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ORIGINAL ARTICLE

# Simultaneous approximation by certain Baskakov–Durrmeyer–Stancu operators

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**Abstract** In the present paper, we establish some direct results in simultaneous approximation for Baskakov–Durrmeyer–Stancu (abbr. BDS) operators $D_n^{(\alpha,\beta)}(f,x)$. We establish point-wise convergence, Voronovskaja type asymptotic formula and an error estimate in terms of second order modulus of continuity of the function.



## 1. Introduction

For $f \in C[0,\infty)$, a new type of Baskakov–Durrmeyer type operator studied by Finta in [2] is defined as

$$D_n(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t)\,dt + p_{n,0}(x) f(0), \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \; b_{n,k}(t) = \frac{1}{B(k,n+1)} \frac{t^{k-1}}{(1+t)^{n+k+1}}. \tag{1.2}$$

These operators are different from the operators studied in [1], [6] and [7]. It is observed that $D_n(f,x)$ reproduce constant as well as linear functions. Gupta et al. [8] estimated point-wise convergence, asymptotic formula and inverse result in simultaneous approximation for the operators (1.1). Govil and Gupta [4] used iterative combinations of such operators to improve the order of approximation. Very recently, Verma et al. [10] considered Baskakov–Durrmeyer–Stancu (abbr. BDS) operators as follows:

$$D_n^{(\alpha,\beta)}(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt + p_{n,0}(x) f\left(\frac{\alpha}{n+\beta}\right), \tag{1.3}$$

where the Baskakov and Beta basis functions are given in (1.2) and the parameters $\alpha$, $\beta$ satisfy the conditions $0 \leqslant \alpha \leqslant \beta$. In [10] authors studied some approximation properties, asymptotic formula and better estimates for these operators.

The aim of the paper is to study pointwise convergence, a Voronovkaja type asymptotic formula and an estimate error in simultaneous approximation by the BDS operators.

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## 2. Preliminary results

In the sequel, we shall need the following results:

**Lemma 1.** [5] *Let $m \in \mathbb{N} \cup 0$. If the mth order is defined as*

$$T_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n} - x\right)^m,$$

*then $T_{n,0}(x) = 1$, $T_{n,1}(x) = 0$ and also there holds the recurrence relation:*

$$T_{n,m+1}(x) = x(1+x)[T\prime_{n,m}(x) + mT_{n,m-1}(x)].$$

*Consequently, we have $T_{n,m}(x) = O(n^{-[(m+1)/2]})$.*

**Lemma 2.** [10] *If we define the central moments as*

$$\mu_{n,m}(x) = D_n^{(\alpha,\beta)}((t-x)^m, x) = \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt + p_{n,0}(x)\left(\frac{\alpha}{n+\beta} - x\right)^m, m \in \mathbb{N}$$

*Then, $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = \frac{\alpha-\beta x}{n+\beta}$ and for $n > m$ we have the following recurrence relation:*

$$(n-m)(n+\beta)\mu_{n,m+1}(x) = nx(1+x)\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right] + [n(\alpha-\beta x) - 2m(\alpha-(n+\beta)x) + mn]\mu_{n,m}(x) + \left[(n+\beta)m\left(\frac{\alpha}{n+\beta} - x\right)^2 - mn\left(\frac{\alpha}{n+\beta} - x\right)\right]\mu_{n,m-1}(x). \quad (2.1)$$

*From the recurrence relation, it can easily be verified that for all $x \in [0,\infty)$, we have*

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

**Remark 1.** From Lemma 2, we get that $D_n^{(\alpha,\beta)}(t^m, x)$ is a polynomial in $x$ of degree exactly $m$, for all $m \in \mathbb{N}^0$. Further $D_n^{(\alpha,\beta)}(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{\alpha^{n-j} n^j}{(n+\beta)^m} D_n(t^j, x)$ and we can write as

$$D_n^{(\alpha,\beta)}(t^m, x) = \frac{n^m(n+m-1)!(n-m)!}{(n+\beta)^m n!(n-1)!} x^m + \frac{mn^{m-1}(n+m-2)!(n-m)!}{(n+\beta)^m n!(n-1)!}[n(m-1) + \alpha(n-m+1)]x^{m-1} + \frac{m(m-1)n^{m-2}\alpha(n+m-3)!(n-m+1)!}{(n+\beta)^m n!(n-1)!} \times \left[n(m-2) + \frac{\alpha(n-m+2)}{2}\right]x^{m-2} + O(n^{-2}).$$

**Lemma 3.** [5] *There exist the polynomials $q_{i,j,r}(x)$ independent of n and k such that*

$$[x(1+x)]^r \frac{d^r}{dx^r}[p_{n,k}(x)] = \sum_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} n^i (k-nx)^j q_{i,j,r}(x) p_{n,k}(x).$$

**Lemma 4.** *Let f be $r-$ times differentiable on $[0,\infty)$ such that $f^{(r-1)}(t) = O(t^\gamma)$, $\gamma > 0$ as $t \to \infty$. Then for $r = 1, 2, \ldots$, we have*

$$\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f, x) = \frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r n!(n-1)!} \sum_{k=0}^{\infty} p_{n+r,k}(x) \times \int_0^{\infty} b_{n-r,k+r} f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

The proof of the above lemma easily follows along the lines of the proof of ([7], Lemma 2.3).

**Definition 1.** The $m$th order modulus of continuity $\omega_m(f, \delta, [a,b])$ for a function continuous on $[a,b]$ is defined by

$$\omega_m(f, \delta, [a,b]) = \sup\left\{|\Delta_h^m f(x)| : |h| \leqslant \delta; x, x+h \in [a,b]\right\}.$$

For $m = 1$, $\omega_m(f, \delta)$ is usual modulus of continuity.

**Definition 2.** Let us assume that $0 < a < a_1 < b_1 < b < \infty$, for sufficiently small $\eta > 0$ the Steklov mean $f_{\eta,2}$ of 2-nd order corresponding to $f \in C_\gamma[a,b]$ and $t \in I_1$ is defined as follows:

$$f_{\eta,2}(t) = \eta^{-2}\int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\left(f(t) - \Delta_h^2 f(t)\right)dt_1 dt_2$$

where $h = (t_1 + t_2)/2$ and $\Delta_h^2$ is the second order forward difference operator with step length $h$. For $f \in C[a,b]$, $f_{\eta,2}$ satisfy the following properties ([9]):

(1) $f_{\eta,2}$ has continuous derivatives up to order 2 over $[a_1, b_1]$;
(2) $\|f_{\eta,2}\|_{C[a_1,b_1]} \leqslant C\omega_r(f, \eta, [a,b]), r = 1, 2;$
(3) $\|f - f_{\eta,2}\|_{C[a_1,b_1]} \leqslant C\omega_2(f, \eta, [a,b]);$
(4) $\|f_{\eta,2}\|_{C[a_1,b_1]} \leqslant C\eta^{-2}\|f\|_{C[a,b]};$
(5) $\|f_{\eta,2}\|_{C[a_1,b_1]} \leqslant C\|f\|_\gamma,$

where $C$'s are certain constants which are different in each occurrence and are independent of $f$ and $\eta$.

**Lemma 5.** [3] *Let $f \in C[a,b]$.* Then,

$$\left\|f_{\eta,2k}^{(i)}\right\|_{C[a,b]} \leqslant C_i\left\{\|f_{\eta,2k}\|_{C[a,b]} + \|f_{\eta,2k}^{(2k)}\|_{C[a,b]}\right\}, \quad i = 1, 2, \ldots, 2k-1,$$

*where $C_i'$s are certain constants independent of f.*

## 3. Direct results

This section deals with the direct results, we establish here pointwise approximation, asymptotic formula and error estimations in simultaneous approximation.

We denote $C_\gamma[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\gamma), \gamma > 0\}$. It can be easily verified that the operators $D_n^{(\alpha,\beta)}(f, x)$ are well defined for $f \in C_\gamma[0,\infty)$.

**Theorem 1.** *Let $\alpha$, $\beta$ be two parameters satisfying the conditions $0 \leqslant \alpha \leqslant \beta$. If $r \in \mathbb{N}, f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty}\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f, x) = f^{(r)}(x). \quad (3.1)$$

*Further, if $f^{(r)}$ exists and continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then (3.1) holds uniformly in $[a,b]$.*

**Proof.** By Taylor's expansion of $f$, we have

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

$$\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f,x)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left[D_n^{(\alpha,\beta)}\right]^{(r)}((t-x)^i,x)+\left[D_n^{(\alpha,\beta)}\right]^{(r)}(\varepsilon(t,x)(t-x)^r,x)=:I_1+I_2.$$

In view of Remark 1, we have

$$I_1=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}[D_n^{(\alpha,\beta)}]^{(r)}(t^j,x)=\frac{f^{(r)}(x)}{r!}\left(\frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r n!(n-1)!}r!\right)=f^{(r)}(x)\left(\frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r n!(n-1)!}\right)\to f^{(r)}(x)\text{ as }n\to\infty.$$

Next, we estimate $I_2$ by using Lemma 3, we have

$$I_2=\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\frac{q_{i,j,r}(x)}{x^r(1+x)^r}\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^j\times\int_0^\infty b_{n,k}(t)\varepsilon(t,x)\left(\frac{nt+\alpha}{n+\beta}-x\right)^r dt+(-1)^r\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\varepsilon(0,x)\left(\frac{\alpha}{n+\beta}-x\right)^r$$

$$|I_2|\leqslant\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\times\int_0^\infty b_{n,k}(t)|\varepsilon(t,x)|\left|\frac{nt+\alpha}{n+\beta}-x\right|^r dt+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}|\varepsilon(0,x)|\left|\frac{\alpha}{n+\beta}-x\right|^r=:I_3+I_4.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$, for a given $\varepsilon>0$ there exist $\delta>0$ such that $|\varepsilon(t,x)|$ whenever $|t-x|<\delta$, further if $\lambda$ is any integer $\geqslant\max\{\gamma,r\}$ then we find a constant $K>0$ such that $|\varepsilon(t,x)|\left|\frac{nt+\alpha}{n+\beta}-x\right|^r\leqslant K\left|\frac{nt+\alpha}{n+\beta}-x\right|^\gamma$. Thus

$$I_3=C_1\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\left\{\int_{|t-x|<\delta}\varepsilon b_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^r dt+\int_{|t-x|\geqslant\delta}Kb_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^\gamma dt\right\}=:I_5+I_6.$$

Applying Schwarz inequality for the integration and summation we have

$$I_5\leqslant\varepsilon C_1\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\left(\int_0^\infty b_{n,k}(t)dt\right)^{\frac12}\times\left(\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2r}dt\right)^{\frac12}\leqslant\varepsilon C_1\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\left(\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^{2j}\right)^{\frac12}\times\left(\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2r}dt\right)^{\frac12},$$

as $\int_0^\infty b_{n,k}(t)dt=1$. Making use of Lemma 2, we get

$$\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^{2j}=n^{2j}\left[\sum_{k=1}^{\infty}p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2j}-(1+x)^{-n}(-x)^{2j}\right]=n^{2j}[O(n^{-j})+O(n^{-s})]\text{ (for any }s>0)=O(n^{-j}).\tag{3.2}$$

Also, by using Lemma 2 and arguing as above, we have

$$\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2r}dt=O(n^{-r}).\tag{3.3}$$

Thus

$$I_5\leqslant\varepsilon C_1\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\cdot O(n^{j/2})\cdot O(n^{-r/2})=\varepsilon O(1).$$

Next, using Schwarz inequality for the integration and summation, in view of (3.2) and (3.3), we have

$$I_6\leqslant C_2\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\times\int_{|t-x|\geqslant\delta}b_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^\gamma dt\leqslant C_2\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\left(\int_{|t-x|\geqslant\delta}b_{n,k}(t)dt\right)^{\frac12}\times\left(\int_{|t-x|\geqslant\delta}b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2\gamma}dt\right)^{\frac12}$$

$$\leqslant C_2\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\left(\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^{2j}\right)^{\frac12}\times\left(\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2\gamma}dt\right)^{\frac12}=\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\cdot O(n^{j/2})\cdot O(n^{-m/2})=O(n^{(r-m)/2})=o(1),$$

where $m$ is an integer $\geqslant\gamma$. Thus due to the arbitrariness of $\varepsilon$, it follows that $I_3=o(1)$. Also, $I_4\to 0$ as $n\to\infty$ and hence

$I_2 = o(1)$. Combining the estimates $I_1$ and $I_2$ we obtain the desired result (3.1).

This completes the proof of theorem. □

**Theorem 2.** *Let $f \in C_\gamma[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ admitting the derivative of order $(r+2)$ at a fixed $x \in (0,\infty)$. Let $f(t) = O(t^\gamma)$ as $t \to \infty$ for some $\gamma > 0$, then we have*

$$\lim_{n\to\infty} n\left(\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f,x) - f^{(r)}(x)\right) = r(r-1-\beta)f^{(r)}(x) + [r(1+2x)+\alpha-\beta x]f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x). \tag{3.4}$$

**Proof.** By Taylor's expansion of $f$, we have

$$f(t) = \sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = o((t-x)^\delta)$ as $t \to \infty$ for some $\delta > 0$,

Using Lemma 4, we can write

$$\begin{aligned} n\left[\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f,x) - f^{(r)}(x)\right] &= n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}[D_n^{(\alpha,\beta)}]^{(r)}((t-x)^i,x) - f^{(r)}(x)\right] \\ &\quad + n\left[\left[D_n^{(\alpha,\beta)}\right]^{(r)}(\varepsilon(t,x)(t-x)^{r+2},x)\right] \\ &=: I_1 + I_2. \end{aligned}$$

By Lemma 2 and Remark 1, we have

$$\begin{aligned} I_1 &= n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}\left[D_n^{(\alpha,\beta)}\right]^{(r)}(t^j,x) - nf^{(r)}(x) \\ &= \frac{f^{(r)}(x)}{r!}n\left[\left[D_n^{(\alpha,\beta)}\right]^{(r)}(t^r,x) - (r!)\right] \\ &\quad + \frac{f^{(r+1)}(x)}{(r+1)!}n\Big\{(r+1)(-x)\left[D_n^{(\alpha,\beta)}\right]^{(r)}(t^r,x) \\ &\quad + \left[D_n^{(\alpha,\beta)}\right]^{(r)}(t^{r+1},x)\Big\} + \frac{f^{(r+2)}(x)}{(r+2)!}n\Big\{\frac{(r+2)(r+1)}{2}x^2[D_n^{(\alpha,\beta)}]^{(r)}(t^r,x) \\ &\quad + (r+2)(-x)\left[D_n^{(\alpha,\beta)}\right]^{(r)}(t^{r+1},x) + \left[D_n^{(\alpha,\beta)}\right]^{(r)}(t^{r+2},x)\Big\} \\ &= n\left[\frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r n!(n-1)!} - 1\right]f^{(r)}(x) \\ &\quad + n\frac{f^{(r+1)}(x)}{(r+1)!}\Big\{(r+1)(-x)\frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r n!(n-1)!}r! \\ &\quad + \frac{n^{r+1}(n+r)!(n-r-1)!}{(n+\beta)^{r+1}n!(n-1)!}(r+1)!x \\ &\quad + \frac{(r+1)n^r(n+r-1)!(n-r-1)!}{(n+\beta)^{r+1}n!(n-1)!}\{nr+\alpha(n-r)\}r!\Big\} \\ &\quad + n\frac{f^{(r+2)}(x)}{(r+2)!}\Big\{\frac{(r+2)(r+1)}{2}x^2\frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r n!(n-1)!}r! \\ &\quad - (r+2)x\Big(\frac{n^{r+1}(n+r)!(n-r-1)!}{(n+\beta)^{r+1}n!(n-1)!}(r+1)!x \\ &\quad + \frac{(r+1)n^r(n+r-1)!(n-r-1)!}{(n+\beta)^{r+1}n!(n-1)!}\{nr+\alpha(n-r)\}r!\Big) \\ &\quad + \frac{n^{r+2}(n+r+1)!(n-r-2)!}{(n+\beta)^{r+2}n!(n-1)!}\frac{(r+2)!}{2}x^2 \\ &\quad + \frac{(r+2)n^{r+1}(n+r)!(n-r-2)!}{(n+\beta)^{r+2}n!(n-1)!}\{n(r+1) \\ &\quad + \alpha(n-r-1)\}(r+1)!x \\ &\quad + \frac{(r+2)(r+1)n^r\alpha(n+r-1)!(n-r-1)!}{(n+\beta)^{r+2}n!(n-1)!}\{nr+\frac{\alpha(n-r)}{2}\}r! \end{aligned}$$

Now the coefficients of $f^{(r)}(x)$, $f^{(r+1)}(x)$ and $f^{(r+2)}(x)$ in the above expression are respectively $r(r-1-\beta)$, $r(1+2x)+\alpha-\beta x$ and $x(1+x)$ respectively, which follow by using induction hypothesis on $r$ and taking the limits as $n \to \infty$. Hence in order to prove (3.4), it suffices to show that $[x(1+x)]^r I_2 \to 0$ as $n \to \infty$, which follows on proceeding along the lines in the estimation of $I_2$ as done in Theorem 1. □

**Theorem 3.** *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $0 < a < a_1 < b_1 < b < \infty$. Then for $n$ sufficiently large, we have*

$$\left\|\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f,.) - f^{(r)}\right\|_{C[a_1,b_1]} \leqslant C_1\omega_2(f^{(r)}, n^{-1/2}, [a_1,b_1]) + C_2 n^{-k}\|f\|_\gamma,$$

*where $C_1 = C_1(r)$ and $C_2 = C_2(r,f)$.*

**Proof.** we can write

$$\begin{aligned} \left\|\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f,.) - f^{(r)}\right\|_{C[a_1,b_1]} &\leqslant \left\|\left[D_n^{(\alpha,\beta)}\right]^{(r)}((f-f_{\eta,2}),.)\right\|_{C[a_1,b_1]} \\ &\quad + \left\|\left[D_n^{(\alpha,\beta)}\right]^{(r)}(f_{\eta,2},.) - f_{\eta,2}^{(r)}\right\|_{C[a_1,b_1]} \\ &\quad + \left\|f^{(r)} - f_{\eta,2}^{(r)}\right\|_{C[a_1,b_1]} \\ &= S_1 + S_2 + S_3. \end{aligned}$$

Since $f_{\eta,2}^{(r)} = (f^{(r)})_{\eta,2}$, hence by property (3) of the Steklov mean, we get

$$S_3 \leqslant C_1\omega_2(f^{(r)}, \eta, [a,b]).$$

Next, using Theorem 2 and Lemma 5, we get

$$S_2 \leqslant C_2 n^{-1}\sum_{i=r}^{2+r}\left\|f_{\eta,2}^{(i)}\right\|_{C[a,b]} \leqslant C_4 n^{-1}\left\{\|f_{\eta,2}\|_{C[a,b]} + \left\|f_{\eta,2}^{(2+r)}\right\|_{C[a,b]}\right\}.$$

By applying properties (2) and (4) of Steklov mean, we obtain

$$S_2 \leqslant C_4 n^{-1}\{\|f\|_\gamma + \eta^{-2}\omega_2(f^{(r)}, \eta, [a,b])\}.$$

Finally, we estimate $S_1$ choosing $a^*, b^*$ satisfying the condition $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Also let $\chi(t)$ denotes the characteristic function on the interval $[a^*, b^*]$, then

$$\begin{aligned} S_1 &\leqslant \left\|\left[D_n^{(\alpha,\beta)}\right]^{(r)}(\chi(t)(f(t) - f_{\eta,2}(t)),.)\right\|_{C[a_1,b_1]} \\ &\quad + \left\|\left[D_n^{(\alpha,\beta)}\right]^{(r)}((1-\chi(t))(f(t) - f_{\eta,2}(t)),.)\right\|_{C[a_1,b_1]} \\ &= S_4 + S_5. \end{aligned}$$

By Lemma 4, we have

$$\left[D_n^{(\alpha,\beta)}\right]^{(r)}(\chi(t)(f(t) - f_{\eta,2}(t)),x) = \frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r n!(n-1)!}\sum_{k=0}^{\infty}p_{n+r,k}(x) \int_0^\infty b_{n-r,k+r}(t)\chi(t)\left[f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right) - f_{\eta,2}^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)\right]dt.$$

Hence,

$$\left\|\left[D_n^{(\alpha,\beta)}\right]^{(r)}(\chi(t)(f(t) - f_{\eta,2}(t)),.)\right\|_{C[a_1,b_1]} \leqslant C_5\left\|f^{(r)} - f_{\eta,2}^{(r)}\right\|_{C[a^*,b^*]}.$$

Now for $x \in [a_1, b_1]$ and $t \in [0,\infty)\backslash[a^*, b^*]$, we choose a $\delta > 0$ satisfying $\left|\frac{nt+\alpha}{n+\beta} - x\right| \geqslant \delta$. By Lemma 3 and Schwarz inequality, we have

$$\begin{aligned}
I &= |[D_n^{(\alpha,\beta)}]^{(r)}((1-\chi(t))(f(t)-f_{\eta,2}(t)),x)| \\
&\leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j \\
&\quad\times\int_0^{\infty} b_{n,k}(t)((1-\chi(t))\left|f\left(\frac{nt+\alpha}{n+\beta}\right)-f_{\eta,2}\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt \\
&\quad+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}((1-\chi(t))\left|f\left(\frac{\alpha}{n+\beta}\right)-f_{\eta,2}\left(\frac{\alpha}{n+\beta}\right)\right| \\
&\leqslant C_6\|f\|_{\gamma}\left\{\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\int_{|t-x|<\delta}b_{n,k}(t)dt\right. \\
&\quad\left.+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\right\} \\
&\leqslant C_6\|f\|_{\gamma}\left\{\delta^{-2s}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\left(\int_0^{\infty}b_{n,k}(t)dt\right)^{1/2}\right. \\
&\quad\left.\times\left(\int_0^{\infty}b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{4s}dt\right)^{1/2}+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\right\} \\
&\leqslant C_6\|f\|_{\gamma}\delta^{-2s}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\left\{\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^{2j}-(1+x)^{-n-r}(-nx)^{2j}\right\}^{1/2} \\
&\quad\times\left(\int_0^{\infty}b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{4s}dt\right)^{1/2}+\|f\|_{\gamma}\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}
\end{aligned}$$

Hence by making the use of Lemmas 1 and 2, we get

$$I \leqslant C_7\|f\|_{\gamma} \leqslant \delta^{-2m}O(n^{(i+j/2-s)}) \leqslant C_7 n^{-q}\|f\|_{\gamma}, \quad q = s - r/2,$$

where the last term vanishes as $n \to \infty$. Now choosing $m > 0$ satisfying $q \geqslant k$, we have

$$I \leqslant C_7 n^{-1}\|f\|_{\gamma}.$$

Therefore by property (3) of Steklov mean, we obtain

$$S_1 \leqslant C_9\omega_2(f^{(r)}, \eta, [a,b]) + C_7 n^{-1}\|f\|_{\gamma}.$$

Choosing $\eta = n^{-1/2}$ the theorem follows. □

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