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ORIGINAL ARTICLE

A note on fixed point theorems for fuzzy mappings

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Abstract In this paper, a common fixed point theorem for contractive type fuzzy mappings in a complete metric space is proved due to Cho (2005) [1]. Further an example is given for the results of Cho (2005) [1, Theorem 3.1] and Park and Jeong (1997) [2, Theorem 3.2] which are not satisfying the condition "for all $x, y \in X$" and have a fixed point.

AMS MATHEMATICS SUBJECT CLASSIFICATION: 47H10; 54H25

1. Introduction and preliminaries

Let (X, d) be a metric space. Fixed points for multivalued mapping $T : X \to 2^X$ are defined as $x \in Tx$ for some $x \in X$. Let $\mathcal{CB}(X)$ denote the set of all nonempty closed and bounded subsets of X. A multivalued mapping $T : X \to \mathcal{CB}(X)$ is called a contraction mapping if there exists $q \in (0, 1)$ such that

$$H(T(x), T(y)) \leqslant qd(x, y) \quad \text{forall } x, y \in X,$$

where the Hausdroff metric $H(A, B)$ on $\mathcal{CB}(X)$ is given by

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$
$$\text{where } d(x, C) = \inf_{y\in C} d(x, y)$$

for any nonempty closed and bounded subsets A, B and C of X and for any point $x \in X$.

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A fuzzy set in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ are called the grade of membership of x in A. Let $\mathcal{F}(X)$ be the collection of all fuzzy sets on X and let ${}^{\alpha}A = \{x \in X : A(x) \geqslant \alpha\}$ denote the α-cut of $A \in \mathcal{F}(X)$.The zero-cut of A is defined as the closure of the set $\{x \in X : A(x) > 0\}$.

A mapping F from X to $\mathcal{F}(Y)$ is called a fuzzy mapping if for each $x \in X, F(x)$ is a fuzzy set on Y and $F(x)(y)$ denotes the degree of membership of y in $F(x)$. Let X be a metric linear space and let $\mathcal{W}(X)$ denote the set of all fuzzy sets on X such that each of its α-cut is a nonempty compact and convex subset (approximate quantity)of X. A fuzzy mapping F from X to $\mathcal{W}(X)$ is called a fuzzy contraction mapping if there exists $q \in (0, 1)$ such that

$$D(F(x), F(y)) \leqslant qd(x, y) \text{ for each } x, y \in X,$$
$$\text{where } D(A, B) = \sup_{\alpha} H({}^{\alpha}A, {}^{\alpha}B)$$

Define $p_\alpha(A, B) = \inf_{x\in {}^{\alpha}A, y\in {}^{\alpha}B} d(x, y)$ and $p(A, B) = \sup_\alpha p_\alpha (A, B)$ for any fuzzy sets $A, B \in \mathcal{W}(X)$.

It is known that p_α is non-decreasing function of α.

Heilpern [3] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction

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mappings which is a fuzzy analogue of the fixed point theorem of Nadler [4]. Bose and Sahani [5] extended the Heilpern's result for a pair of generalized fuzzy contraction mappings. Marudai and Srinivasan [6] generalized the Heilpern's result using the Nadler's result. They also obtained a nontrivial generalization of the Nadler's fixed point theorem for fuzzy contraction mappings under weaker settings. Further Vijayaraju and Marudai [7] generalize the result of Bose and Mukerjee [8] for contractive type fuzzy mappings in complete metric spaces. The significance of these results is assuming "each of its α-cut of fuzzy set is nonempty closed and bounded subset of X" instead of approximate quantity of X". Akbar Azam and Muhammad Arshad [9] proved the result of Vijayaraju and Marudai [7, Theorem 3.1] is incomplete and corrected the proof in right direction. In this paper a common fixed point theorem for contractive type fuzzy mappings in complete metric space due to Cho [1] is proved by using the concept of Vijayaraju and Marudai [7]. Further an example is given for the results of Cho [1, Theorem 3.1] and Park and Jeong [2, Theorem 3.2] which are not satisfying the condition "for all $x, y \in X$" and have a fixed point.

2. Main results

The following lemma due to Nadler [4] is the main key of our result.

Lemma 2.1 [4]. *Let (X, d) be a metric space and $A, B \in \mathcal{CB}(X)$, then for each $a \in A, k > 0$ there exists an element $b \in B$ such that $d(a, b) \leqslant H(A, B) + k$.*

Cho [1] and Park and Jeong [2] proved some fixed point theorems for fuzzy mappings from X to $W(X)$ under the contractive type conditions in complete metric space. The following example shows that the condition "for all $x, y \in X$" fails for the results [1, Theorem 3.1] and [2, Theorem 3.2].

Theorem 2.2 [1]. *Let $F, G : X \to W(X)$ be fuzzy mappings satisfying the following condition: There exists $k \in (0, 1)$ such that*

$$D(Fx, Gy) \leqslant \frac{k}{\sqrt{2}}\{p(x, Fx)p(y, Gy) + p(y, Gy)d(x, y)\}^{\frac{1}{2}} \qquad (*)$$

for all $x, y \in X$. Then F and G have a common fixed point.

Theorem 2.3 [2]. *Let $F, G : X \to W(X)$ be fuzzy mappings satisfying the following condition: There exists $k \in (0, 1)$ such that*

$$D(Fx, Gy) \leqslant k\{p(x, Fx)p(y, Gy)\}^{\frac{1}{2}} \qquad (**)$$

for all $x, y \in X$. Then F and G have a common fixed point.

Example 2.4. Let $X = [0, 1]$. For $x, y \in X, d(x, y) = |x - y|, \alpha \in (0, 1]$. Define $F, G : X \to W(X)$ by

$$F(0)(z) = \begin{cases} 1, & z = 0 \\ \frac{1}{2}, & 0 < z \leqslant 1/50 \\ 0, & z > 1/50 \end{cases} \quad G(0)(z) = \begin{cases} 1, & z = 0 \\ 1/4, & 0 < z \leqslant 1/100 \\ 0, & z > 1/100 \end{cases}$$

$$F(x)(z) = \begin{cases} \alpha, & 0 \leqslant z \leqslant x/25 \\ \frac{\alpha}{2}, & x/25 < z \leqslant x/10 \\ 0, & z > x/10 \end{cases} \quad G(x)(z) = \begin{cases} \alpha, & 0 \leqslant z \leqslant x/20 \\ \frac{\alpha}{2}, & x/20 < z \leqslant x/10 \\ 0, & z > x/10 \end{cases}$$

Here ${}^1F(x) = {}^1G(x) = \{0\}$ and ${}^{\alpha}F(x) = [0, x/25]$ and ${}^{\alpha}G(x) = [0, x/20]$

$$D(F(x), G(y)) = sup_{\alpha} H({}^{\alpha}F(x), {}^{\alpha}G(y)) = |x/20 - y/25|$$
$$\leqslant \frac{k}{\sqrt{2}}[|x - x/25|.|y - y/20| + |y - y/20||x - y|]^{\frac{1}{2}}$$

For $x = y, F$ and G satisfy all the conditions of Theorem 2.2 and 0 is the common fixed point of F and G.

For $x \neq y$, the condition $(*)$ fails for taking the values $x = 1, y = 0$.

Similarly the condition $(**)$ of Theorem 2.3 fails also.

From the above example, we observe that Theorem 2.2 holds for assuming the condition for all $x \in X$ and for all nonzero values of y in X and Theorem 2.3 holds for assuming the condition for all nonzero values $x, y \in X$.

Next a common fixed theorem for fuzzy mappings is proved due to Cho [1].

Theorem 2.5 [1]. *Let $F, G : X \to W(X)$ be fuzzy mappings satisfying the following condition: There exist $\alpha, \beta > 0$ such that $\alpha + \beta < 1$ and*

$$D(Fx, Gy) \leqslant \frac{\alpha p(y, Gy))[(1 + p(x, Fx))p(x, Fx)]^{\frac{1}{2}}}{1 + 2d(x, y)} + \beta d(x, y),$$

for all $x, y \in X$. Then F and G have a common fixed point.

Theorem 2.6. *Let (X, d) be a complete metric space and let F_1 and F_2 be fuzzy mappings from X to $\mathcal{F}(X)$ satisfying the following condition:*

(i) *For each $x, y \in X$, there exists $\alpha(x), \alpha(y) \in (0, 1]$ such that ${}^{\alpha(x)}F_1(x)$ and ${}^{\alpha(y)}F_2(y)$ are nonempty closed bounded subsets of X.*

(ii)

$$H({}^{\alpha(x)}F_1(x), {}^{\alpha(y)}F_2(y))$$
$$\leqslant \frac{a_1 d(y, {}^{\alpha(y)}F_2(y))[\{1 + d(x, {}^{\alpha(x)}F_1(x))\}d(x, {}^{\alpha(x)}F_1(x))]^{\frac{1}{2}}}{1 + 2d(x, y)} + a_2 d(x, y),$$

where $a_1, a_2 > 0$ and $a_1 + a_2 < 1$.

Then there exists $z \in X$ such that $z \in {}^{\alpha(z)}F_1(z) \cap {}^{\alpha(z)}F_2(z)$.

Proof. Let $x_0 \in X$. Then by condition (i), there exists $\alpha_1 \in (0, 1]$ such that ${}^{\alpha_1}F_1(x_0)$ is a nonempty closed bounded subset of X.

Choose $x_1 \in {}^{\alpha_1}F_1(x_0)$.

For this x_1, there exists $\alpha_2 \in (0, 1]$ such that ${}^{\alpha_2}F_2(x_1)$ is a nonempty closed bounded subset of X. Since ${}^{\alpha_1}F_1(x_0)$ and ${}^{\alpha_2}F_2(x_1)$ are nonempty closed bounded subsets of X and by Lemma 2.1, there exists $x_2 \in {}^{\alpha_2}F_2(x_1)$ such that

$$
\begin{aligned}
d(x_1,x_2) &\leqslant H({}^{\alpha_1}F_1(x_0),{}^{\alpha_2}F_2(x_1))+a_2\\
&\leqslant \frac{a_1 d(x_1,{}^{\alpha_2}F_2(x_1))[\{1+d(x_0,{}^{\alpha_1}F_1(x_0))\}d(x_0,{}^{\alpha_1}F_1(x_0))]^{\frac{1}{2}}}{1+2d(x_0,x_1)}+a_2 d(x_0,x_1)+a_2\\
&\leqslant \frac{a_1 d(x_1,x_2)[\{1+d(x_0,x_1)\}d(x_0,x_1)]^{\frac{1}{2}}}{1+2d(x_0,x_1)}+a_2 d(x_0,x_1)+a_2\\
&\leqslant \frac{a_1 d(x_1,x_2)[2\{1+d(x_0,x_1)\}d(x_0,x_1)]^{\frac{1}{2}}}{1+2d(x_0,x_1)}+a_2 d(x_0,x_1)+a_2\\
&\leqslant \frac{a_1 d(x_1,x_2)[1+2d(x_0,x_1)]}{1+2d(x_0,x_1)}+a_2 d(x_0,x_1)+a_2\\
&= a_1 d(x_1,x_2)+a_2 d(x_0,x_1)+a_2.
\end{aligned}
$$

Therefore $d(x_1,x_2)\leqslant \frac{a_2}{1-a_1}d(x_0,x_1)+\frac{a_2}{1-a_1}=kd(x_0,x_1)+k$, where $k=\frac{a_2}{1-a_1}$.

For this x_2, there exists $\alpha_3\in(0,1]$ such that ${}^{\alpha_3}F_1(x_2)$ is a nonempty closed bounded subset of X.

Since ${}^{\alpha_3}F_1(x_2)$ and ${}^{\alpha_2}F_2(x_1)$ are nonempty closed bounded subsets of X, there exists $x_3\in{}^{\alpha_3}F_1(x_2)$ such that

$$d(x_2,x_3)\leqslant H({}^{\alpha_3}F_1(x_2),{}^{\alpha_2}F_2(x_1))+ka_2$$

and we obtain

$$
\begin{aligned}
d(x_2,x_3)&\leqslant kd(x_1,x_2)+k^2\\
&\leqslant k^2 d(x_0,x_1)+2k^2.
\end{aligned}
$$

Continuing this process there exists sequence $\{x_n\}$ of X such that

$x_{2n+1}\in{}^{\alpha_{2n+1}}F_1(x_{2n})$ and $x_{2n+2}\in{}^{\alpha_{2n+2}}F_2(x_{2n+1})$ and

$$
\begin{aligned}
d(x_{2n+1},x_{2n+2})&\leqslant H({}^{\alpha_{2n+1}}F_1(x_{2n}),{}^{\alpha_{2n+2}}F_2(x_{2n+1}))+k^{2n}a_2\\
&\leqslant kd(x_{2n},x_{2n+1})+k^{2n+1}\leqslant\ldots\\
&\leqslant k^{2n+1}d(x_0,x_1)+(2n+1)k^{2n+1}.
\end{aligned}
$$

It follows that for each $n=1,2,3\ldots$

$$d(x_n,x_{n+1})\leqslant k^n d(x_0,x_1)+nk^n.$$

Since $k<1$, it follows from Cauchy's root test that $\sum nk^n$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in X, then there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$.

$$
\begin{aligned}
d(z,{}^{\alpha(z)}F_2(z)&\leqslant d(z,x_{2n+1})+d(x_{2n+1},{}^{\alpha(z)}F_2(z))\\
&\leqslant d(z,x_{2n+1})+H({}^{\alpha_{2n+1}}F_1(x_{2n}),{}^{\alpha(z)}F_2(z))\\
&\leqslant d(z,x_{2n+1})+\frac{a_1 d(z,{}^{\alpha(z)}F_2(z))[\{1+d(x_{2n},{}^{\alpha_{2n+1}}F_1(x_{2n})\}d(x_{2n},{}^{\alpha_{2n+1}}F_1(x_{2n}))]^{\frac{1}{2}}}{1+2d(x_{2n},z)}\\
&\quad + a_2 d(x_{2n},z)\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Hence $z\in{}^{\alpha(z)}F_2(z)$.

Similarly $z\in{}^{\alpha(z)}F_1(z)$.

Therefore $z\in{}^{\alpha(z)}F_1(z)\cap{}^{\alpha(z)}F_2(z)$. □

Corollary 2.7. *Let (X,d) be a complete metric space and let F_1 and F_2 be fuzzy mappings from X to $\mathcal{F}(X)$ satisfying the condition(i) of Theorem 2.6 and*

$$H({}^{\alpha(x)}F_1(x),{}^{\alpha(y)}F_2(y))\leqslant\frac{a_1 d(y,{}^{\alpha(y)}F_2(y))[d(x,{}^{\alpha(x)}F_1(x))]^{\frac{3}{4}}}{1+2d(x,y)}+a_2 d(x,y),$$

where $a_1,a_2>0$ and $a_1+a_2<1$. Then there exists $z\in X$ such that $z\in{}^{\alpha(z)}F_1(z)\cap{}^{\alpha(z)}F_2(z)$.

Example 2.8. Let $X=[0,1]$. For $x,y\in X, d(x,y)=|x-y|$ and $\alpha,\beta\in(0,1]$. Define $F_1,F_2:X\to\mathcal{F}(X)$ by

For $x=0$,

$$F_1(0)(z)=\begin{cases}1, & z=0\\ \frac{1}{2}, & 0<z\leqslant 1/50\\ 0, & z>1/50\end{cases}\quad F_2(0)(z)=\begin{cases}1, & z=0\\ 1/4, & 0<z\leqslant 1/100\\ 0, & z>1/100\end{cases}$$

For $x\neq 0$,

$$F_1(x)(z)=\begin{cases}\alpha, & 0\leqslant z<x/15\\ \frac{\alpha}{2}, & x/15\leqslant z\leqslant x/2\\ 0, & z>x/2\end{cases}\quad F_2(x)(z)=\begin{cases}\beta, & 0\leqslant z<x/8\\ \frac{\beta}{4}, & x/8\leqslant z\leqslant x/2\\ 0, & z>x/2\end{cases}$$

Here ${}^{\alpha}F_1(0)={}^{\alpha}F_2(0)=\{0\}$ if $\alpha=1$ and

$${}^{\frac{\alpha}{2}}F_1(x)={}^{\frac{\beta}{4}}F_2(x)=[0,x/2]$$

For $x=y, H({}^{\alpha(x)}F_1(x),{}^{\alpha(y)}F_2(y))=0$.

For $x\neq y$,

$$H({}^{\alpha(x)}F_1(x), {}^{\alpha(y)}F_2(y)) < \frac{1/5|y - y/2|[(1 + |x - x/2|)|x - x/2|]^{\frac{1}{2}}}{1 + 2|x - y|} + 3/4|x - y|$$

Therefore F and G satisfy all the conditions of Theorem 2.6 for taking the values $a_1 = 1/5$ and $a_2 = 3/4$ and 0 is the common fixed point of F and G.

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