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ORIGINAL ARTICLE

Characterization of distributions by conditional expectation of record values

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Abstract A family of continuous probability distributions has been characterized by two conditional expectations of record statistics conditioned on a non-adjacent record value. Besides various deductions, this work extends the result of Lee [8] in which Pareto distribution has been characterized.

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1. Introduction

Characterizations of distributions through conditional expectations of record values have been considered among others by Nagaraja [1], Franco and Ruiz [2], Wu and Lee [3], Raqab [4], Athar et al. [5], Gupta and Ahsanullah [6].

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed continuous random variables with the distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$. Let $X_{u(s)}$ be the s-th upper record value, then the conditional *pdf* of $X_{u(s)}$ given $X_{u(r)} = x$, $1 \leqslant r < s$ is Ahsanullah [7]

$$f(X_{u(s)}|X_{u(r)} = x) = \frac{1}{\Gamma(s-r)}\left[-\ln \overline{F}(y) + \ln \overline{F}(x)\right]^{s-r-1} \frac{f(y)}{\overline{F}(x)}, \quad x < y, \tag{1.1}$$

where $\overline{F}(x) = 1 - F(x)$.

Lee [8] has characterized Pareto distribution by conditional expectation of two records $X_{u(s)}$ and $X_{u(r)}$ conditioned on $X_{u(m)}$ for all $s > r \geqslant m$, where $s = r+1$, $r+2$ and r + 3. In this paper we have characterized a general class of distributions $\overline{F}(x) = [ah(x) + b]^c$ by the conditional expectation of $X_{u(s)}$ and $X_{u(r)}$ conditioned on $X_{u(m)}$ for all $s > r \geqslant m$, thus extending the results of Lee [8].

2. Characterization results

Theorem 2.1. *Let X be an absolutely continuous random variable with the df $F(x)$ and the pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Then for $m \leqslant r < s$*

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$$E\left[h\left(X_{u(s)}\right)|X_{u(m)}=x\right]=a^{*}E\left[h\left(X_{u(r)}\right)|X_{u(m)}=x\right]+b^{*} \tag{2.1}$$

if and only if

$$\overline{F}(x)=[ah(x)+b]^{c}, \tag{2.2}$$

where $a^{*}=\left(\frac{c}{c+1}\right)^{s-r}$ *and* $b^{*}=-\frac{b}{a}(1-a^{*})$.

Proof. In view of the Athar et al. [5], we have

$$E\left[h\left(X_{u(s)}\right)|X_{u(m)}=x\right]=a_{1}^{*}h(x)+b_{1}^{*}, \tag{2.3}$$

where,

$$a_{1}^{*}=\left(\frac{c}{c+1}\right)^{s-m} \quad \text{and} \quad b_{1}^{*}=-\frac{b}{a}\left(1-a_{1}^{*}\right)$$

and

$$E\left[h\left(X_{u(r)}\right)|X_{u(m)}=x\right]=a_{2}^{*}h(x)+b_{2}^{*} \tag{2.4}$$

where

$$a_{2}^{*}=\left(\frac{c}{c+1}\right)^{r-m} \quad \text{and} \quad b_{2}^{*}=-\frac{b}{a}\left(1-a_{2}^{*}\right).$$

Using (2.3) and (2.4), it is easy to establish (2.1).

For sufficiency part, we have

$$\begin{aligned}&\frac{1}{\Gamma(s-m)}\int_{x}^{\beta}h(y)\left[-\ln\overline{F}(y)+\ln\overline{F}(x)\right]^{s-m-1}f(y)dy\\&\quad=a^{*}\frac{1}{\Gamma(r-m)}\int_{x}^{\beta}h(y)\left[-\ln\overline{F}(y)+\ln\overline{F}(x)\right]^{r-m-1}f(y)dy\\&\qquad+b^{*}\overline{F}(x)\end{aligned} \tag{2.5}$$

Differentiate both the sides of (2.5) w.r.t. x, to get

$$\begin{aligned}&-\frac{(s-m-1)}{\Gamma(s-m)}\int_{x}^{\beta}h(y)\left[-\ln\overline{F}(y)+\ln\overline{F}(x)\right]^{s-m-2}\frac{f(x)}{\overline{F}(x)}f(y)dy\\&\quad=-a^{*}\frac{(r-m-1)}{\Gamma(r-m)}\int_{x}^{\beta}h(y)\left[-\ln\overline{F}(y)+\ln\overline{F}(x)\right]^{r-m-2}\\&\qquad\times\frac{f(x)}{\overline{F}(x)}f(y)dy-b^{*}f(x).\end{aligned}$$

after noting that if $B=\int_{u(x)}^{v(x)}f(x,y)dy$ then

$$\frac{\partial B}{\partial x}=f(x,v)\frac{\partial v}{\partial x}-f(x,u)\frac{\partial u}{\partial x}+\int_{u(x)}^{v(x)}\frac{\partial f(x,y)}{\partial x}dy.$$

Therefore,

$$\begin{aligned}&\frac{1}{\Gamma(s-m-1)}\int_{x}^{\beta}h(y)\left[-\ln\overline{F}(y)+\ln\overline{F}(x)\right]^{s-m-2}f(y)dy\\&\quad=a^{*}\frac{1}{\Gamma(r-m-1)}\int_{x}^{\beta}h(y)\left[-\ln\overline{F}(y)+\ln\overline{F}(x)\right]^{r-m-2}f(y)dy\\&\qquad+b^{*}\overline{F}(x).\end{aligned}$$

Similarly, differentiating $(r-m-1)$ times both the sides w.r.t. x, we get

$$\begin{aligned}&\frac{1}{\Gamma(s-r)}\int_{x}^{\beta}h(y)\left[-\ln\overline{F}(y)+\ln\overline{F}(x)\right]^{s-r-1}\frac{f(y)}{\overline{F}(x)}dy\\&\quad=a^{*}h(x)+b^{*}=g_{s|r}(x).\end{aligned}$$

Using the result (Khan et al. [9]),

$$E\left[h\left(X_{u(s)}\right)|X_{u(r)}=x\right]=g_{s|r}(x)$$

we get,

$$\overline{F}(x)=e^{-\int_{\alpha}^{x}A(t)dt}$$

where

$$A(t)=\frac{g'_{s|r}(t)}{g_{s|r}(t)-g_{s|r+1}(t)}=-\frac{ach'(t)}{[ah(t)+b]} \quad \text{and} \quad \lim_{x\to\beta}\int_{\alpha}^{x}A(t)dt=\infty.$$

Thus,

$$\overline{F}(x)=[ah(x)+b]^{c}$$

and hence the theorem. □

Remark 2.1. At $r=m$, $h(x)=x$, we get the result as obtained by Franco and Ruiz [2,10], Athar et al. [5], Ahsanullah and Wesolowski [11], Dembińska and Wesolowski [12], Khan and Alzaid [13].

Remark 2.2. Lee [8] has obtained characterization result for Pareto distribution

$$\overline{F}(x)=x^{-\theta}, \quad x>1, \quad \theta>0, \quad \theta\neq 1,$$

which can be obtained by putting $a=1$, $b=0$, $c=-\theta$, $h(x)=x$ at $s=r+1$, $r+2$ and $r+3$ in the Theorem 2.1.

Remark 2.3. At $a=-\frac{a}{c}, b=1, c\to\infty$

$$a^{*}=1, \quad b^{*}=\frac{(s-r)}{a},$$

$\overline{F}(x)=e^{-ah(x)}$, $a>0$, reduces to the result as obtained by Khan et al. [14].

3. Examples based on the distribution function $F(x)=1-[a\,h(x)+b]^{c}$

Proper choice of a, b and $h(x)$ characterize the distributions as given below:

Distribution	$F(x)$	a	b	c	$h(x)$
Power function	$a^{-p}x^p$ $0 < x \leqslant a$	$-a^{-p}$	1	1	x^p
Pareto	$1 - a^p x^{-p}$ $a \leqslant x < \infty$	a^{-1} a^{-q}	0 0	$-p$ $-p/q$	$x,\ p \neq 1$ $x^q,\ q > 0, p \neq q$
Beta of the first kind	$1 - (1 - x)^p$ $0 \leqslant x \leqslant 1$	1 -1	0 1	p/q p	$(1 - x)^q,\ q > 0$ x
Weibull	$1 - e^{-\theta x^p}$ $0 \leqslant x < \infty$	1	0	θ/q	$e^{-qx^p}, q > 0$
Inverse Weibull	$e^{-\theta x^{-p}}$ $0 \leqslant x < \infty$	-1	1	1	$e^{-\theta x^{-p}}$
Burr type II	$[1 + e^{-x}]^{-k}$ $-\infty < x < \infty$	-1	1	1	$(1 + e^{-x})^{-k}$
Burr type III	$[1 + x^{-c}]^{-k}$ $0 \leqslant x < \infty$	-1	1	1	$(1 + x^{-c})^{-k}$
Burr type IV	$\left[1 + \left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$ $0 \leqslant x \leqslant c$	-1	1	1	$\left[1 + \left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$
Burr type V	$[1 + c\ e^{-tanx}]^{-k}$ $-\pi/2 \leqslant x \leqslant \pi/2$	-1	1	1	$[1 + c\ e^{-tanx}]^{-k}$
Burr type VI	$[1 + c\ e^{-ksinhx}]^{-k}$ $-\infty < x < \infty$	-1	1	1	$[1 + c\ e^{-ksinhx}]^{-k}$
Burr type VII	$2^{-k}(1 + tanhx)^k$ $-\infty < x < \infty$	-2^{-k}	1	1	$[1 + tanhx]^k$
Burr type VIII	$\left(\frac{2}{\pi} tan^{-1} e^x\right)^k$ $-\infty < x < \infty$	$-\left(\frac{2}{\pi}\right)^k$	1	1	$\left(tan^{-1} e^x\right)^k$
Burr type X	$\left(1 - e^{-x^2}\right)^k$ $0 < x < \infty$	-1	1	1	$\left(1 - e^{-x^2}\right)^k$
Burr type XI	$\left(x - \frac{1}{2\pi} sin 2\pi x\right)^k$ $0 \leqslant x \leqslant 1$	-1	1	1	$\left(x - \frac{1}{2\pi} sin 2\pi x\right)^k$
Burr type XII	$1 - (1 + \theta x^p)^{-m}$ $0 \leqslant x < \infty$	θ	1	$-m$	$x^p,\ m \neq 1$
Cauchy	$\frac{1}{2} + \frac{1}{\pi} tan^{-1} x$ $-\infty < x < \infty$	$-\frac{1}{\pi}$	$\frac{1}{2}$	1	tan^{-1}x

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