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Generalized vector equilibrium problem with pseudomonotone mappings

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Abstract In this paper, we consider different types of pseudomonotone set-valued mappings and establish some connections between these pseudomonotone mappings. Further, by using these pseudomonotone mappings, we establish some existence results for generalized vector equilibrium problem.

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1. Introduction and preliminaries

Let K be a nonempty subset of a real topological vector space X and let a bifunction defined as $f : K \times K \to \mathbb{R}$ with $f(x,x) = 0$ for all $x \in K$. The equilibrium problem studied by Blum and Oettli [1], deals with the existence of $x \in K$ such that $f(x,y) \geqslant 0$ for all $y \in K$. The vector equilibrium problem is obtained by considering the bifunction f with values in an ordered topological vector space. Most of the work on existence of solutions for equilibrium problems are based on generalized monotonicity, which represents some algebraic properties assumed on the bifunction f and their extension to the vector case, see, for example, [2–4]. In recent years, a number of authors have proposed many important generalizations of monotonicity such as pseudomonotonicity, relaxed monotonicity which play an important role in certain applications of mathematical programming as well as in economic theory, see for example, [5–11] and references therein. One type of pseudomonotone operators was introduced by Karamardian [7] in 1976 in the single-valued case. This pseudomonotonicity notion is sometimes called algebraic, in order to avoid confusion with the one introduced by Brezis [12] in 1968. Even for real-valued functions, it is clear that these two pseudomonotonicity concepts are different.

Let X, Y be Hausdorff topological vector spaces; let $K \subset X$ be a nonempty closed convex set and l et $P : K \to 2^Y$ be a set-valued mapping such that P is closed and convex cone (i.e., if $\lambda P \subset P$, for all $\lambda > 0$ and $P + P \subset P$) with int $P \neq \emptyset$. Let $\phi : X \times Y \to Y$ be a bifunction such that $\sup_{f \in T(x)} \phi(x,f) \notin -\text{int } P$. In this paper we consider the following generalized vector equilibrium problem (for short, GVEP): Find $x \in K$ such that

$$\sup_{f \in T(x)} \phi(y,f) \notin -\text{int } P, \quad \forall\, y \in K. \tag{1.1}$$

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In this paper we consider different types of pseudomonotone set-valued mappings in a very general setting and establish some connections between these pseudomonotone mappings. Further, we prove Minty's Lemma. By using Minty's Lemma and KKM theorem, we establish some existence theorems for generalized vector equilibrium problems. The concepts and results presented in this paper improve and extend the many existence results given in [5,6,8,13].

We recall some concepts and results which are needed in sequel.

Definition 1.1. A mapping $f : K \times K \to Y$ is called hemicontinuous, if for any $x, y, z \in K, t \in (0, 1)$, the mapping $t \to \langle f(x + t(y - x)), z\rangle$ is continuous at 0^+.

Definition 1.2. A mapping $T : K \to 2^Y$ is said to be upper semicontinuous on the segments of K if the mapping $t \to T((1 - t)x + ty)$ is upper semicontinuous at 0, for every $x, y \in K$.

Definition 1.3. A mapping $F : K \to Y$ is said to be P-convex, if for any $x, y \in K$ and $\lambda \in [0, 1]$,

$$F(\lambda x + (1 - \lambda)y) \in \lambda F(x) + (1 - \lambda)F(y) - P.$$

Lemma 1.1. *Let (Y, P) be an ordered topological vector space with a closed and convex cone P with int $P \neq \emptyset$. Then for all $x, y, z \in Y$, we have*

(i) *$y - z \in -int\ P$ and $y \notin -int\ P \Rightarrow z \notin -int\ P$;*
(ii) *$y - z \in -P$ and $y \notin -int\ P \Rightarrow z \notin -int\ P$.*

Definition 1.4. Let B be a convex compact subset of K. A mapping $\phi : K \times K \to Y$ is said to be coercive with respect to B, if there exits $x_0 \in B$ such that

$$\sup_{f\in T(x_0)} \phi(y, f) \in -\text{int } P.$$

Definition 1.5. A mapping $\phi : K \times K \to Y$ is said to be affine in first argument if for any $x_i \in K$ and $\lambda_i \geqslant 0, (1 \leqslant i \leqslant n)$, with $\sum_{i=1}^n \lambda_i = 1$ and any $y \in K$,

$$\phi\left(\sum_{i=1}^n \lambda_i x_i, y\right) = \sum_{i=1}^n \lambda_i \phi(x_i, y).$$

Theorem 1.1 [14]. *Let E be a topological vector space; K be a nonempty subset of E and let $G : K \to 2^E$ be a KKM mapping such that $G(x)$ is closed for each $x \in K$ and is compact for at least one $x \in K$, then $\bigcap_{x\in K} G(x) \neq \emptyset$.*

2. Existence results for generalized equilibrium problem

Now we will give the following concepts and results which are used in the sequel.

Definition 2.1. The mapping $\phi : X \times Y \to Y$ with respect to T, where $T : K \to 2^Y$, is said to be

(i) A-pseudomonotone, if for every $x, y \in K$,

$$\sup_{f\in T(x)} \phi(y, f) \notin -\text{int } P \text{ implies } \sup_{g\in T(y)} \phi(x, g) \notin \text{int } P;$$

(ii) B-pseudomonotone, if for every $x \in K$ and for every net $\{x_i\} \subset K$, with $x_i \to x$

$$\liminf_i \sup_{f\in T(x_i)} \phi(x, f_i) \notin -\text{int } P$$

implies that for every $y \in K$ there exists $f(y) \in T(x)$ such that

$$\limsup\ \phi(y, f_i) - \phi(y, f(y)) \notin \text{int } P;$$

(iii) C-pseudomonotone, if $x, y \in K$ and $\{x_i\} \subset K$, with $x_i \to x$,

$$\sup_{f\in T(x_i)} \phi((1-t)y + tx, f) \notin -\text{int } P, \text{ for all } t \in [0,1], \text{ for all } i \in I$$

implies $\sup_{f\in T(x)} \phi(y, f) \notin -\text{int } P$.

Now, we establish some results among above defined pseudomonotone mappings.

Proposition 2.1. *Let X, Y be a topological vector space. Let $K \subset X$ be a nonempty closed convex subset of X. Let $T : K \to 2^Y$ be a set-valued mapping. Let $\Phi : X \times Y \to Y$ is A-pseudomonotone, upper semicontinuous and P-convex in first argument, also graph $Y \setminus \{-int\ P\}$ is closed, then ϕ is C-pseudomonotone.*

Proof. For each $y \in K$, define set-valued mapping $F, G : K \to 2^K$ by

$$F(y) := \{x \in K : \sup_{f\in T(x)} \phi(y, f) \notin -\text{int } P\}, \quad \forall\ y \in K.$$

$$G(y) := \{x \in K : \sup_{g\in T(y)} \phi(x, g) \notin \text{int } P\}, \quad \forall\ y \in K.$$

In order to prove the C-pseudomonotonicity of ϕ, we have to show that for each line segment L, we have

$$\overline{\bigcap_{y\in K\cap L} F(y)} \cap L \subset \overline{\bigcap_{y\in K\cap L} G(y)} \cap L \subset \bigcap_{y\in K\cap L} G(y) \cap L = \bigcap_{y\in K\cap L} F(y) \cap L$$

The first inclusion is directly followed by A-pseudomontonicity of ϕ.

Next, we prove the second inclusion. Let $x \in \overline{\bigcap_{y\in K\cap L} G(y)} \cap L$ and $x_\alpha \to x$ such that $x_\alpha \in \bigcap_{y\in K\cap L} G(y)$. Hence $\sup_{g\in T(y)} \phi(x_\alpha, g) \notin \text{int } P$. Since ϕ is upper semicontinuous in first argument and $Y \setminus \{\text{int } P\}$ is closed, preceding inclusion implies that $\sup_{g\in T(y)} \phi(x, g) \notin \text{int } P$, that is $x \in \bigcap_{y\in K\cap L} G(y) \cap L$.

Next, we define the family of sets to characterize the C-pseudomonotone mappings.

Let for each $z \in K$,

$$Q(z) = \{x \in K : \sup_{f\in T(x)} \phi(z, f) \notin -\text{int } P\}. \qquad \square$$

Proposition 2.2. *The mapping $\phi : X \times Y \to Y$ is C-pseudomonotone and affine in the first argument if and only if, for every $x, y \in K$,*

$$\mathrm{cl}\left(\bigcap_{z\in[x,y]} Q(z)\right) \cap [x,y] = \left(\bigcap_{z\in[x,y]} Q(z)\right) \cap [x,y].$$

Proof. It is obvious that

$$\left(\bigcap_{z\in[x,y]} Q(z)\right) \cap [x,y] \subset \mathrm{cl}\left(\bigcap_{z\in[x,y]} Q(z)\right) \cap [x,y].$$

Next, it is enough to prove that, for all $y \in K, x \in \mathrm{cl}\left(\bigcap_{z\in[x,y]} Q(z)\right)$ implies $x \in \bigcap_{z\in[x,y]} Q(z)$. Let $x \in \mathrm{cl}\left(\bigcap_{z\in[x,y]} Q(z)\right)$, then there exists a net $\{x_i\}, x_i \in \mathrm{cl}\left(\bigcap_{z\in[x,y]} Q(z)\right)$ with $x_i \to x$. From the definition of set $Q(z)$, it means that $x_i \in K$ and $\sup_{f\in T(x_i)} \phi((1-t)y + tx, f) \notin -\mathrm{int}\ P$, for all $t \in [0,1]$, for all $i \in I$ and from D-pseudomonotonicity we get,

$$\sup_{f\in T(x)} \phi(y,f) \notin -\mathrm{int}\ P.$$

Since ϕ is affine in first argument, it follows from $\sup_{f\in T(x)} \phi(x,f) \notin -\mathrm{int}\ P$, that $\sup_{f\in T(x)} \phi((1-t)y + tx, f) \notin -\mathrm{int}\ P$ that is $x \in \bigcap_{t\in[0,1]} Q((1-t)y + tx)$ or $x \in \bigcap_{z\in[x,y]} Q(z)$.

Conversely, let $x, y \in K, \{x_i\} \subset K$ with $x_i \to x$ and $\sup_{f\in T(x_i)} \phi((1-t)y + tx, f) \notin -\mathrm{int}\ P$, for all $t \in [0,1]$, for all $i \in I$.

This means that $x_i \in \bigcap_{z\in[x,y]} Q(z)$, so that $x \in \mathrm{cl}\left(\bigcap_{z\in[x,y]} Q(z)\right) \cap [x,y] = \left(\bigcap_{z\in[x,y]} Q(z)\right) \cap [x,y]$. We get $x \in Q((1-t)y + tx)$ for every $t \in [0,1]$. In particular, for $t = 0$, $x \in Q(y)$, which implies

$$\sup_{f\in T(x)} \phi(y,f) \notin -\mathrm{int}\ P. \qquad \square$$

Remark 2.1. Converse part of above proposition can also be assumed P-convexity instead of affineness.

First, we prove following Minty's type Lemma.

Lemma 2.1. *Let X, Y be topological space and let $K \subset X$ be nonempty closed convex subset of X. Let $T : K \to 2^Y$ be a set-valued mapping. Let $\phi : X \times Y \to Y$ be A-pseudomonotone and hemicontinuous in second argument and P-convex in first argument, then following two problems are equivalent:*

(i) *Find $x \in K$ such that $\sup_{f\in T(x)} \phi(y,f) \notin -int\ P$, for all $y \in K$.*
(ii) *Find $x \in K$ such that $\sup_{g\in T(y)} \phi(x,g) \notin int\ P$, for all $y \in K$.*

Proof. By A-pseudomonotonicity of ϕ, it is obvious that problem (i) implies problem (ii). Suppose x is not a solution of problem (i). Then there exists $\hat{y} \in K$ such that,

$$\sup_{f\in T(x)} \phi(\hat{y},f) \in -\mathrm{int}\ P. \tag{2.1}$$

Let $x_\alpha := x + \alpha(\hat{y} - x) \in K$ as K is convex, for all $\alpha \in [0,1]$.

For any $\alpha \in [0,1]$, define a mapping $H : [0,1] \to 2^Y$ such that

$$H(\alpha) = \{\sup_{f\in T(x_\alpha)} \phi(\hat{y},\hat{f})\}.$$

By inclusion (2.1), $H(0) \subset -\mathrm{int}\ P$. By hemicontinuity, there exists $\hat{\alpha} \in (0,1]$, such that for any $\alpha \in (0,\hat{\alpha})$, $\sup_{f^\star\in T(x_\alpha)} \phi(\hat{y},f^\star) \in -\mathrm{int}\ P$.

By the P-convexity of ϕ, we have for any $f^\star \in T(x_\alpha)$,

$$\begin{aligned}\phi(x_\alpha,f^\star) &= \phi(\alpha\hat{y} + (1-\alpha)x,\ f^\star)\\ &\in \alpha\phi(\hat{y},f^\star) + (1-\alpha)\phi(x,\ f^\star) - P\end{aligned}$$

Therefore, we get

$$\sup_{f^\star\in T(x_\alpha)} \phi(x_\alpha,f^\star) \in \alpha \sup_{f^\star\in T(x_\alpha)} \phi(\hat{y},f^\star) + (1-\alpha) \sup_{f^\star\in T(x_\alpha)} \phi(x,f^\star) - P$$

$$\sup_{f^\star\in T(x_\alpha)} \phi(x_\alpha,f^\star) - (1-\alpha) \sup_{f^\star\in\ T(x_\alpha)} \phi(x,f^\star) \in \alpha \sup_{f^\star\in T(x)} \phi(\hat{y},f^\star) - P$$
$$\in -\mathrm{int}\ P - P \subset -\mathrm{int}\ P.$$

Since $\sup_{f^\star\in T(x_\alpha)} \phi(x_\alpha,f^\star) \notin -\mathrm{int}\ P$, above inclusion implies that $\sup_{f^\star\in T(x_\alpha)} \phi(x,f^\star) \notin -\mathrm{int}\ P$, which is contradiction to our assumption (ii). Hence (i) is equivalent to problem (ii).

We prove following existence theorem. $\square$

Theorem 2.1. *Let $K \subset X$ be a nonempty closed convex subset of X. Let $\phi : X \times Y \to Y$ be A-pseudomonotone, hemicontinuous in the second argument with respect to T, where $T : K \to 2^Y$ is set-valued mapping and P-convex in first argument, coercive with respect to the compact subset $B \subset K$. If for each $x \in K, \phi$ is upper semicontinuous in first argument of B and graph of $Y \setminus \{-int\ P\}$ is closed with respect to B. Then GVEP (1.1) has a solution.*

Proof. For each $y \in K$, define set-valued mapping $F, G : K \to 2^K$ by

$$F(x) := \{x \in K : \sup_{f\in T(x)} \phi(y,f) \notin -\mathrm{int}\ P\}$$

$$G(x) := \{x \in K : \sup_{g\in T(y)} \phi(x,g) \notin \mathrm{int}\ P\}, \quad \text{for all } y \in K.$$

First, we claim that F is a KKM mapping. Indeed, let $\{x_1, \ldots, x_n\}$ be a finite subsets of K and suppose $x \in \mathrm{conv}\{x_1, \ldots, x_n\}$ be arbitrary. Then $x = \sum_{j=1}^n \lambda_j x_j, \lambda_j \geqslant 0$ and $\sum_{i=1}^n \lambda_j = 1$. Suppose, if possible $x = \sum_{j=1}^n \lambda_j x_j \notin \bigcup_{j=1}^n F(x_j)$, then $\sup_{f\in T(x)} \phi(x_j,f) \in -\mathrm{int}\ P$, for every $j = 1, \ldots, n$.

Since ϕ is P-convex in first argument, for a fixed $f \in T(x)$, we have

$$\begin{aligned}\sup_{f\in T(x)} \phi(x,f) = \sup_{f\in T(x)} \phi(\sum_{j=1}^n \lambda_j x_j, f) &\in \sum_{j=1}^n \lambda_j \sup \phi(x_j,f) - P\\ &\in -\mathrm{int}\ P - P \subseteq -\mathrm{int}\ P\end{aligned}$$

which is contradiction to our assumption $\sup_{f\in T(\hat{x})}(\hat{x},f) \notin -\mathrm{int}\ P$. Thus $x = \sum_{j=1}^{n}\lambda_j x_j \in \bigcup_{j=1}^{n} F(x_j)$, that is $\mathrm{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{j=1}^{n} F(x_j)$.

Hence the mapping $\overline{F} : K \to 2^K$, defined by $\overline{F}(x) = \overline{F(x)}$, the closure of $F(x)$, is also KKM mapping. The coercivity of ϕ with respect to B implies that $\overline{F(x_0)} \subset B$. Hence $\overline{F(x_0)}$ is compact. Thus, by Theorem 1.1, it follows that $\bigcap_{x\in K}\overline{F(x)} \neq \emptyset$.

Next, we claim that

$$\bigcap_{x\in K}\overline{F(x)} \subset G(\hat{x}), \quad \text{for all } \hat{x} \in K.$$

Indeed, let $x \in \bigcap_{x\in K}\overline{F(x)}$. Since $\bigcap_{x\in K}\overline{F(x)} \subset B$ (see [13]), then $x \in \bigcap_{x\in K}\overline{F(x)} \cap B$, for all $x \in K$. Let $\hat{x} \in K$ be arbitrary, there exists a net $\{x_\alpha\}$ in $F(\hat{x})$ such that $x_\alpha \to x \in B$, that is

$$\sup_{f\in T(x_\alpha)} \phi(y,f) \notin -\mathrm{int}\ P,$$

which implies, using A-pseudomonotonicity of ϕ

$$\sup_{g\in T(y)} \phi(x_\alpha, g) \notin \mathrm{int}\ P.$$

Since for each $x \in K$, the graph of $Y \setminus \{-\mathrm{int}\ P\}$ is closed, clearly the graph of $Y \setminus \{\mathrm{int}\ P\}$ is also closed.

Since ϕ is upper semicontinuous in first argument, then preceeding inclusion implies that $\sup_{g\in T(y)}\phi(x,g) \notin \mathrm{int}\ P$, that is $x \in F(x)$, for all $x \in K$.

Hence $\bigcap_{x\in K}\overline{F(x)} \subset \bigcap_{x\in K} G(x) \subset B$. Finally, using Lemma 2.1, we get $\bigcap_{x\in K} F(x) = \bigcap_{x\in K} G(x)$. Thus $\bigcap_{x\in K} F(x) \neq \emptyset$, that is, there exists $x \in K$ such that $\sup_{f\in T(x)}\phi(y,f) \notin -\mathrm{int}\ P$.

This completes the proof. □

Now we give some condition in which GVEP (1.1) has at least one solution. We will use following result which is a generalization of the Ky Fan's Lemma.

Lemma 2.2 [15]. *Let E be a topological vector space, $M \subset E$, $F : M \to 2^E$ such that*

(i) cl $F(x_0)$ *is compact for* $x_0 \in M$;
(ii) *for every* $x_1, x_2, \ldots, x_n \in M$, $\mathrm{conv}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$;
(iii) *for each $x \in M$, the intersection of $F(x)$ with any finite dimensional subspace of E is closed;*
(iv) *for every line segment L of M;*

$$\mathrm{cl}\left(\bigcap_{x\in M\cap L} F(x)\right) \cap L = \left(\bigcap_{x\in M\cap L} F(x)\right) \cap L$$

Then $\bigcap_{x\in M} F(x) \neq \emptyset$. If M is convex, closed and $F(x) \subset M$ for every $x \in M$, then the hypothesis (iv) can be replaced with:

(iv′) *for every line segment L of M;*

$$\mathrm{cl}\left(\bigcap_{x\in L} F(x)\right) \cap L = \left(\bigcap_{x\in L} F(x)\right) \cap L$$

Now by using above lemma we have following result.

Theorem 2.2. *Let $K \subset X$ be a nonempty, closed convex subset of X and $T : K \to 2^Y$ be a set-valued mapping and let $\phi : X \times Y \to Y$ be a mapping with condition $\sup_{f\in T(x)}\phi(x,f) \in -\mathrm{int}\ P$. Suppose that*

(a) *ϕ is C-pseudomonotone with respect to T;*
(b) *there exists a compact subset $B \subset X$ and $z_o \in K$ such that $\sup_{f\in T(x)}\phi(z_0, f) \in -\mathrm{int}\ P$, for every $x \in K \setminus B$;*
(c) *for every finite dimensional subspace Z of X, ϕ is upper semicontinuous and hemicontinuous in second argument with respect to T;*
(d) *ϕ is P-convex in first argument and $T(x)$ is compact for every $x \in K$ such that $Y \setminus \{\text{-int } P\}$ is closed.*

Then GVEP (1.1) has at least one solution.

Proof. Let $F(z) = \{x \in K : \sup_{f\in T(x)}\phi(z,f)\}$, we check the hypothesis of Lemma 2.2. □

(i) We have that $F(z_0) \subset B$ (if there exists $x \in F(z_0)$ and $x \notin B$ then $\sup_{f\in T(x)}\phi(z,f) \notin -\mathrm{int}\ P$ and $x \in K \setminus B$, which is a contradiction.)Therefore $\mathrm{cl}F(z_o) \subset B$, and B being compact, $\mathrm{cl}F(z_0)$ is compact.
(ii) Assume that $x_1, \ldots, x_n \in K$. Now let us consider on contrary that there exists $\lambda_1, \lambda_2, \ldots, \lambda_n \geqslant 0$, with $\sum_{j=1}^{n}\lambda_j = 1$ such that $\hat{x} = \sum_{j=1}^{n}\lambda_j x_j \notin F(x_i)$, for every $i = 1, \ldots, n$, which means that

$$\sup_{f\in T(\hat{x})} \phi(x_i, f) \in -\mathrm{int}\ P, \quad \text{for every } i = 1, \ldots, n.$$

For a fixed $f \in T(\hat{x})$, we have

$$\sup_{f\in T(\hat{x})} \phi(\hat{x}, f) = \sup_{f\in T(\hat{x})} \phi\left(\sum_{j=1}^{n}\lambda_j x_j, f\right) \in \sum_{j=1}^{n}\lambda_j \sup \phi(x_j, f) - P$$
$$\in -\mathrm{int}\ P - P \subseteq -\mathrm{int}\ P$$

which is contradiction to our assumption $\sup_{f\in T(\hat{x})} \phi(\hat{x}, f) \notin -\mathrm{int}\ P$.
(iii) Let Z be a finite dimensional subspace of X. We want to prove that $Z \cap F(z)$ is closed. Let $z \in K$

$$F(z) \cap Z = \{x \in K \cap Z : \sup_{f\in T(x)} \phi(z,f) \in Y \setminus \{-\mathrm{int}\ P\}.$$

Let $\{x_\alpha\}$ be a net in $F(z) \cap Z$ such that $x_\alpha \to x$. Since $K \cap Z$ is closed, $Y \setminus \{-\mathrm{int}\ P\}$ is closed graph and upper semicontinuous in second argument, then it follows that $x \in K \cap Z$, which follows that $x \in F(z) \cap Z$.
(iv) It follows directly from Proposition 2.2 and Remark 2.1.

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