-2r)u_{12}-u_{11}+r(u_{22}+u_{02})$$

$$=(2-2r)\frac{1}{2}\beta\sin(\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$+\frac{1}{2}r\beta\sin 2\Delta x\pi\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$-\frac{1}{2}\beta\sin(\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$u(2,3)=u_{23}$$

$$=(2-2r)u_{22}-u_{21}+r(u_{32}+u_{12})$$

$$=(2-2r)\frac{1}{2}\beta\sin(2\Delta x\pi)[[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^2-2]$$

$$+\frac{1}{2}r\beta\sin 3\Delta x\pi\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$+\frac{1}{2}r\beta\sin\Delta x\pi\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$-\frac{1}{2}\beta\sin(2\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$u(3,3)=u_{33}$$

$$=(2-2r)u_{32}-u_{31}+r(u_{42}+u_{22})$$

$$=\frac{1}{2}(2-2r)\beta\sin(3\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$+\frac{1}{2}r\beta\sin 4\Delta x\pi\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$+\frac{1}{2}r\beta\sin 2\Delta x\pi\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$-\frac{1}{2}\beta\sin(3\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

Finally:

$$u(k,3)=u_{k3}=(2-2r)\beta\sin(k\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$-\tfrac{1}{2}\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)}\right]$$

$$+\tfrac{1}{2}r\beta[\sin(k+1)\Delta x\pi]\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$+\tfrac{1}{2}r\beta[\sin(k-1)\Delta x\pi]\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$=\tfrac{1}{2}(2-2r)\beta\sin(k\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$+\tfrac{1}{2}r\beta\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$[\sin(k+1)\Delta x\pi+\sin(k-1)\Delta x\pi]-\tfrac{1}{2}\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$=\tfrac{1}{2}(2-2r)\beta\sin(k\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$+r\beta\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$[\sin(k\Delta x\pi)\cos(\Delta x\pi)]-\tfrac{1}{2}\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$=\tfrac{1}{2}\beta\sin(k\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]$$

$$[2-2r+2r\cos(\Delta x\pi)]-\tfrac{1}{2}\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$=\tfrac{1}{2}\beta\sin(k\Delta x\pi)\left[\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^2-2\right]\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

$$-\tfrac{1}{2}\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]$$

Then:

$$u_{k3}=\frac{1}{2}\left[\left[\beta\sin(k\Delta x\pi)[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}]^3\right.\right.$$

$$\left.\left.-3\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]\right]\right]$$

Finally, the numerical solution for this problem is:

$$u_k^n=\frac{1}{2}\left[\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n\right.$$

$$\left.-n\beta\sin(k\Delta x)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{(n-2)}\right]$$

We can prove that:

(i)$u_k^n\xrightarrow{m.s}u$

Proof. Since:

$$\underset{\substack{n\to\infty\\k\to\infty}}{l.i.m}\, u_k^n = \underset{\substack{\Delta x,\Delta t\to 0\\k\Delta x, n\Delta t\to x,t}}{l.i.m}\, u_k^n = u \text{ (if and only if)} \lim_{\substack{n\to\infty\\k\to\infty}} E\left|u_k^n - u\right|^2 = 0 \text{ or}$$

$$\lim_{\substack{\Delta x,\Delta t\to 0\\k\Delta x, n\Delta t\to x,t}} E\left|u_k^n - u\right|^2 = 0$$

$$u_k^n - u = \tfrac{1}{2}\left[\beta\sin(k\Delta x\pi)[2+2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}]^n - n\beta\sin(k\Delta x)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{(n-2)}\right] - \beta\sin\pi x\cos 2\pi x\left|u_k^n - u\right|^2$$

$$= \left|\tfrac{1}{2}\left[\beta\sin(k\Delta x\pi)[2-2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}]^n - n\beta\sin(k\Delta x)\left[2-2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{(n-2)}\right] - \beta\sin\pi x\cos 2\pi x\right|^2$$

$$= \tfrac{1}{4}\left[\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n - n\beta\sin(k\Delta x)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}\right]^{(n-2)}\right]^2$$

$$-\left[\beta\sin(k\Delta x\pi)\left[2+2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}\right]^n - n\beta\sin(k\Delta x)[2+2r\sum_{i=1}^{\infty}(-1)^i\tfrac{(\Delta x\pi)^{2i}}{(2i)!}]^{(n-2)}\right][\beta\sin\pi x\cos 2\pi x] + [\beta\sin\pi x\cos 2\pi x]^2$$

Then:

$$E\left|u_k^n - u\right|^2 = \frac{1}{4}E\left[\beta\sin(k\Delta x\pi)\left[2+2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}\right]^n - n\beta\sin(k\Delta x)\left[2+2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}\right]^{(n-2)}\right]^2$$

$$-E\left[\beta\sin(k\Delta x\pi)\left[2+2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}\right]^n - n\beta\sin(k\Delta x)\left[2+2\Delta y^2\sum_{i=1}^{\infty}(-1)^i\frac{(\Delta x\pi)^{2i}}{\Delta x^2(2i)!}\right]^{(n-2)}\right]$$

$$[\beta\sin\pi x\cos 2\pi x] + E[\beta^2\sin^2\pi x\cos^2 2\pi x]$$

At time $t = (n+1)\Delta t$ and by taking the limit as $\Delta x \to 0$, $\Delta t \to 0$, $E(\beta) \to 0$

And $(k\Delta x, n\Delta t) \to (x, t)$

We obtain: $E\left|u_k^n - u\right|^2 \to 0$ Then: $u_k^n \overset{m.s}{\to} u$ □

8. Conclusions

This paper solves some RPDEs in mean square sense using random finite difference scheme. Mean square consistency of the random difference scheme for (RPDE) is established. Sufficient conditions for the mean square stability of the proposed numerical solution are given.

The random parabolic, elliptic and hyperbolic partial differential equations can be solved numerically using the random difference method (with three points) in mean square sense. The convergence of the solution scheme to the exact one is proved. (with three points) Some numerical examples are solved to illustrate the method of analysis.

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