Emenogu
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Investigating the effect of loading on the governing equation at the split region for a semi-infinite crack in an orthotropic material under antiplane loading

Ndubueze G. Emenogu*

*Correspondence: nemenogu@gmail.com

Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria

Abstract

Most often there is a great disparity between experimental results and analytic results. Before now under great disparity, researchers kept suspecting the experimental procedures without investigating whether their analytic solution actually satisfy the governing equation. When a body in a plane is under loading, the loading splits the plane into regions, and the governing equation must be satisfied at these regions for the derived solution to be true. In this paper, I considered a homogeneous infinite orthotropic material containing a semi-infinite crack. A longitudinal shear load of magnitude Q is applied on the crack front. The displacement field in closed form is obtained. A verification of this solution at the split regions is carried out and shown to satisfy the governing differential equation.

Keywords: Closed-form solution, Semi-infinite crack, Infinite orthotropic material

Mathematics Subject Classification: 74B05, 74B10, 74E10

Introduction

The study of how cracks grow and propagate in orthotropic materials is important in the design of components for improved fracture toughness. The determination of the state of stress near the crack front differs from the usual problems of determining stress concentration in a body because the linear theory of elasticity and the linearized boundary conditions lead to infinite stresses and infinite stress gradients at the end of a thin cut as investigated by Li [1]. Consequently, the main interest is to determine the intensity of the stress field surrounding the crack tip, by finding the stress intensity factor (SIF), a dimensional quantity, with the dimension kgf/mm$^{3/2}$. The stress intensity factor may be found by analytical methods, numerical methods or a combination of both. Analytical methods have been used to develop fracture mechanics and have delivered the basic equations for crack tip stress and displacement fields, which may serve as the starting point for many other solutions. Analytical methods try to satisfy the boundary conditions exactly and result in closed-form solutions. They are useful for investigating crack problems in the case of infinite bodies. When a problem is not readily amenable to an analytical solution, numerical procedures may be used to obtain an approximate solution.

Many researchers have tried to determine the stress intensity factors near the tip of a crack in an orthotropic material. Sih and Chen [2] investigated the problem of cracks moving in a finite orthotropic strip under tearing action. By application of Schwarz–Christoffel transformation and the complex variable theory, closed-form solutions of the stress intensity factors were obtained and the effect of strip width on the dynamic stresses was examined. It was found that the SIF increased monotonically with decreasing strip width and the effect becomes more pronounced at higher crack velocity. Georgiadis and Theocaris [3] solved the problem of steady-state elastodynamic crack problem by the method of a complex variable. Tait and Moodie [4] considered the problem of a finite-length crack moving with constant velocity in an orthotropic strip under antiplane shear stress using a complex variable method. Also using the complex variable method, the problem of a cracked orthotropic strip under antiplane stresses or displacements was solved by Georgiadis [5]. The values for the stress and SIF at the crack tip were obtained. Xiangfa and Yuri's [6] obtained an analytic closed solution for the problem of a mode III edge crack between two bonded semi-infinite non-homogeneous elastic strips. Using the conformal mapping technique and dislocation solution, an antiplane displacement potential for the interacting crack was constructed. Employing this displacement potential, SIF and the energy release rate for the edge crack are obtained. Sih and Chen [7] investigated the problem of Griffith crack in an orthotropic layer subjected to antiplane shear. The problem was solved numerically to obtain the values of the SIF. An orthotropic strip containing a finite-length crack was investigated by Singh et al. [8] using the integral transform technique. They obtained the values of SIF. Danyluk and Singh [9] used an integral transform technique for the plane problem of a crack of fixed length moving at a constant velocity in the same direction as the surfaces of an orthotropic solid. Exact solutions for the stress intensity factor were obtained. Similarly, an integral transform technique was used by Singh et al. [10] to solve the elastodynamic problem of a crack at the interface of two bonded dissimilar orthotropic solids. Closed-form expressions for the SIF were obtained. Das [11] studied an orthotropic elastic layer with punches. Using numerical techniques, an expression for the SIF was obtained. Itou [12] used the Fourier transform technique and the Schmidt method to determine the stresses around a moving finite crack with a constant velocity in an elastic layer between two elastic half-planes. The value of the stress intensity factor is obtained numerically. Rizk [13] investigated an orthotropic semi-infinite plate containing a crack under thermal shock. The Fourier transform technique and the expansion method were used to solve the problem. The results show the effect of material orthotropy on the stress intensity factors. A rectangular cracked bi-material consisting of two dissimilar orthotropic elastic media was analyzed by Xian and Xiang [14]. By employing the Fourier series method and Lobatto–Chebyshev method, the values of the stress intensity factors were obtained. The results show that the SIF depends on the material properties as well as the geometry of the configuration. A non-homogenous orthotropic material containing multiple interfacial cracks was investigated by Matbuly and Nassar [15]. Applying finite Fourier transforms and Gauss–Chebyshev integration formulae, the stress intensity factors are determined in closed-form expressions. Chen et al. [16] considered the out-of-plane elasticity crack problem for an orthotropic strip with mixed boundary condition. Using analytical methods, the stress intensity factor at the crack tip was evaluated. Mode III problem of a cracked orthotropic strip containing a Volterra-type screw dislocation was carried out by Monfared et al. [17]. He constructed an integral equation using the distributed dislocation technique which was solved numerically leading to SIF. Mousavi and Fariborz [18] studied the stress distribution in a graded orthotropic solid containing a screw dislocation under time-harmonic deformation. Employing numerical methods, they obtained the dislocation density function on the crack surfaces and stress intensity factors of cracks. The dynamic behavior of moving cracks in a non-homogeneous orthotropic half-plane under antiplane loading was considered by Nourazar and Ayatollahi [19]. Applying the Galilean transformation, the governing wave equation was converted to an equation independent of time. Finally employing the complex Fourier transform and the distributed dislocation technique, the value of the screw dislocation was obtained. The solution was employed to derive integral equations leading to the stress intensity factor.

The problem of two finite-length cracks moving in an orthotropic layer under antiplane loading was carried out by Singh et al. [20]. Using Fourier transforms, the analysis of the problem was reduced to solving a system of integral equations. An analytical solution of these integral equations was obtained, leading to an exact expression for the stress intensity factors.

The aim of this study

Many numerical research works have either been abandoned or completely discarded because the simulated results are at great variance with the analytic results. The analytic results most times are never subjected to test to ascertain that they satisfy the given governing equation before using them as a benchmark to checkmate the validity of the numerical results. The aim of this study is to ensure that analytic solutions are subjected to thorough verification in line with the governing equations before they can be a benchmark for comparison and further usage in the derivation of other results.

Significance of the study

This study is very vital in the sense that every numerical result gotten in the literature is compared with the analytical result which serves as a benchmark for the validity of the numerical simulations. In this sense, a well-carried-out experimental result may be seen to be in error when compared with an analytic result whose result was not verified as to satisfy the governing equation. These flaws may put a clog in the wheel sciences and research in general. Thus, this study will benefit the academia, researchers of different fields of endeavor, students and the like in ensuring that exact solutions are really exact and not misleading.

Formulation of the problem

In this paper, we consider the antiplane problem of an infinite elastic orthotropic material containing an infinite crack with its tip referred to a moving coordinate system (x', y', z'). The semi-infinite crack occupies the region defined by $-\infty < x' < 0$. A pair of longitudinal shear loads of magnitude Q is applied along the cracked surface on an interval $[-a\,,\,-d]$

of length L. A circular crack breaker (stop hole) of radius b is introduced at the center of the orthotropic material which is at the origin of a fixed coordinate system (x, y, z). Figure 1 illustrates the configuration of the problem under consideration. Suppose that, at time $t = 0$, the crack tip starts to move with constant velocity along the x'-direction and ends up at the crack breaker, attaining a displacement $v\,t$. Suppose also that the disturbance due to the load is antiplane so that it creates an only out-of-plane displacement $w(x, y, t)$ and stresses $\sigma_{xz}(x, y)$ and $\sigma_{yz}(x, y)$ in the z-direction. The problem is to investigate the behavior of the elastic fields at the split regions.

In line with antiplane strain condition, the displacement components (u, v, w) reduce to $(0, 0, w)$ where $w = w(x', y', t')$. Consequently, the only nonzero stress components are $\sigma_{x'z}$ and $\sigma_{y'z}$ which are given by

$$\sigma_{x'z} = c_{44}\frac{\partial w}{\partial x'},\ \sigma_{y'z=}c_{55}\frac{\partial w}{\partial y'} \tag{1}$$

where c_{44} and c_{55} are the shear moduli in the x' and y' directions.

Satisfying the equation of motion given by

$$\frac{\partial \sigma_{x'z}}{\partial x'} + \frac{\partial \sigma_{y'z}}{\partial y'} = \gamma \frac{\partial^2 w}{\partial t^2} \tag{2}$$

leads to the wave equation in two dimensions for w as

$$\text{Governing equation}: \ \frac{\partial^2 w}{\partial x'^2} + \frac{1}{\eta^2}\frac{\partial^2 w}{\partial y'^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{3}$$

where $\eta = \left(\frac{c_{44}}{c_{55}}\right)^{\frac{1}{2}}$, $c = \left(\frac{c_{44}}{\gamma}\right)^{\frac{1}{2}}$ is the wave speed with γ as the mass density of the material.

The corresponding boundary conditions (BCS) for this situation can be stated as

$$\text{BCS}: \ \sigma_{zx'}(x, 0) = \begin{cases} \pm Q, & a \le x \le d \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

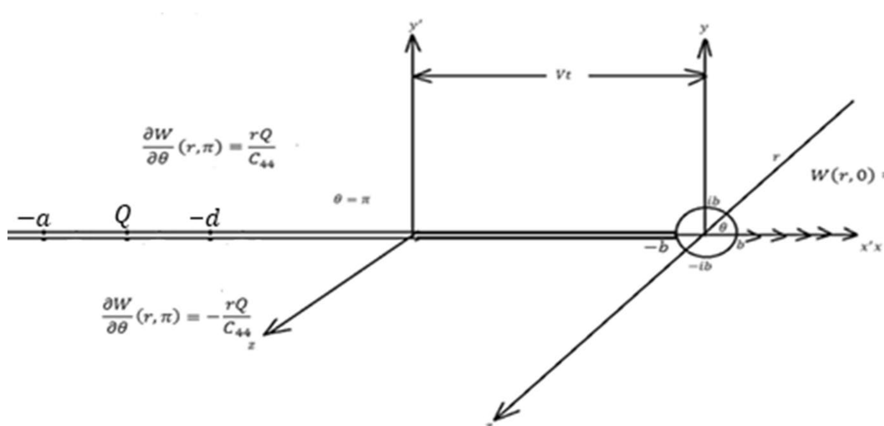

Fig. 1 Geometry of the problem

$$\sigma_{zy'}(b,0) = 0,\ b > 0 \tag{5}$$

$$w(x,0) = 0,\ x > 0. \tag{6}$$

We seek to investigate the displacement fields at the point $(x, 0, 0)$ on the boundary of the crack breaker in order to establish its conformity with the governing equation at that region.

Solution of the problem

For a crack moving with a constant velocity v, it is convenient to introduce the Galilean transformation [4]

$$x = x' - vt,\ y = \eta y',\ t' = t \tag{7}$$

to suppress the time component. The equation of motion becomes

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{8}$$

In polar coordinates (r, θ), the governing boundary problem takes the form

$$\frac{\partial^2 w(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\theta)}{\partial \theta^2} = 0 \quad r \geq b, -\pi < \theta < \pi \tag{9}$$

$$\frac{\partial w}{\partial \theta}(r, \pm\pi) = \begin{cases} \pm\frac{rQ}{c_{44}} & a \leq r \leq d \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$w(r,0) = 0 r > b \tag{11}$$

$$\frac{\partial w}{\partial r}(b,\theta) = 0, 0 < \theta < \pi \tag{12}$$

To make the problem analyzable by method of integral transform, the original z-plane of analysis is transformed onto ϕ-plane with a semi-infinite crack terminating at the origin by the holomorphic function (Figs. 2, 3).

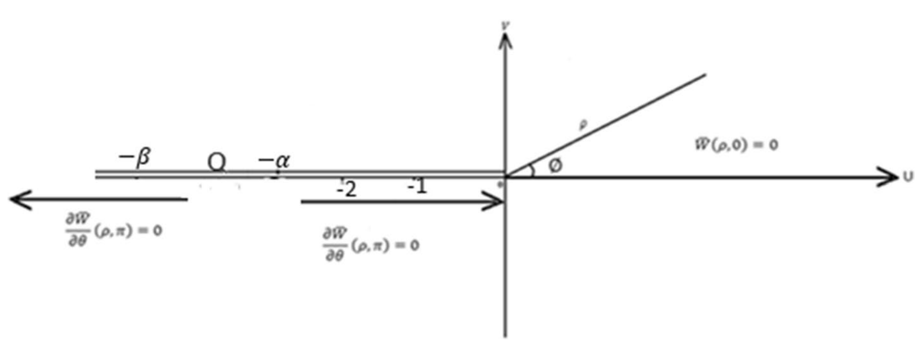

Fig. 2 Transformed configuration of the original problem

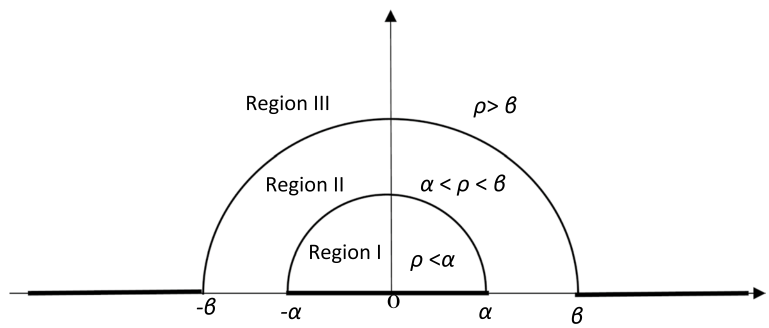

Fig. 3 Split regions in the upper half-plane

$$\varphi(z) = \frac{1}{2}\left(\frac{z}{b} + \frac{b}{z}\right) - 1 \tag{13}$$

Setting $\varphi(z) = \rho e^{i\phi}, z = re^{i\theta}$ and using the conformality condition

$$w(r, \theta) = W(\rho, \phi) \tag{14}$$

the boundary value problem in terms of $W(\rho, \phi)$ becomes

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \emptyset^2}\right) W(\rho, \phi) = 0 \ \rho > 0, \ 0 \leq \emptyset \leq \pi \tag{15}$$

$$\frac{\partial W(\rho, \pi)}{\partial \emptyset} = \begin{cases} \frac{bQ\rho}{C_{44}}\left[\frac{(\rho-1)}{\sqrt{\rho(\rho-2)}} + 1\right] & a \leq r \leq d \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

$$\frac{\partial W(b, \emptyset)}{\partial \emptyset} = 0 \quad 0 \leq \emptyset \leq \pi \tag{17}$$

$$W(\rho, \emptyset) = 0. \tag{18}$$

To solve the transformed problem, we use the Mellin integral transform method. The Mellin transform of $W(\rho, \phi)$ is defined by

$$\overline{W}(s, \phi) = \int_0^\infty W(\rho, \phi)\rho^{s-1}\partial\rho \tag{19}$$

Applying the Mellin transform to Eqs. (15)–(18), we obtain

$$\overline{W}(s, \emptyset) = \frac{bQ}{C_{44}}\mathcal{F}(\beta, \alpha; s)\frac{\sin \emptyset s}{s \cos \pi s} \tag{20}$$

where

$$\mathcal{F}(\beta,\alpha;s)=\int_{\alpha}^{\beta}\left(\rho^{s}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}-\rho^{s-1}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}+\rho^{s}\right)\partial\rho \tag{21}$$

Using the inverse Mellin transform, we obtain

$$W(\rho,\emptyset)\ \frac{bQ}{C_{44}}\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\mathcal{F}(\beta,\alpha;s)\frac{\sin\emptyset s}{s\cos\pi s}\rho^{-s}\mathrm{d}s \tag{22}$$

The quantity $\mathcal{F}(\beta,\alpha;s)$ given in Eq. (21) can be solved by the use of the convergent series

$$(1-t)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}c_k t^k\ |t|<1 \tag{23}$$

where the coefficients are defined by

$$c_k=\frac{(2k)!}{2^{2k}(k!)^2} \tag{24}$$

Term-by-term evaluation of the three terms in Eq. (21) and further simplification of Eq. (22) using residue theorem and Jordan lemma give the displacement field as

$$W(\rho,\emptyset)=\frac{bQ}{C_{44}}\left\{I_{\beta}^{(1)}-I_{\beta}^{(2)}+I_{\beta}^{(3)}\right\}-\frac{bQ}{C_{44}}\left\{I_{\alpha}^{(1)}-I_{\alpha}^{(2)}+I_{\alpha}^{(3)}\right\} \tag{25}$$

where for $\rho<\beta$

$$\begin{aligned}
I_{\beta}^{(1)}(\rho,\phi)=&-c_0 2^0\beta^1\sin\phi\left(\frac{\rho}{\beta}\right)\\
&+c_0 2^0\frac{\beta}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}\\
&-c_1 2^1\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n-\frac{1}{2}\right)^2}\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}\\
&-c_2 2^2\frac{1}{\pi\beta}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}\\
&-c_3 2^3\frac{1}{\pi\beta^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}\\
&-c_4 2^4\frac{1}{\pi\beta^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}-\\
&-c_5 2^5\frac{1}{\pi\beta^4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}\\
&-c_6 2^6\frac{1}{\pi\beta^5}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{9}{2}\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}-\cdots
\end{aligned} \tag{26}$$

$$
\begin{aligned}
-I_\beta^{(2)}(\rho,\phi) = {} & c_0 2^0 \beta^0 \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n-\frac{1}{2}\right)^2} \left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}} \\
& + c_1 2^1 \frac{1}{\pi\beta} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} \left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}} \\
& + c_2 2^2 \frac{1}{\pi\beta^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} \left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}} \\
& + c_3 2^3 \frac{1}{\pi\beta^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} \left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}} - \cdots
\end{aligned}
\tag{27}
$$

$$
I_\beta^{(3)}(\rho,\phi) = -\beta \sin\phi \left(\frac{\rho}{\beta}\right) + \frac{\beta}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)} \left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}. \tag{28}
$$

For $\rho > \beta$

$$
\begin{aligned}
I_\beta^{(1)}(\rho,\phi) = {} & \frac{\beta}{\pi} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(n-\frac{1}{2}\right)\phi \left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} \\
& - \phi + \frac{1}{\pi} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(n-\frac{1}{2}\right)\phi \left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^2} \\
& + \frac{c_2 2^2}{\beta} \sin\phi \left(\frac{\rho}{\beta}\right)^{-1} + c_2 2^2 \frac{1}{\pi\beta} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(n-\frac{1}{2}\right)\phi \left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} \\
& + \frac{c_3 2^3}{\beta^2} \frac{\sin 2\phi}{2} \left(\frac{\rho}{\beta}\right)^{-2} + c_3 2^3 \frac{1}{\pi\beta^2} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(n-\frac{1}{2}\right)\phi \left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} \\
& + \frac{c_4 2^4}{\beta^3} \frac{\sin 3\phi}{3} \left(\frac{\rho}{\beta}\right)^{-3} + c_4 2^4 \frac{1}{\pi\beta^3} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(n-\frac{1}{2}\right)\phi \left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)} \\
& + \frac{c_5 2^5}{\beta^4} \frac{\sin 4\phi}{4} \left(\frac{\rho}{\beta}\right)^{-4} + c_5 2^5 \frac{1}{\pi\beta^4} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(n-\frac{1}{2}\right)\phi \left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{9}{2}\right)\left(n-\frac{1}{2}\right)} \\
& + \frac{c_6 2^6}{\beta^5} \frac{\sin 5\phi}{5} \left(\frac{\rho}{\beta}\right)^{-5} + \cdots
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
-I_{\beta}^{(2)}(\rho,\phi) = \phi - \frac{1}{\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^{2}} \\
- \frac{c_1 2^1}{\beta}\sin\phi\left(\frac{\rho}{\beta}\right)^{-1} - \frac{c_1 2^1}{\pi\beta}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} \\
- \frac{c_2 2^2}{\beta^2}\frac{\sin 2\phi}{2}\left(\frac{\rho}{\beta}\right)^{-2} - \frac{c_2 2^2}{\pi\beta^2}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} \\
- \frac{c_3 2^3}{\beta^3}\frac{\sin 3\phi}{3}\left(\frac{\rho}{\beta}\right)^{-3} - \frac{c_3 2^3}{\pi\beta^3}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)} \\
\frac{c_4 2^4}{\beta^3}\frac{\sin 4\phi}{4}\left(\frac{\rho}{\beta}\right)^{-4} - \frac{c_4 2^4}{\pi\beta^4}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{9}{2}\right)\left(n-\frac{1}{2}\right)} \\
- \frac{c_5 2^5}{\beta^5}\frac{\sin 5\phi}{5}\left(\frac{\rho}{\beta}\right)^{-5} - \frac{c_5 2^5}{\pi\beta^5}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{11}{2}\right)\left(n-\frac{1}{2}\right)} \\
- \frac{c_6 2^6}{\beta^6}\frac{\sin 6\phi}{6}\left(\frac{\rho}{\beta}\right)^{-6} - \cdots
\end{aligned}
\tag{30}
$$

$$
I_{\beta}^{(3)}(\rho,\phi) = \frac{\beta}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}. \tag{31}
$$

For $\rho < \alpha$

$$
\begin{aligned}
I_{\alpha}^{(1)}(\rho,\phi) = -\sin\phi\rho + \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^{2}} \\
- c_2 2^2\frac{1}{\pi\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} - c_3 2^3\frac{1}{\pi\alpha^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} \\
- c_4 2^4\frac{1}{\pi\alpha^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} - c_5 2^5\frac{1}{\pi\alpha^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)\left(n-\frac{1}{2}\right)} \\
- c_6 2^6\frac{1}{\pi\alpha^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)\left(n-\frac{1}{2}\right)} - \cdots
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
-I_{\alpha}^{(2)}(\rho,\phi) &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^{2}} \\
&+ c_2 2^2\frac{1}{\pi\alpha^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} \\
&+ c_3 2^3\frac{1}{\pi\alpha^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} \\
&+ c_4 2^4\frac{1}{\pi\alpha^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{(n+\frac{7}{2})\left(n-\frac{1}{2}\right)} \\
&+ c_5 2^5\frac{1}{\pi\alpha^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)\left(n-\frac{1}{2}\right)} \\
&+ c_8 2^8\frac{1}{\pi\alpha^8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{15}{2}\right)\left(n-\frac{1}{2}\right)} \\
&+ c_9 2^9\frac{1}{\pi\alpha^9}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{17}{2}\right)\left(n-\frac{1}{2}\right)} + \cdots
\end{aligned}
\tag{33}
$$

$$
I_{\alpha}^{(3)}(\rho,\phi) = -\sin\phi\rho + \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)}. \tag{34}
$$

For $\rho > \alpha$

$$
\begin{aligned}
I_{\alpha}^{(1)}(\rho,\phi) &= \frac{\alpha}{\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} \\
&- \phi + \frac{1}{\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^{2}} + \frac{c_2 2^2}{\alpha}\sin\phi\left(\frac{\rho}{\alpha}\right)^{-1} \\
&+ c_2 2^2\frac{1}{\pi\alpha}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} + \frac{c_3 2^3}{\alpha^2}\frac{\sin 2\phi}{2}\left(\frac{\rho}{\alpha}\right)^{-2} \\
&+ c_3 2^3\frac{1}{\pi\alpha^2}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} + \frac{c_4 2^4}{\alpha^3}\frac{\sin 3\phi}{3}\left(\frac{\rho}{\alpha}\right)^{-3} \\
&+ c_4 2^4\frac{1}{\pi\alpha^3}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{(n-\frac{7}{2})\left(n-\frac{1}{2}\right)} + \frac{c_5 2^5}{\alpha^4}\frac{\sin 4\phi}{4}\left(\frac{\rho}{\alpha}\right)^{-4} \\
&+ c_5 2^5\frac{1}{\pi\alpha^4}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{9}{2}\right)\left(n-\frac{1}{2}\right)} + \frac{c_6 2^6}{\alpha^5}\frac{\sin 5\phi}{5}\left(\frac{\rho}{\alpha}\right)^{-5} + \cdots
\end{aligned}
\tag{35}
$$

$$\begin{aligned}
-I_{\alpha}^{(2)}(\rho,\phi) = \phi &- \frac{1}{\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^2} - \frac{c_1 2^1}{\alpha}\sin\phi\left(\frac{\rho}{\alpha}\right)^{-1} \\
&- \frac{c_1 2^1}{\pi\alpha}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{c_2 2^2}{\alpha^2}\frac{\sin 2\phi}{2}\left(\frac{\rho}{\alpha}\right)^{-2} \\
&- \frac{c_2 2^2}{\pi\alpha^2}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{c_3 2^3}{\alpha^3}\frac{\sin 3\phi}{3}\left(\frac{\rho}{\alpha}\right)^{-3} \\
&- \frac{c_3 2^3}{\pi\alpha^3}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{c_4 2^4}{\alpha^3}\frac{\sin 4\phi}{4}\left(\frac{\rho}{\alpha}\right)^{-4} \\
&- \frac{c_4 2^4}{\pi\alpha^4}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{9}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{c_5 2^5}{\alpha^5}\frac{\sin 5\phi}{5}\left(\frac{\rho}{\alpha}\right)^{-5} - \cdots
\end{aligned} \tag{36}$$

$$I_{\alpha}^{(3)}(\rho,\phi) = \frac{\alpha}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(n-\frac{1}{2}\right)\phi}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}} \tag{37}$$

From Eq. (24), we obtain the following values of c_k

$$\begin{aligned}
&c_0 = 1,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{8},\ c_3 = \frac{5}{16},\ c_4 = \frac{35}{128},\ c_5 = \frac{63}{256} \\
&c_6 = \frac{231}{1024},\ c_7 = \frac{429}{2048},\ c_8 = \frac{6435}{32,768},\ c_9 = \frac{12155}{65,536},\ c_{10} = \frac{46,189}{262,144}.
\end{aligned} \tag{38}$$

Since the loading split the upper half $\rho\phi$-plane into three regions denoted by R_1, R_{11} and R_{111} defined as follows $R_1 = \{(\rho,\phi)/\,0 < \rho < \alpha \quad, 0 < \phi < \pi\}$, $R_{11} = \{(\rho,\phi)/\alpha < \rho < \beta\ ,\ 0 < \phi < \pi\}$ and $R_{111} = \{(\rho,\phi)/\beta < \rho < \infty,\ 0 < \phi < \pi\}$, we verify that our result satisfies the governing equation at these three regions.

Results and discussion

Verification of the solution in the three regions

Region 1 $\rho < \alpha,\ \rho < \beta, 0 \le \phi \le \pi.$

So far, we have gotten the displacement field which is the basis for determining other elastic fields such as stress field and stress intensity factor. To avoid having a misleading result in the end, we subject this result to thorough verifications at the three regions. Does it satisfy the governing equation

$$\frac{\partial^2 W}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial\phi^2} = 0 \tag{39}$$

and the boundary condition

$$W(\rho, 0) = 0, \quad 0 < \rho < \alpha \tag{40}$$

$$\frac{\partial W}{\partial \phi}(\rho, \pi) = 0, \quad 0 < \rho < \alpha \tag{41}$$

Substituting Eqs. (26–28) and (32–34) in to Eq. (25), we have

$$\begin{aligned}
\frac{\partial W(\rho,\phi)}{\partial \phi} = \frac{bQ}{c_{44}} \Bigg\{ & \frac{2\beta}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)} \\
& + \left(c_1 2^1 - c_2 2^2\right) \frac{1}{\pi\tau} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} \\
& + \left(c_2 2^2 - c_3 2^3\right) \frac{1}{\pi\beta^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)} \\
& + \left(c_3 2^3 - c_4 2^4\right) \frac{1}{\pi\beta^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)} \\
& + \left(c_4 2^4 - c_5 2^5\right) \frac{1}{\pi\beta^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)} \\
& + \left(c_5 2^5 - c_6 2^6\right) \frac{1}{\pi\beta^5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)} \\
- \frac{bQ}{c_{44}} \Bigg\{ & \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)} \\
& + \left(c_1 2^1 - c_2 2^2\right) \frac{1}{\pi\alpha} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} \\
& + \left(c_2 2^2 - c_3 2^3\right) \frac{1}{\pi\alpha^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)} \\
& + \left(c_3 2^3 - c_4 2^4\right) \frac{1}{\pi\alpha^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)} \\
& + \left(c_4 2^4 - c_5 2^5\right) \frac{1}{\pi\alpha^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)} \\
& + \left(c_5 2^5 - c_6 2^6\right) \frac{1}{\pi\alpha^5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)}
\end{aligned} \tag{42}$$

Therefore,

$$W(\rho, 0) = 0 \tag{43}$$

Differentiating Eq. (42), we have

$$\begin{aligned}
\frac{\partial W(\rho,\phi)}{\partial \phi} = \frac{bQ}{c_{44}} \Bigg\{ & \frac{2\beta}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)} \\
& + \left(c_1 2^1 - c_2 2^2\right) \frac{1}{\pi\tau} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} \\
& + \left(c_2 2^2 - c_3 2^3\right) \frac{1}{\pi\alpha^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)} \\
& + \left(c_3 2^3 - c_4 2^4\right) \frac{1}{\pi\beta^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)} \\
& + \left(c_4 2^4 - c_5 2^5\right) \frac{1}{\pi\beta^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)} \\
& + \left(c_5 2^5 - c_6 2^6\right) \frac{1}{\pi\beta^5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)} \\
& - \frac{bQ}{c_{44}} \Bigg\{ \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)} \\
& + \left(c_1 2^1 - c_2 2^2\right) \frac{1}{\pi\alpha} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} \\
& + \left(c_2 2^2 - c_3 2^3\right) \frac{1}{\pi\alpha^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)} \\
& + \left(c_3 2^3 - c_4 2^4\right) \frac{1}{\pi\alpha^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)} \\
& + \left(c_4 2^4 - c_5 2^5\right) \frac{1}{\pi\alpha^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)} \\
& + \left(c_5 2^5 - c_6 2^6\right) \frac{1}{\pi\alpha^5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)}
\end{aligned} \tag{44}$$

Therefore,

$$\frac{\partial W}{\partial \phi}(\rho, \pi) = 0, \quad 0 < \rho < \alpha \tag{45}$$

Solving to derive other terms of Eq. (39), we have

$$\begin{aligned}
W(\rho,\phi) &= \frac{bQ}{c_{44}}\left\{ \frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)}\right.\\
&\quad -\frac{1}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\\
&\quad \left. -\frac{1}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots\right\}\\
&= \frac{bQ}{c_{44}}\left\{ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)}\right.\\
&\quad -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\\
&\quad \left. -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots\right\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial W(\rho,\phi)}{\partial\phi} &= \frac{bQ}{c_{44}}\left\{ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)}\right.\\
&\quad -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)}\\
&\quad \left. -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}
\end{aligned}$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial \phi^2} = \frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$\left.+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}$$

$$\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial \phi^2} = \frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\,\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\,\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$\left.+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}$$

$$\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{3}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}$$

$$\frac{1}{\rho}\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial\rho^2} = \frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}$$

Hence, for $\frac{bQ}{c_{44}}\left(I_\beta^{(1)}-I_\beta^{(2)}+I_\beta^{(3)}\right)$

$$\frac{\partial^2 W}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial\phi^2}$$
$$=\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}$$
$$+\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$\left.-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots}\right\}$$
$$+\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.$$
$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$\left.+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}=0$$

Similarly, for $\frac{bQ}{c_{44}}\left(I_\alpha^{(1)}-I_\alpha^{(2)}+I_\alpha^{(3)}\right)$

$$\begin{aligned}
&\frac{\partial^2 W}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2}\\
&=\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.\\
&-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}\\
&\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}\\
&+\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.\\
&-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}\\
&\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}\\
&+\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}\right.\\
&+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}\\
&\left.+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\cdots\right\}=0
\end{aligned} \tag{46}$$

Therefore, for $W(\rho,\phi)=\frac{bQ}{c_{44}}\left\{I_{\beta}^{(1)}-I_{\beta}^{(2)}+I_{\beta}^{(3)}\right\}-\frac{bQ}{c_{44}}\left\{I_{\alpha}^{(1)}-I_{\alpha}^{(2)}+I_{\alpha}^{(3)}\right\}$

$$\frac{\partial^2 W}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2}=0+0=0 \tag{47}$$

satisfying the governing equation.

Satisfaction of the governing equation for $\alpha<\rho<\beta$, $0\leq\phi\leq\pi$ (region II)

We now prove that Eq. (39) satisfies both the governing Laplace equation

$$\frac{\partial^2 W}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2}=0 \tag{48}$$

and the boundary conditions

$$W(\rho,0)=0\,,\qquad \alpha<\rho<\beta \tag{49}$$

$$\frac{\partial W}{\partial\phi}(\rho,\pi)=\frac{bQ\rho}{c_{44}}\left[\frac{(\rho-1)}{\sqrt{\rho(\rho-2)}}+1\right],\quad \alpha<\rho<\beta,\ \rho>2 \tag{50}$$

Now using Eqs. (35–37) for $\rho>\alpha$ (region II)

$$\begin{aligned}W(\rho,\phi)=&\frac{bQ}{c_{44}}\left\{\frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right.\\&+\frac{1}{2}\sin\phi\rho^{-1}+\frac{1}{2\pi\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}\\&+\frac{\sin 2\phi}{2}\rho^{-2}+\frac{1}{\pi\alpha^{2}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}\\&+\frac{5}{8}\frac{\sin 3\phi}{3}\rho^{-3}+\frac{15}{8\pi\alpha^{3}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots\end{aligned} \tag{51}$$

Therefore,

$$W(\rho,0)=0 \tag{52}$$

Rewriting Eq. (51), we have

$$\begin{aligned}W(\rho,\phi)=&\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right.\\&+\frac{1}{2}\sin\phi\rho^{-1}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}\\&+\frac{\sin 2\phi}{2}\rho^{-2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}\\&+\frac{15}{8}\frac{\sin 3\phi}{3}\rho^{-3}+\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots\end{aligned} \tag{53}$$

Differentiating Eq. (53), we have

$$\frac{\partial W(\rho,\phi)}{\partial \phi} = \frac{bQ}{c_{44}}\left\{ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} + \frac{1}{2}\cos\phi\rho^{-1} \right. \\ \left. + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)} + \cos 2\phi\rho^{-2} \right. \tag{54}$$

Therefore,

$$\frac{\partial W(\rho,\pi)}{\partial \phi} = \frac{bQ}{c_{44}}\left\{ -\frac{1}{2}\rho^{-1} + \rho^{-2} - \frac{15}{8}\rho^{-3} + \cdots \right\} \tag{55}$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial \phi^2} = \frac{bQ}{c_{44}}\left\{ -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} - \frac{1}{2}\sin\phi\rho^{-1} \right. \\ - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)} - 2\sin 2\phi\rho^{-2} \\ - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)} - \frac{45}{8}\sin 3\phi\rho^{-3} \\ \left. - \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots \right. \tag{56}$$

$$\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial \phi^2} = \frac{bQ}{c_{44}}\left\{ -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} - \frac{1}{2}\sin\phi\rho^{-3} \right. \\ - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)} - \sin 2\phi\rho^{-4} \\ - \frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} - \frac{45}{8}\sin 3\phi\rho^{-5} \\ \left. - \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots \right. \tag{57}$$

$$\begin{aligned}\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\Bigg\{ & \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{1}{2}\sin\phi\rho^{-2} \\ & + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} - \sin 2\phi\rho^{-3} \\ & \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{15}{8}\sin 3\phi\rho^{-4} \\ & + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)} + \cdots\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{\partial^2 W(\rho,\phi)}{\partial \rho^2} = \frac{bQ}{c_{44}}\Bigg\{ & \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}} + \sin\phi\rho^{-3} \\ & + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)} + 3\sin 2\phi\rho^{-4} \\ & + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} + \frac{15}{2}\sin 3\phi\rho^{-5} \\ & + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{1}{\rho}\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\Bigg\{ & -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} - \frac{1}{2}\sin\phi\rho^{-3} \\ & - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)} - \sin 2\phi\rho^{-4} \\ & - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} - \frac{15}{8}\sin 3\phi\rho^{-5} \\ & + \frac{5}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots\end{aligned} \tag{60}$$

$$\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2} = \frac{bQ}{c_{44}}\Bigg\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$-\frac{1}{2}\sin\phi\rho^{-3} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}$$
$$-\sin 2\phi\rho^{-4} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}$$
$$-\frac{45}{8}\sin 3\phi\rho^{-5} - \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots \tag{61}$$

Therefore,

$$\frac{\partial^2 W}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial\phi^2} = \frac{bQ}{c_{44}}\Bigg\{\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}} + \sin\phi\rho^{-3}$$
$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}$$
$$+3\sin 2\phi\rho^{-4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}$$
$$+\frac{15}{2}\sin 3\phi\rho^{-5} + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots$$
$$+\frac{bQ}{c_{44}}\Bigg\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$-\frac{1}{2}\sin\phi\rho^{-3} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}$$
$$-\sin 2\phi\rho^{-4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}$$
$$-\frac{15}{8}\sin 3\phi\rho^{-5} + -\frac{5}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots$$
$$+\frac{bQ}{c_{44}}\Bigg\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}$$
$$-\frac{1}{2}\sin\phi\rho^{-3} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}$$
$$-\sin 2\phi\rho^{-4} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}$$
$$-\frac{45}{8}\sin 3\phi\rho^{-5} - \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} = 0 \tag{62}$$

Now for $\rho < \beta$ (region II)

$$\begin{aligned}W(\rho,\phi) = \frac{bQ}{c_{44}}\Bigg\{&-2\sin\phi\rho + \frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}\\&-\frac{1}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\\&-\frac{1}{\pi\tau^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}\\&+\frac{15}{8\pi\tau^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}\\&-\frac{7}{2\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}\\&+\frac{63}{8\pi\beta^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots\end{aligned} \tag{63}$$

$$\begin{aligned}\frac{\partial W(\rho,\phi)}{\partial\phi} = \frac{bQ}{c_{44}}\Bigg\{&-2\cos\phi\rho + \frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)}\\&-\frac{1}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)}\\&-\frac{1}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)}\\&+\frac{15}{8\pi\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)}\\&-\frac{7}{2\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)}\\&+\frac{63}{8\pi\beta^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)}+\cdots\end{aligned} \tag{64}$$

Therefore,

$$\frac{\partial W(\rho,\pi)}{\partial\phi} = \frac{bQ}{c_{44}}\{2\rho\} \tag{65}$$

$$
\begin{aligned}
\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2} = \frac{bQ}{c_{44}}\Bigg\{ & 2\sin\phi\rho - \frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)} \\
& + \frac{1}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} \\
& + \frac{1}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)} \\
& - \frac{15}{8\pi\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)} \\
& + \frac{7}{2\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)} \\
& - \frac{63}{8\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)} + \cdots
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2} = \frac{bQ}{c_{44}}\Bigg\{ & 2\sin\phi\rho^{-1} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)} \\
& + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)} \\
& + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} \\
& + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)} \\
& + \frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)} \\
& - \frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)} + \cdots
\end{aligned} \tag{67}
$$

$$\begin{aligned}\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\Bigg\{ & -2\sin\phi\rho + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{\frac{3}{2}-n}\rho^{n-\frac{3}{2}}}{\left(\frac{3}{2}-n\right)} \\ & -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} \\ & -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{3}{2}\right)} \\ & +\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{5}{2}\right)} \\ & -\frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{7}{2}\right)} \\ & +\frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{9}{2}\right)} + \cdots\end{aligned} \tag{68}$$

$$\begin{aligned}\frac{1}{\rho}\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\Bigg\{ & -2\sin\phi\rho^{-1} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)} \\ & -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)} \\ & -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} \\ & +\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)} + \cdots\Bigg\}\end{aligned} \tag{69}$$

$$\begin{aligned}\frac{\partial^2 W(\rho,\phi)}{\partial \rho^2} = \frac{bQ}{c_{44}}\Bigg\{ & -\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\,\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}} \\ & -\frac{\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)} \\ & -\frac{\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} \\ & +\frac{\frac{15}{8\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)} \\ & -\frac{\frac{7}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)} - \cdots\Bigg\}\end{aligned} \tag{70}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 W}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2} = \frac{bQ}{c_{44}}\Bigg\{ & -\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\,\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}} \\
& -\frac{\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)} \\
& -\frac{\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} \\
& +\frac{\frac{15}{8\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)} \\
& -\frac{\frac{7}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)} \\
& +\frac{bQ}{c_{44}}\Bigg\{-2\sin\phi\rho^{-1}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)} \\
& -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)} \\
& -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} \\
& +\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)}+\cdots\Bigg\} \\
& +\frac{bQ}{c_{44}}\Bigg\{2\sin\phi\rho^{-1}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)} \\
& +\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)} \\
& +\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} \\
& +\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)} \\
& +\frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)} \\
& -\frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)}\Bigg\}=0
\end{aligned} \tag{71}$$

Conversion of the boundary condition for region II to series form

Recall the boundary condition for region II

$$\frac{\partial W}{\partial \phi}(\rho, \pi) = \frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\sqrt{\rho((\rho-2))}} + \rho\right], \quad \alpha < \rho < \beta, \quad \rho > 2 \tag{72}$$

We convert the above equation to series form using the formula

$$(1-t)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k t^k \tag{73}$$

Now,

$$\begin{aligned}\frac{\partial W}{\partial \phi}(\rho, \pi) &= \frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\sqrt{\rho(\rho-2)}} + \rho\right] = \frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\rho^{\frac{1}{2}}(\rho-2)^{\frac{1}{2}}} + \rho\right] \\ &= \frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\rho^{\frac{1}{2}}(\rho-2)^{\frac{1}{2}}} + \rho\right] = \frac{bQ}{c_{44}}\left[\rho^{\frac{1}{2}}(\rho-1)(\rho-2)^{-\frac{1}{2}} + \rho\right] \\ &= \frac{bQ}{c_{44}}\left[\rho^{\frac{1}{2}}\rho(\rho-2)^{-\frac{1}{2}} - \rho^{\frac{1}{2}}(\rho-2)^{-\frac{1}{2}} + \rho\right] \\ &= \frac{bQ}{c_{44}}\left[\rho^{\frac{1}{2}}\rho\rho^{-\frac{1}{2}}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} - \rho^{\frac{1}{2}}\rho^{-\frac{1}{2}}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} + \rho\right] \\ &= \frac{bQ}{c_{44}}\left[\rho\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} - \left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} + \rho\right]\end{aligned} \tag{74}$$

But

$$(1-t)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k t^k \Rightarrow \left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k \left(\frac{2}{\rho}\right)^k = \sum_{k=0}^{\infty} c_k 2^k \rho^{-k}$$

Hence,

$$\rho\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k 2^k \rho^{1-k} \tag{75}$$

Therefore,

$$\begin{aligned}
\frac{\partial W}{\partial \phi}(\rho,\pi) &= \frac{bQ}{c_{44}}\left[\rho\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} - \left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}} + \rho\right] \\
&= \frac{bQ}{c_{44}}\left[\sum_{k=0}^{\infty} c_k 2^k \rho^{1-k} - \sum_{k=0}^{\infty} c_k 2^k \rho^{-k} + \rho\right] \\
&= \frac{bQ}{c_{44}}\left[\left(c_0 2^0 \rho - c_0 2^0 \rho^0\right) + \left(c_1 2^1 \rho^0 - c_1 2^1 \rho^{-1}\right)\right. \\
&+\left(c_2 2^2 \rho^{-1} - c_2 2^2 \rho^{-2}\right) + \left(c_3 2^3 \rho^{-2} - c_3 2^3 \rho^{-3}\right) \\
&+\left.\left(c_4 2^4 \rho^{-3} - c_4 2^4 \rho^{-4}\right) + \cdots + \rho\right] \\
&= \frac{bQ}{c_{44}}\left[(\rho-1) + \left(1-\rho^{-1}\right) + \left(\frac{3}{8}\times 4\rho^{-1} - \frac{3}{8}\times 4\rho^{-2}\right)\right. \\
&+\left.\left(\frac{5}{16}\times 8\rho^{-2} - \frac{5}{16}\times 8\rho^{-3}\right) + \left(\frac{35}{128}\times 16\rho^{-3} - \frac{35}{128}\times 16\rho^{-4}\right) + \cdots + \rho\right] \\
&= \frac{bQ}{c_{44}}\left[(\rho-1) + \left(1-\rho^{-1}\right) + \left(\frac{3}{2}\rho^{-1} - \frac{3}{2}\rho^{-2}\right)\right. \\
&+\left.\left(\frac{5}{2}\rho^{-2} - \frac{5}{2}\rho^{-3}\right) + \left(\frac{35}{8}\rho^{-3} - \frac{35}{8}\rho^{-4}\right) + \cdots + \rho\right] \\
&= \frac{bQ}{c_{44}}\left[2\rho + \frac{1}{2}\rho^{-1} + \rho^{-2} + \frac{15}{8}\rho^{-3} + \cdots\right]
\end{aligned} \tag{76}$$

From region II ($\rho > \alpha$)

$$\frac{\partial W(\rho,\pi)}{\partial \phi} = \frac{bQ}{c_{44}}\left\{-\frac{1}{2}\rho^{-1} + \rho^{-2} - \frac{15}{8}\rho^{-3} + ...\right\}\ ,\rho > \alpha \tag{77}$$

From region II ($\rho < \beta$)

$$\frac{\partial W(\rho,\pi)}{\partial \phi} = \frac{bQ}{c_{44}}\{2\rho\},\ \rho < \beta \tag{78}$$

Hence by superposition principle

$$\begin{aligned}
\frac{\partial W(\rho,\pi)}{\partial \phi} &= \frac{bQ}{c_{44}}\{2\rho\} - \frac{bQ}{c_{44}}\left\{-\frac{1}{2}\rho^{-1} + \rho^{-2} - \frac{15}{8}\rho^{-3} + \cdots\right\} \\
&= \frac{bQ}{c_{44}}\left\{2\rho + \frac{1}{2}\rho^{-1} - \rho^{-2} + \frac{15}{8}\rho^{-3} + \cdots\right\}
\end{aligned} \tag{79}$$

which satisfies the boundary condition in series form.

Satisfaction of the governing equation for $\rho > \beta$, $\rho > \alpha$, $0 \le \phi \le \pi$ (region III)
For region III, it is not a difficult algebra to show that

$$W(\rho,\phi) = \frac{bQ}{c_{44}}\left\{I_\beta^{(1)} - I_\beta^{(2)} + I_\beta^{(3)}\right\} - \frac{bQ}{c_{44}}\left\{I_\alpha^{(1)} - I_\alpha^{(2)} + I_\alpha^{(3)}\right\} \tag{80}$$

satisfies both the governing Laplace equation

$$\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2} = 0 \tag{81}$$

and the boundary condition

$$W(\rho,0)=0, \quad \rho>\beta \tag{82}$$

$$\frac{\partial W}{\partial \phi}(\rho,\pi)=0, \quad \rho>\beta. \tag{83}$$

For $\frac{bQ}{c_{44}}\left(I_\beta^{(1)}-I_\beta^{(2)}+I_\beta^{(3)}\right)$

$$\begin{aligned}
W(\rho,\phi)=\frac{bQ}{c_{44}}\left(I_\beta^{(1)}-I_\beta^{(2)}+I_\beta^{(3)}\right)=\frac{bQ}{c_{44}}\Bigg\{&\frac{\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\\
&-\phi+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^2}+\frac{3}{2}\sin\phi\rho^{-1}\\
&+\frac{3}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{5}{2}\frac{\sin 2\phi}{2}\rho^{-2}\\
&+\frac{5}{2\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{35}{8}\frac{\sin 3\phi}{3}\rho^{-3}\\
&+\frac{35}{8\pi\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots\\
&+\phi-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{1}{2}\right)^2}-\sin\phi\rho^{-1}\\
&-\frac{1}{\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}-\frac{3}{2}\frac{\sin 2\phi}{2}\rho^{-2}\\
&-\frac{3}{2\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}-\frac{5}{2}\frac{\sin 3\phi}{3}\rho^{-3}\\
&-\frac{5}{2\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots\\
&+\frac{\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\\
&+\frac{1}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{15}{8}\frac{\sin 3\phi}{3}\rho^{-3}\\
&+\frac{15}{8\pi\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\cdots
\end{aligned} \tag{84}$$

Therefore,

$$w(\rho,0)=0 \tag{85}$$

$$
\begin{aligned}
\frac{\partial W(\rho,\phi)}{\partial \phi} = \frac{bQ}{c_{44}}\Bigg\{ & \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+\frac{1}{2}}\cos\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} \\
& + \frac{1}{2}\cos\phi\rho^{-1} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+\frac{1}{2}}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)} \\
& + \cos 2\phi\rho^{-2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+\frac{1}{2}}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)} \\
& + \frac{15}{8}\cos 3\phi\rho^{-3} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+\frac{1}{2}}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)} \\
& + \frac{15}{8}\cos 3\phi\rho^{-3} + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+\frac{1}{2}}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots
\end{aligned}
\tag{86}
$$

Therefore,

$$
\frac{\partial W(\rho,\pi)}{\partial \phi} = \frac{bQ}{c_{44}}\left\{\frac{5}{2}\rho^{-1} + \rho^{-2} - \frac{15}{8}\rho^{-3} + \cdots\right\}
\tag{87}
$$

$$
\begin{aligned}
\frac{\partial^2 W(\rho,\phi)}{\partial \phi^2} = \frac{bQ}{c_{44}}\Bigg\{ & -\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)} - \frac{1}{2}\sin\phi\rho^{-1} \\
& - \frac{\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)} - 2\sin 2\phi\rho^{-2} \\
& - \frac{\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)} - \frac{45}{8}\sin 3\phi\rho^{-3} \\
& - \frac{\frac{15}{8\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial \phi^2} = \frac{bQ}{c_{44}}\Bigg\{ & -\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} - \frac{1}{2}\sin\phi\rho^{-3} \\
& - \frac{1}{2\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)} - 2\sin 2\phi\rho^{-4} \\
& - \frac{1}{\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} - \frac{45}{8}\sin 3\phi\rho^{-5} \\
& - \frac{15}{8\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots
\end{aligned}
\tag{89}
$$

$$\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\left\{ \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{1}{2}\sin\phi\rho^{-2} \right.$$
$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)} - \sin 2\phi\rho^{-3}$$
$$- \frac{3}{2\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} - \frac{15}{8}\sin 3\phi\rho^{-4}$$

$$\frac{15}{8\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)} + \cdots$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial \rho^2} = \frac{bQ}{c_{44}}\left\{ \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}} + \sin\phi\rho^{-3} \right.$$
$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}$$
$$+ 3\sin 2\phi\rho^{-4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}$$
$$+ \frac{15}{2}\sin 3\phi\rho^{-5} + \frac{15}{8\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots$$

$$\frac{1}{\rho}\frac{\partial W(\rho,\phi)}{\partial \rho} = \frac{bQ}{c_{44}}\left\{ -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} \right.$$
$$-\frac{1}{2}\sin\phi\rho^{-3} - \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}$$
$$- \sin 2\phi\rho^{-4} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} \tag{90}$$
$$- \frac{15}{8}\sin 3\phi\rho^{-5} - \frac{15}{8\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots$$

Therefore,

$$
\begin{aligned}
\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2} = \frac{bQ}{c_{44}}\Bigg\{ & \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}} \\
& + \sin\phi\rho^{-3} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)} \\
& + 3\sin 2\phi\rho^{-4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} \\
& + \frac{15}{2}\sin 3\phi\rho^{-5} + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \cdots \\
& + \frac{bQ}{c_{44}}\Bigg\{ -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} \\
& -\frac{1}{2}\sin\phi\rho^{-3} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)} \\
& - \sin 2\phi\rho^{-4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} \\
& - \frac{15}{8}\sin 3\phi\rho^{-5} - \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} + \ldots \\
& + \frac{bQ}{c_{44}}\Bigg\{ -\frac{2}{\pi\tau}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} \\
& - \frac{1}{2}\sin\phi\rho^{-3} - \frac{1}{2\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)} \\
& - 2\sin 2\phi\rho^{-4} - \frac{1}{\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)} \\
& - \frac{45}{8}\sin 3\phi\rho^{-5} - \frac{15}{8\pi}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)} = 0
\end{aligned}
\tag{91}
$$

Similarly, for $W(\rho,\phi) = \frac{bQ}{c_{44}}\left(I_\alpha^{(1)} + I_\alpha^{(2)} + I_\alpha^{(3)}\right)$

$$\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2} = 0$$

Therefore,

$$\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2} = 0 + 0 = 0 \tag{92}$$

satisfying the governing equation.

Now for $\rho > \beta$

$$\frac{\partial W(\rho,\pi)}{\partial \phi} = \frac{bQ}{c_{44}}\left\{-\frac{1}{2}\rho^{-1} + \rho^{-2} - \frac{15}{8}\rho^{-3} + \cdots\right\}$$

Similarly, for $\rho > \alpha$

$$\frac{\partial W(\rho,\pi)}{\partial \phi}=\frac{bQ}{c_{44}}\left\{-\frac{1}{2}\rho^{-1}+\rho^{-2}-\frac{15}{8}\rho^{-3}+\cdots\right\}$$

Therefore.

$$\frac{\partial W(\rho,\pi)}{\partial \phi}=\left.\frac{\partial W(\rho,\pi)}{\partial \phi}\right|_{\beta}-\left.\frac{\partial W(\rho,\pi)}{\partial \phi}\right|_{\alpha}=0 \tag{93}$$

From the results gotten at the three regions, we can see that our displacement equation satisfies the governing equation in the regions. The relevance of this satisfaction is that suppose we intend to derive the stress field or the stress intensity factor say, we would be fully convinced that our result is not misleading when a numerical analyst decides to use our analytic result to compare his numerical findings. Also in region 2, notice that we use the formula $(1-t)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}c_k t^k$ to convert the boundary condition

$$\frac{\partial W}{\partial \phi}(\rho,\pi)=\frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\sqrt{\rho((\rho-2))}}+\rho\right],\quad \alpha<\rho<\beta,\ \rho>2$$

to series form to obtain our desired result. This is also novel.

Conclusions

An infinite orthotropic material weakened by a semi-infinite crack under longitudinal loading is investigated in this study. The analysis is based upon an integral transform and complex variable techniques. The displacement field derived is shown to satisfy the governing equation at all the regions. With these results, closed-form solutions of elastic fields such as stress, displacement and stress intensity factors can serve as a benchmark for the purpose of judging the accuracy and efficiency of various numerical and approximate techniques as was observed in the extension of this work "closed-form solution for a semi-infinite crack moving in an infinite orthotropic material with a circular crack breaker under antiplane strain." Our result in that paper was seen to agree with similar numerical computation in the literature owing to the fact that our displacement field from which the stress field and the stress intensity factor were obtained satisfies the given governing equation. Hence, there is a need to ensure that every solution to a differential equation satisfies the equation before using it for further analysis, since analytic solutions are the benchmark for judging numerical ones. Therefore, students and researchers alike are to ensure proper verification of their solution at the solution regions so as not to void a correct numerical work that could have helped to save a life.

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NGE did the work and drafted the manuscript. The author read and approved the final manuscript.

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Declarations

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The authors declare that they have no competing interests.

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