Journal of the Egyptian Mathematical Society (2016) **24**, 60–67

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Weighted statistical convergence of order α and its applications

Sanjoy Ghosal

Department of Mathematics, Kalyani Government Engineering College, Kalyani, Nadia 741235, West Bengal, India

Received 30 May 2014; revised 11 July 2014; accepted 9 August 2014
Available online 11 February 2015

KEYWORDS

Weighted statistical convergence;
Strong $(\overline{N}, t_n)$-summable;
Weighted modulus statistical convergence of order α;
Weighted modulus strong cesàro convergence of order α

Abstract The definition of weighted statistical convergence was first introduced by Karakaya and Chishti (2009) [1]. After that the definition was modified by Mursaleen et al. (2012) [2]. But some problems are still there; so it will be further modified in this paper. Using it some newly developed definitions of the convergence of a sequence of random variables in probability have been introduced and their interrelations also have been investigated, and in this way a partial answer to an open problem posed by Das and Savas (2014) [3] has been given.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 40A35; 40G15; 60B10

1. Introduction

The notion of statistical convergence was introduced by Fast [4], Steinhaus [5] and Schonenberg [6] and other authors independently as follows: Let $\mathbb{N}$ denote the set of all natural numbers and $A \subset \mathbb{N}$, then the asymptotic density of A is denoted by $d(A)$ and is defined by

$$d(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : k \in A\}|,$$

provided the limit exists (where the vertical bars denote the cardinality of the enclosed set). A number sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent to x if for every $\epsilon > 0$, the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - x| \geqslant \epsilon\}$ has asymptotic density zero and we write $x_n \xrightarrow{S} x$ or by $S - \lim_{n\to\infty} x_n = x$.

Now the idea of statistical convergence has turned out to be one of the most active areas of research in summability theory after the works of Šalát [7], Fridy [8] and Gürdal [9,10] and it has several generalizations and applications like:

(i) weighted statistical convergence by Karakaya and Chishti [1] (see the paper Mursaleen et al. [2] for modified definition of weighted statistical convergence),
(ii) statistical convergence of order α by Çolak [11] (statistical convergence of order α was also independently introduced by Bhunia et al. [12]),
(iii) λ-statistical convergence of order α by Çolak and Bektaş [13],
(iv) λ-statistical convergence of order α of sequences of function by Et et al. [14],
(v) lacunary statistical convergence of order α by Sengül and Et [15],

E-mail address: sanjoykrghosal@yahoo.co.in
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2014.08.006
1110-256X

(vi) on pointwise and uniform statistical convergences of order α by Cinar et al. [16],
(vii) extremal A-statistical limit points via ideals by Gürdal and Sari [17],
(viii) topology induced by random 2-norms space by Gürdal and Huban [18],
(ix) lacunary $\mathcal{I}$-convergent sequences by Tripathy et al. [19],
(x) probability theory by Ghosal [20] and many other, different fields of mathematics.

In another direction, the history of strong p-Cesàro summability, being longer, is not so clear. As per author's knowledge in [21], it has been shown that if a sequence is strongly p-Cesàro summable (for $0 < p < \infty$) to x, then the sequence must be statistically convergent to the same limit. Both the authors Fast [4] and Schonberg [6] noted that if a bounded sequence is statistically convergent to x, then it is strongly Cesàro summable to x. Furthermore statistically convergent sequences do not form a locally convex FK-space. Maddox noted that strong p-Cesàro summable can be considered as a BK-space if $1 \leqslant p < \infty$ and as a p-normable space if $0 < p < 1$ (see [22,23]).

In particular, in probability theory, a new type of convergence called statistical convergence in probability was introduced in [24], as follows: Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space $\mathcal{W}$ (for each n) with respect to a given class of events $\triangle$ and a given probability function $P : \triangle \to \mathbb{R}$. Then the sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent in probability to a random variable $X : \mathcal{W} \to \mathbb{R}$ if for any $\epsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : P(|X_k - X| \geqslant \epsilon) \geqslant \delta\}| = 0,$$

where the vertical bars denote the cardinality of the enclosed set $\{k \leqslant n : P(|X_k - X| \geqslant \epsilon) \geqslant \delta\}$. In this case we write $X_n \xrightarrow{(S,P)} X$. The class of all sequences of random variables which are statistically convergent in probability is denoted by (S, P). One can also see [25,26] for related works.

Maddox [27] and Ruckle [28] presented the following definition as follows: A modulus function ϕ is a function from $[0,\infty)$ to $[0,\infty)$ such that (i) $\phi(x) = 0$ if and only if $x = 0$, (ii) $\phi(x+y) \leqslant \phi(x) + \phi(y)$, forall $x, y > 0$, (iii) ϕ is increasing, (iv) ϕ is continuous from the right at zero. A modulus function may be bounded or unbounded. Tripathy and Sarma [29] and other authors used modulus function to construct new sequence spaces. Recently Savaş and Patterson [30] have defined and studied some sequence spaces by using a modulus function.

In this paper ideas of two types of convergences of a sequence of random variables in probability, namely,

(i) weighted modulus statistical convergence of order α and
(ii) weighted modulus strong Cesàro convergence of order α have been introduced and the interrelations among them have been investigated. Also their certain basic properties have been studied.

The main object of this paper is to modify the definition of weighted statistical convergence and establish some important theorems related to the modes of convergences (i) and (ii), which will effectively extend and improve all the existing results in this direction [1,2,11,12,21,24,25,27]. Moreover, intend to establish the relations among these two summability notions and in this way, a partial answer to an open problem posed by Das and Savas [3] has been given.

2. Weighted statistical convergence of order α

We first recall the definition of statistical convergence of order α of a sequence of real numbers from [11,12] as follows:

Definition 2.1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent of order α (where $0 < \alpha \leqslant 1$) to a real number x if for every $\epsilon > 0$, such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leqslant n : |x_k - x| \geqslant \epsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{S^\alpha} x$ and the set of all statistically convergent sequences of order α is denoted by S^α.

Karakaya and Chishti [1] first defined the concept of weighted statistical convergence as follows: Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ where $n \in \mathbb{N}$ and $T_n \to \infty$ as $n \to \infty$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted statistically convergent to a real number x if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n} |\{k \leqslant n : t_k|x_k - x| \geqslant \epsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{S_{\overline{N}}} x$.

Mursaleen et al. [2] modified the definition of weighted statistical convergence as follows: Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ where $n \in \mathbb{N}$ and $T_n \to \infty$ as $n \to \infty$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted statistically convergent (or, $S_{\overline{N}}$-convergent) to a real number x if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n} |\{k \leqslant T_n : t_k|x_k - x| \geqslant \epsilon\}| = 0.$$

In this case we write $S_{\overline{N}} - \lim x_n = x$.

Both the above definitions are not well defined in general. This follows from the following example.

Example 2.1. Let $\{x_n\}_{n\in\mathbb{N}}$ be any bounded sequence and $t_n = \frac{1}{n}$ where $n \in \mathbb{N}$. Then $T_n \to \infty$ as $n \to \infty$. It is quite clear that $x_n \xrightarrow{S_{\overline{N}}} x$ and $S_{\overline{N}} - \lim x_n = x$ where x be any real number (for both definitions), i.e., any bounded real sequence $\{x_n\}_{n\in\mathbb{N}}$ is weighted statistically convergent to any real number (if $t_n = \frac{1}{n}$ forall $n \in \mathbb{N}$). Hence both the definitions of weighted statistical convergence are not well defined. So both the definitions of weighted statistical convergence need to be modified.

Now, we are going to modify the definition of weighted statistical convergence as follows:

Definition 2.2. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ forall $n \in \mathbb{N}$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted statistically convergent of order α (where $0 < \alpha \leqslant 1$) to x if for every $\epsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{T_n^\alpha}|\{k\leqslant T_n : t_k|x_k-x|\geqslant\epsilon\}|=0.$$

In this case we write $x_n \xrightarrow{(S_{\overline{N}}^\alpha, t_n)} x$. The class of all weighted statistical convergence sequences of order α is denoted by $(S_{\overline{N}}^\alpha, t_n)$. For $\alpha = 1$, we say that $\{x_n\}_{n\in\mathbb{N}}$ is weighted statistically convergent to x and is denoted by $x_n \xrightarrow{(S_{\overline{N}}^1, t_n)} x$.

Theorem 2.1. *If $x_n \xrightarrow{(S_{\overline{N}}^\alpha, t_n)} x$ and $x_n \xrightarrow{(S_{\overline{N}}^\alpha, t_n)} y$ then $x = y$.*

Proof. If possible let $x \neq y$. Choose $\epsilon = \frac{1}{2}|x-y| > 0$ and $\liminf_{n\to\infty} t_n > \delta > 0$. Then

$$\begin{aligned}T_n^{1-\alpha} &\leqslant \frac{1}{T_n^\alpha}|\{k\leqslant T_n : t_k|x-y|\geqslant\epsilon\delta\}|\\ &\leqslant \frac{1}{T_n^\alpha}\left|\left\{k\leqslant T_n : t_k|x_k-x|\geqslant\frac{\epsilon\delta}{2}\right\}\right|\\ &\quad+\frac{1}{T_n^\alpha}\left|\left\{k\leqslant T_n : t_k|x_k-y|\geqslant\frac{\epsilon\delta}{2}\right\}\right|,\end{aligned}$$

which is impossible because the right hand limit is equal to zero but not left hand limit. Hence the result.

If $t_n = 1\ \forall n\in\mathbb{N}$, then statistical convergence of order α and weighted statistical convergence of order α are same. So other than this condition, if we assume that, the statistical convergence of order α is a subset (or superset) of weighted statistical convergence of order α holds, then by Theorem 2.2 and Example 2.2 and 2.3, our assumption will not correct. □

Theorem 2.2. *Let $\lim_{n\to\infty}\frac{t_{n+1}}{T_n^\alpha} = 0$ and $x_n \xrightarrow{(S_{\overline{N}}^\alpha, t_n)} x$ then $x_n \xrightarrow{S^\alpha} x$.*

Proof. Let $x_n \xrightarrow{(S_{\overline{N}}^\alpha, t_n)} x$, $\liminf_{n\to\infty} t_n > c > 0$ and n be a sufficiently large number, then there exists a positive integer m such that $T_m < n \leqslant T_{m+1}$. Then for $\epsilon > 0$,

$$\begin{aligned}\frac{1}{n^\alpha}|\{k\leqslant n : |x_k-x|\geqslant\epsilon\}| &\leqslant \frac{1}{T_m^\alpha}|\{k\leqslant T_{m+1} : t_k|x_k-x|\geqslant c\epsilon\}|\\ &= \frac{1}{T_m^\alpha}|\{k\leqslant T_m : t_k|x_k-x|\geqslant c\epsilon\}|+\frac{t_{m+1}}{T_m^\alpha}.\end{aligned}$$

Since $T_m \to \infty$ as $n\to\infty$, it follows that $\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k\leqslant n : |x_k-x|\geqslant\epsilon\}| = 0$ and consequently $x_n \xrightarrow{S^\alpha} x$.

The following example shows that in general the converse of Theorem 2.2 is not true, as well as the first part of the Theorem 2.3 (a) [2] (i.e., every statistically convergent sequence is $S_{\overline{N}}$-statistically convergent) is not true. □

Example 2.2. Let the sequence $\{x_n\}_{n\in\mathbb{N}}$ be defined by

$$x_n = \begin{cases}1 \text{ if } n = m^2 \text{ where } m\in\mathbb{N},\\ \frac{1}{\sqrt{n}} \text{ if } n \neq m^2 \text{ where } m\in\mathbb{N}.\end{cases}$$

It is quite clear that $\{x_n\}_{n\in\mathbb{N}}$ is statistically convergent sequence of order α to 0, but not weighted statistically convergent sequence of order α to 0 (if we choose $t_n = n$ for all $n\in\mathbb{N}$ and $\frac{1}{2} < \alpha \leqslant 1$).

The following example shows that weighted statistical convergence does not imply statistical convergence.

Example 2.3. Let $t_n = 2^{n-1}\ \forall n\in\mathbb{N}$, then $T_n = 2^n - 1$. We consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ is

$$x_n = \begin{cases}1 & \text{if } n \text{ is the first } \left[(\sqrt{2})^{m-1}\right] \text{ integers in the interval}\\ & (T_{m-1}, T_m] \text{ where } m\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases}$$

Then for each $\epsilon > 0$

$$\begin{aligned}&\frac{1}{T_n}|\{k\leqslant T_n : t_k|x_k-0|\geqslant\epsilon\}|\\ &\quad\leqslant \frac{1}{2^n-1}\{1+(1.5)^1+(1.5)^2+\cdots+(1.5)^{n-1}\}\\ &\quad\leqslant \left(\frac{1.5}{2}\right)^n\frac{2\left(1-\left(\frac{1}{1.5}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)^n\right)}.\end{aligned}$$

Let n be a sufficiently large number, then there exists a positive integer m such that $T_m < n \leqslant T_{m+1}$. Then

$$\frac{1}{n}|\{k\leqslant n : |x_k-0|\geqslant\epsilon\}| > \frac{1}{n}\{1+(1.1)^1+(1.1)^2+\cdots+(1.1)^{m-2}\}$$

$$> \frac{1}{n}(1.1)^{m-1}-1 > \frac{1}{n}(1.1)^{\frac{n}{\log 2}-2}-1\ \left(\text{since } m > \frac{n}{\log 2}-1\right),$$

which shows that $\{x_n\}_{n\in\mathbb{N}}$ is weighted statistically convergent to 0 but not statistically convergent to 0.

Remark 2.1. In general the symmetric difference between the set of statistically convergent sequences and the set of weighted statistically convergent sequences is non-empty (if $t_n \neq 1\,\forall n\in\mathbb{N}$).

The following example shows that in general the set $(S_{\overline{N}}^\alpha, t_n)\cap m$ (where $0 < \alpha \leqslant 1$), is not a closed subset of m (the set of all bounded real sequences endowed with the superior norm).

Example 2.4. Let $t_n = n, n\in\mathbb{N}, 0 < \alpha \leqslant 1$ and $x = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right\}$ then

$$x^{(n)} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0\ldots\} \in (S_{\overline{N}}^\alpha, t_n)\cap m$$

(where $n = 1, 2, 3, \ldots$).

So $\lim_{n\to\infty}||x^{(n)}-x|| = \lim_{n\to\infty}\frac{1}{n+1} = 0$. Then $\frac{1}{T_n^\alpha}|\{k\leqslant T_n : t_k|\frac{1}{k}-a|\geqslant\epsilon\}| = T_n^{1-\alpha}$ where a is any constant and

$$\epsilon = \begin{cases}\frac{|1-a|}{2} \text{ if } a\neq 1\\ \frac{1}{4} \text{ if } a = 1.\end{cases}$$

So $x \notin (S_{\overline{N}}^\alpha, t_n)\cap m$. This shows that the set $(S_{\overline{N}}^\alpha, t_n)\cap m$ is not a closed subset of m.

3. Weighted strong Cesàro convergence of order α

We first recall the definition of weighted strong Cesàro convergence (or strong $(\overline{N}, t_n)$-summable) of a sequence of real numbers from [1,2] as follows:

Definition 3.1. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ where $n \in \mathbb{N}$ and $T_n \to \infty$ as $n \to \infty$. Then the sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted strong Cesàro convergence (or strongly $(\overline{N}, t_n)$-summable) to a real number x if

$$\lim_{n\to\infty} \frac{1}{T_n} \sum_{k=1}^{n} t_k |x_k - x| = 0.$$

In this case we write $x_n \xrightarrow{|\overline{N},t_n|} x$. The set of all strongly $(\overline{N}, t_n)$-summable real sequences is denoted by $|\overline{N}, t_n|$.

Next we introduce the definition of weighted strong Cesàro convergence of order α (or strongly $(\overline{N}, t_n)$-summable of order α) of a sequence of real numbers as follows:

Definition 3.2. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ where $n \in \mathbb{N}$ and $T_n \to \infty$ as $n \to \infty$. Then the sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted strong Cesàro convergence of order α (or strongly $(\overline{N}, t_n)$-summable of order α) (where $0 < \alpha \leqslant 1$) to a real number x if

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} \sum_{k=1}^{n} t_k |x_k - x| = 0.$$

In this case we write $x_n \xrightarrow{|\overline{N},t_n|^\alpha} x$. The set of all strongly $(\overline{N}, t_n)$-summable sequences of order α is denoted by $|\overline{N}, t_n|^\alpha$. For $\alpha = 1$, $|\overline{N}, t_n|^\alpha$ is denoted by $|\overline{N}, t_n|$.

Theorem 3.1. *The set $|\overline{N}, t_n| \cap m$ is a closed subset of m (the set of all bounded real sequences endowed with the superior norm).*

Proof. Let $x^{(n)} = \{x_j^{(n)}\}_{j\in\mathbb{N}} \in |\overline{N}, t_n| \cap m$ (where $n = 1, 2, 3, \ldots$), $\lim_{n\to\infty} x^{(n)} = x (= \{x_j\}_{j\in\mathbb{N}})$ in m and for each n, $x^{(n)} \xrightarrow{|\overline{N},t_n|} a_n$. Then $\lim_{n\to\infty} ||x^{(n)} - x|| = 0$. We shall prove that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to a real number a and $x \xrightarrow{|\overline{N},t_n|} a$.

For $\epsilon > 0$, there exists a positive real number n_0 such that

$$|x_j^{(k)} - x_j^{(r)}| < \frac{\epsilon}{3} \forall k, r \geqslant n_0 \ \& \ j = 1, 2, 3, \ldots$$

$$\frac{1}{T_n} \sum_{j=1}^{n} t_j |x_j^{(k)} - a_k| < \frac{\epsilon}{3} \text{ and } \frac{1}{T_n} \sum_{j=1}^{n} t_j |x_j^{(r)} - a_r| < \frac{\epsilon}{3} \forall n \geqslant n_0.$$

Now $|a_k - a_r| \leqslant |x_j^{(k)} - a_k| + |x_j^{(k)} - x_j^{(r)}| + |x_j^{(r)} - a_r|$.

Then for $k, r \geqslant n_0$

$$|a_k - a_r| \leqslant \frac{1}{T_{n_0}} \sum_{j=1}^{n_0} t_j |x_j^{(k)} - a_k| + \frac{\epsilon}{3} + \frac{1}{T_{n_0}} \sum_{j=1}^{n_0} t_j |x_j^{(r)} - a_r| < \epsilon.$$

So $\{a_n\}_{n\in\mathbb{N}}$ fulfill the Cauchy's condition for convergence and hence there exists a real number a such that $\lim_{n\to\infty} a_n = a$.

For next part, let $\epsilon > 0$ so there exists a natural number n_1 such that

$$|a_r - a| < \frac{\epsilon}{4}, \ |x_j - x_j^{(r)}| < \frac{\epsilon}{4} \text{ for } r \geqslant n_1$$

and $$\frac{1}{T_n} \sum_{j=1}^{n} t_j |x_j^{(r)} - a_r| < \frac{\epsilon}{2} \forall n \geqslant n_1.$$

Now for arbitrary $j \in \mathbb{N}$ we have

$$|x_j - a| \leqslant |x_j^{(n_1)} - x_j| + |x_j^{(n_1)} - a_{n_1}| + |a_{n_1} - a| < \frac{\epsilon}{2} + |x_j^{(n_1)} - a_{n_1}|$$

$$\Rightarrow \frac{1}{T_n} \sum_{j=1}^{n} t_j |x_j - a| \leqslant \frac{\epsilon}{2} + \frac{1}{T_n} \sum_{j=1}^{n} t_j |x_j^{(n_1)} - a_{n_1}| < \epsilon \text{ for } n \geqslant n_1.$$

This shows that $|\overline{N}, t_n| \cap m$ is a closed subset of m.

Then it is easy to show that $\frac{1}{T_n} \sum_{j=1}^{n} t_j |x_j - a| < \epsilon \forall n \geqslant n_1$.

The following example shows that in general the set $|\overline{N}, t_n|^\alpha \cap m$ (where $0 < \alpha < 1$), is a not closed subset of m (the set of all bounded real sequences endowed with the superior norm). □

Example 3.1. Let $t_n = n, n \in \mathbb{N}, 0 < \alpha \leqslant \frac{1}{2}$ and $x = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right\}$ then

$$x^{(n)} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0 \ldots\} \in |\overline{N}, t_n|^\alpha \cap m$$

(where $n = 1, 2, 3, \ldots$).

So $\lim_{n\to\infty} ||x^{(n)} - x|| = \lim_{n\to\infty} \frac{1}{n+1} = 0$. Then $\frac{1}{T_n^\alpha} \sum_{j=1}^{n} t_j |\frac{1}{j} - a| \geqslant \frac{2^\alpha n^{(1-2\alpha)} |1-a|}{(1+\frac{1}{n})^\alpha} \geqslant c$ where a and c are respectively any constant and positive constant. So $x \notin |\overline{N}, t_n|^\alpha \cap m$. This shows that the set $|\overline{N}, t_n|^\alpha \cap m$ is not a closed subset of m. Similarly it can be shown that the result is true for $\frac{1}{2} < \alpha < 1$.

4. Applications in probability

First we like to introduce the definition of weighted modulus statistical convergence of order α of a sequence of real numbers as follows:

Definition 4.1. Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ forall $n \in \mathbb{N}$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus statistical convergence of order α (where $0 < \alpha \leqslant 1$) to a real numbers x if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} |\{k \leqslant T_n : t_k \phi(|x_k - x|) \geqslant \epsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{(S_{\overline{N}}^\alpha, \phi, t_n)} x$ and the class of all weighted modulus statistical convergence sequences of order α is denoted by $(S_{\overline{N}}^\alpha, \phi, t_n)$. If $\phi(x) = x, x \in [0, \infty)$ then weighted modulus statistical convergence of order α reduces to weighted statistical convergence of order α.

Now we like to introduce the definition of weighted modulus statistical convergence of order α in probability of random variables as follows:

Definition 4.2. Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ forall $n \in \mathbb{N}$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus statistical convergence of order α (where $0 < \alpha \leqslant 1$) in probability to a random variable $X : \mathcal{W} \to \mathbb{R}$ if for any $\epsilon, \delta > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} |\{k \leqslant T_n : t_k \phi(P(|X_k - X| \geqslant \epsilon)) \geqslant \delta\}| = 0.$$

In this case, $X_n \xrightarrow{(S_{\bar{N}}^\alpha, P^\phi, t_n)} X$ and the class of all weighted modulus statistical convergence sequences of order α in probability is denoted by $(S_{\bar{N}}^\alpha, P^\phi, t_n)$.

It is very obvious that if $X_n \xrightarrow{(S_{\bar{N}}^\alpha, P^\phi, t_n)} X$ and $X_n \xrightarrow{(S_{\bar{N}}^\alpha, P^\phi, t_n)} Y$ then $P\{X = Y\} = 1$, for any α, β.

The following example shows that there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables which is $(S_{\bar{N}}^\beta, P^\phi, t_n)$ to a random variable X but it is not $(S_{\bar{N}}^\alpha, P^\phi, t_n)$ to X, for $0 < \alpha < \beta \leqslant 1$.

Example 4.1. Let $\frac{r}{s}$ be a rational number between α and β. Let the probability density function of X_n be given by,

$$f_n(x) = \begin{cases} 1 \text{ where } 0 < x < 1 \\ 0 \text{ otherwise}, \text{ if } n = [m^{\frac{s}{r}}] \text{ for any } m \in \mathbb{N}, \end{cases}$$

$$f_n(x) = \begin{cases} \frac{nx^{n-1}}{2^n} \text{ where } 0 < x < 2 \\ 0 \text{ otherwise}, \text{ if } n \neq [m^{\frac{s}{r}}] \text{ for any } m \in \mathbb{N}. \end{cases}$$

Let $0 < \epsilon, \delta < 1, t_n = n, \forall n \in \mathbb{N}, \quad \phi(x) = \sqrt{x}, \forall x \in [0, \infty)$ then

$$P(|X_n - 2| \geqslant \epsilon) = \begin{cases} 1 \text{ if } n = [m^{\frac{s}{r}}] \text{ for any } m \in \mathbb{N}, \\ \left(1 - \frac{\epsilon}{2}\right)^n \text{ if } n \neq [m^{\frac{s}{r}}] \text{ for any } m \in \mathbb{N}. \end{cases}$$

Now we have the following inequalities,

$$\frac{1}{T_n^\beta} |\{k \leqslant T_n : t_k \phi(P(|X_k - 2| \geqslant \epsilon)) \geqslant \delta\}| \leqslant 2^\beta \frac{(1 + \frac{1}{n})^{\frac{2r}{s}}}{n^{2(\beta - \frac{r}{s})}}$$

and

$$\frac{1}{5} n^{2(\frac{r}{s} - \alpha)} - 1 \leqslant \frac{n^{\frac{2r}{s}} - 2^{\frac{r}{s}}}{2^{\frac{r}{s}} n^{2\alpha} (1 + \frac{1}{n})^\alpha} \leqslant \frac{1}{T_n^\alpha} |\{k \leqslant T_n : t_k \phi(P(|X_k - 2| \geqslant \epsilon)) \geqslant \delta\}|$$

So we have $X_n \xrightarrow{(S_{\bar{N}}^\beta, P^\phi, t_n)} 2$ but $\{X_n\}_{n\in\mathbb{N}} \notin (S_{\bar{N}}^\alpha, P^\phi, t_n)$.

The following example shows that there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables which is weighted statistical convergence of order α to a random variable X but it is not weighted modulus statistical convergence of order α.

Example 4.2. Let the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 1\} \text{ with p.m.f } P(X_n = -1) = P(X_n = 0) \\ \text{if } n = m^2 \text{ where } m \in \mathbb{N}, \\ \{0, 1\} \text{ with p.m.f, } P(X_n = 0) = 1 - \frac{1}{n^2}, \\ P(X_n = 1) = \frac{1}{n^2}, \text{ if } n \neq m^2 \text{ where } m \in \mathbb{N}. \end{cases}$$

Let $0 < \epsilon, \delta < 1, \quad \frac{1}{2} < \alpha \leqslant 1, t_n = 2n, \forall n \in \mathbb{N}$ and $\phi(x) = \sqrt{x}, \forall x \in [0, \infty) \Rightarrow T_n = n^2 + n \forall n \in \mathbb{N}$. Then

$$\frac{1}{T_n^\alpha} |\{k \leqslant T_n : t_k P(|X_k - 0| \geqslant \epsilon) \geqslant \delta\}| \leqslant \frac{\sqrt{1 + \frac{1}{n}}}{(1 + \frac{1}{n})^\alpha} \frac{1}{n^{(2\alpha - 1)}}$$

and

$$\frac{1}{T_n^\alpha} |\{k \leqslant T_n : t_k \phi(P(|X_k - 0| \geqslant \epsilon)) \geqslant \delta\}| \geqslant T_n^{1-\alpha}$$

So $\{X_n\}_{n\in\mathbb{N}} \in (S_{\bar{N}}^\alpha, P, t_n)$ but not in $(S_{\bar{N}}^\alpha, P^\phi, t_n)$.

Theorem 4.1. *Let $0 < \alpha \leqslant \beta \leqslant 1$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. If $X_n \xrightarrow{(S_{\bar{N}}^\alpha, P^\phi, t_n)} X$ and $P(|X| \geqslant \alpha) = 0$ for some positive real number α, then $g(X_n) \xrightarrow{(S_{\bar{N}}^\beta, P^\phi, t_n)} g(X)$.*

Proof. Since g is uniformly continuous on $[-\alpha, \alpha]$, then in this interval, for each $\epsilon > 0$ there exists δ such that

$$|g(x_n) - g(x)| < \epsilon \text{ if } |x_n - x| < \delta.$$

It follows that

$$\phi(P(|g(X_n) - g(X)| \geqslant \epsilon)) \leqslant \phi(P(|X_n - X| \geqslant \delta)).$$

Then for $\eta > 0$,

$$\frac{1}{T_n^\beta} |\{k \leqslant T_n : t_k \phi(P(|g(X_k) - g(X)| \geqslant \epsilon)) \geqslant \eta\}| \leqslant \frac{1}{T_n^\alpha} |\{k \leqslant T_n : t_k \phi(P(|X_k - X| \geqslant \delta)) \geqslant \eta\}|.$$

Hence the result. □

Corollary 4.1. *Let $0 < \alpha \leqslant \beta \leqslant 1$, $X_n \xrightarrow{(S_{\bar{N}}^\alpha, P^\phi, t_n)} x$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $g(X_n) \xrightarrow{(S_{\bar{N}}^\beta, P^\phi, t_n)} g(x)$.*

Proof is straight forward, so omitted.

The following example shows that in general the converse of Theorem 4.1 (or, Corollary 4.1) is not true.

Example 4.3. Consider a sequence of random variables: $X_n \in \{a, b\}$ with p.m.f $P(X_n = a) = P(X_n = b) \forall n \in \mathbb{N}$.

Choose $g(x) = (x - a)(x - b) \forall x \in \mathbb{R}$.

For any modulus function ϕ and $\{t_n\}_{n\in\mathbb{N}}$ be any sequence of real numbers such that $\liminf t_n > 0$ it is easy to get $g(X_n) \xrightarrow{(S_{\bar{N}}^\alpha, P^\phi, t_n)} g(a)$ but $\{X_n\}_{n\in\mathbb{N}}$ is not $(S_{\bar{N}}^\alpha, P^\phi, t_n)$ to c (where c be any real number)

Now we like to introduce the definitions of weighted modulus strong Cesàro convergence of order α of a sequence of real numbers and weighted modulus strong Cesàro convergence of order α in probability of a sequence of random variables as follows:

Definition 4.3. Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ where $n \in \mathbb{N}$ and $T_n \to \infty$ as $n \to \infty$. Then the sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus strong Cesàro convergence of order α (where $0 < \alpha \leqslant 1$) to a real number x if

$$\lim_{n\to\infty}\frac{1}{T_n^\alpha}\sum_{k=1}^{n}t_k\phi(|x_k-x|)=0.$$

In this case we write $x_n \xrightarrow{(\overline{N}^\alpha,\phi,t_n)} x$.

Definition 4.4. Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1>0$ and $T_n=t_1+t_2+\cdots+t_n$ where $n\in\mathbb{N}$ and $T_n\to\infty$ as $n\to\infty$. Then the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus strong Cesàro convergence of order α (where $0<\alpha\leqslant 1$) in probability to a random variable X if for any $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{T_n^\alpha}\sum_{k=1}^{n}t_k\phi(P(|X_k-X|\geqslant\epsilon))=0.$$

In this case, $X_n \xrightarrow{(\overline{N}^\alpha,P^\phi,t_n)} X$ and the class of all weighted modulus strong Cesàro convergence sequences of order α in probability is denoted by $(\overline{N}^\alpha,P^\phi,t_n)$.

Theorem 4.2. *If* $X_n \xrightarrow{(\overline{N}^\alpha,P^\phi,t_n)} X$ *and* $X_n \xrightarrow{(\overline{N}^\beta,P^\phi,t_n)} Y$ *then* $P\{X=Y\}=1$, *for any* α,β *where* $0<\alpha,\beta\leqslant 1$.

Proof. Without any loss of generality assume that $\beta\leqslant\alpha$. If possible let $P\{X=Y\}\neq 1$. Then there exists a positive real number ϵ such that $P(|X-Y|\geqslant\epsilon)>0$. Then

$$0<\phi(P(|X-Y|\geqslant\epsilon))\leqslant\frac{1}{T_n^\alpha}\sum_{k=1}^{n}t_k\phi(P(|X_k-X|\geqslant\frac{\epsilon}{2}))+\frac{1}{T_n^\beta}\sum_{k=1}^{n}t_k\phi(P(|X_k-Y|\geqslant\frac{\epsilon}{2}))$$

which is impossible because the right hand limit is equal to zero. Hence the result. □

Lemma 4.1 (*[27,28]*). *Let* ϕ *be any modulus function and* $0<\delta<1$. *Then* $\phi(x)\leqslant 2\phi(1)\delta^{-1}|x|$, *where* $|x|\geqslant\delta$.

Theorem 4.3. *Let* $0<\alpha\leqslant\beta\leqslant 1$ *and* $\{t_n\}_{n\in\mathbb{N}}$ *be a bounded sequence of positive real numbers such that* $\limsup_{n\to\infty}\frac{n}{T_n^\beta}<\infty$. *Then* $\lim_{n\to\infty}\frac{1}{T_n^\alpha}\sum_{k=1}^{n}t_kP(|X_k-X|\geqslant\epsilon)=0$ *implies* $X_n \xrightarrow{(\overline{N}^\beta,P^\phi,t_n)} X$.

Proof. Let $t_n\leqslant M_1$, $\forall n\in\mathbb{N}$ and $\limsup_{n\to\infty}\frac{n}{T_n^\beta}=M_2$. For any $\epsilon>0$ there exists a positive real number δ with $0<\delta<1$ such that $\phi(x)<\epsilon\,\forall x\in[0,\delta]$. Then,

$$\begin{aligned}\frac{1}{T_n^\beta}\sum_{k=1}^{n}t_k\phi(P(|X_k-X|\geqslant\epsilon))&=\frac{1}{T_n^\beta}\sum_{\substack{k=1\\P(|X_k-X|\geqslant\epsilon)\geqslant\delta}}^{n}t_k\phi(P(|X_k-X|\geqslant\epsilon))\\&\quad+\frac{1}{T_n^\beta}\sum_{\substack{k=1\\P(|X_k-X|\geqslant\epsilon)<\delta}}^{n}t_k\phi(P(|X_k-X|\geqslant\epsilon))\\&\leqslant\frac{2\phi(1)}{\delta T_n^\alpha}\sum_{k=1}^{n}t_kP(|X_k-X|\geqslant\epsilon)+M_1(M_2+1)\epsilon,\end{aligned}$$

(by using the Lemma 4.1). Since ϵ is arbitrary, so the result follows. □

Theorem 4.4. *Let* $0<\alpha\leqslant\beta\leqslant 1$. *Then* $(\overline{N}^\alpha,P^\phi,t_n)\subset(\overline{N}^\beta,P^\phi,t_n)$. *This inclusion is strict for any* $\alpha<\beta$.

Proof. The first part of this theorem is straightforward, so omitted. For the second part we will give an example

Let c be a rational number between 2α and 2β. We consider a sequence of random variables:

$$X_n\in\begin{cases}\{-1,1\}\text{ with p.m.f }P(X_n=1)=P(X_n=-1),\\ \text{if }n=\left[m^{\frac{1}{c}}\right]\quad\text{where}\quad m\in\mathbb{N},\\ \{0,1\}\text{ with p.m.f }P(X_n=0)=1-\frac{1}{n^8}\text{ and}\\ P(X_n=1)=\frac{1}{n^8},\text{ if }n\neq\left[m^{\frac{1}{c}}\right]\quad\text{where}\quad m\in\mathbb{N},\end{cases}$$

where $[x]$ is the greatest integer not greater than x. Then we have, for $0<\epsilon,\ \delta<1, t_n=n, \forall n\in\mathbb{N}$ and $\phi(x)=\sqrt{x}$, $\forall x\in[0,\infty)$.

$$\lim_{n\to\infty}\frac{[n^c]-1}{(n+1)^{2\alpha}}\leqslant\lim_{n\to\infty}\frac{1}{T_n^\alpha}\sum_{k=1}^{n}t_k\phi(P(|X_k-0|\geqslant\epsilon))$$

$$\lim_{n\to\infty}\frac{1}{T_n^\beta}\sum_{k=1}^{n}t_k\phi(P(|X_k-0|\geqslant\epsilon))\leqslant\lim_{n\to\infty}\left[\frac{n^c+1}{n^{2\beta}}+\frac{1}{n^{2\beta}}\left(\frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{n^3}\right)\right]$$

This shows that $X_n \xrightarrow{(\overline{N}^\beta,P^\phi,t_n)} 0$ but not $(\overline{N}^\alpha,P^\phi,t_n)$ to 0.

In the following, the relationships between $(S_{\overline{N}}^\alpha,P^\phi,t_n)$ and $(\overline{N}^\alpha,P^\phi,t_n)$ are investigated. □

Theorem 4.5. *If* $0<\alpha\leqslant\beta\leqslant 1, \liminf_{n\to\infty}t_n>0$ *and* $\liminf_{n\to\infty}\frac{n}{T_n}\geqslant 1$, *then* $(\overline{N}^\alpha,P^\phi,t_n)\subset(S_{\overline{N}}^\beta,P^\phi,t_n)$.

Proof. Let $\epsilon,\delta>0$. Then

$$\begin{aligned}\frac{1}{T_n^\alpha}\sum_{k=1}^{n}t_k\phi(P(|X_k-X|\geqslant\epsilon))&\geqslant\frac{1}{T_n^\alpha}\sum_{k=1}^{[T_n]}t_k\phi(P(|X_k-X|\geqslant\epsilon))\\&\geqslant\frac{\delta}{T_n^\beta}|\{k\leqslant T_n:t_k\phi(P(|X_k-X|\geqslant\epsilon))\geqslant\delta\}|.\end{aligned}$$

Hence the result follows. □

For bounded sequence of real numbers, statistical convergence is equivalent to strongly Cesàro summable (see [4,6,21]). But in weighted convergence, for bounded sequence weighted statistical convergence of order α may not equivalent to weighted strongly Cesàro summable of order α see Examples 4.4 and 4.5 (since $\phi(P(A))$ is a real numbers and is bounded forall $A\subset\mathcal{W}$). In fact none of the cases occur i.e., weighted statistical convergence is not subset nor superset of weighted strong Cesàro summable.

The following example shows that, the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is $(S_{\overline{N}}^\beta,P^\phi,t_n)$ to X but it is not $(\overline{N}^\alpha,P^\phi,t_n)$ to X where $0<\alpha\leqslant\beta\leqslant 1$.

Example 4.4. For $\alpha\neq\beta$: Choose $\alpha=\frac{1}{5}$ and $\beta=\frac{1}{2}$. Let the probability density function of X_n be given by,

$$f_n(x) = \begin{cases} 1 \text{ where } 0 < x < 1, \\ 0 \text{ otherwise, if } n = \left[m^{\frac{20}{9}}\right], \quad \text{where } m \in \mathbb{N} \end{cases}$$

$$f_n(x) = \begin{cases} \frac{nx^{n-1}}{2^n} \text{ where } 0 < x < 2, \\ 0 \text{ otherwise, if } n \neq \left[m^{\frac{20}{9}}\right], \quad \text{where } m \in \mathbb{N} \end{cases}$$

Take $t_n = n, \forall n \in \mathbb{N}, \phi(x) = \sqrt{x}, \forall x \in [0, \infty)$ and $0 < \epsilon, \delta < 1$. Now we have the inequality,

$$\frac{1}{T_n^\beta}|\{k \leqslant T_n : t_k\phi(P(|X_k - 2| \geqslant \epsilon)) \geqslant \delta\}| \leqslant \frac{2\sqrt{2}}{\sqrt[10]{n}}.$$

So we have $X_n \xrightarrow{(S_{\overline{N}}^\beta, P^\phi, t_n)} 2$.

Another inequality is,

$$\frac{\sqrt[20]{n}}{\sqrt[5]{2}} \leqslant \frac{1}{T_n^\alpha}\sum_{k=1}^n t_k\phi(P(|X_k - 2| \geqslant \epsilon))$$

So this inequality shows that X_n is not $(\overline{N}^\alpha, P^\phi, t_n)$ summable to 2.

For $\alpha = \beta$: Let $t_n = n, \forall n \in \mathbb{N}$, $\phi(x) = \sqrt{x}, \forall x \in [0, \infty)$ and a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 1\} \text{ with probability } \frac{1}{2}, \text{ if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}, \\ \{0, 1\} \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^3}, P(X_n = 1) = \frac{1}{n^3}, \\ \text{if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}. \end{cases}$$

Let $0 < \epsilon < 1$, then,

$$P(|X_n - 0| \geqslant \epsilon) = \begin{cases} 1 \text{ if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}, \\ \frac{1}{n^3} \text{ if } n \neq \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}. \end{cases}$$

This implies $X_n \xrightarrow{(S_{\overline{N}}^\alpha, P^\phi, t_n)} 0$ for all $0 < \alpha \leqslant 1$.

Now let $H = \{n \in \mathbb{N} : n \neq \{T_m\}^{T_m} \text{ where } m \in \mathbb{N}\}$. Now we have the inequality,

$$\begin{aligned} &\sum_{k=1}^n t_k\phi(P(|X_k - 0| \geqslant \epsilon)) \\ &= \sum_{\substack{k=1 \\ k\in H}}^n t_k\phi(P(|X_k - 0| \geqslant \epsilon)) + \sum_{\substack{k=1 \\ k\notin H}}^n t_k\phi(P(|X_k - 0| \geqslant \epsilon)) \\ &> \sum_{\substack{k=1 \\ k\in H}}^n \frac{1}{\sqrt{k}} + \sum_{\substack{k=1 \\ k\notin H}}^n 1 > \sum_{k=1}^n \frac{1}{\sqrt{k}} \\ &> \sqrt{n} \left(\text{Since we know that } \sum_{k=1}^n \frac{1}{\sqrt{k}} > \sqrt{n} \forall n \geqslant 2 \right) \\ &\Rightarrow \frac{1}{T_n^\alpha}\sum_{k=1}^n t_k\phi(P(|X_k - 0| \geqslant \epsilon)) > \frac{\sqrt{n}}{(n+1)^{2\alpha}} \end{aligned}$$

This inequality shows that $\{X_n\}_{n\in\mathbb{N}}$ is not $(\overline{N}^\alpha, P^\phi, t_n)$ summable to 0 of order α in probability for $0 < \alpha \leqslant \frac{1}{4}$.

Theorem 4.6. *Let* $0 < \alpha \leqslant \beta \leqslant 1, \{t_n\}_{n\in\mathbb{N}}$ *be a bounded sequence of real numbers such that* $\liminf_{n\to\infty} t_n > 0$ *and* $\limsup_{n\to\infty} \frac{n}{T_n^\beta} < \infty$. *Then* $(S_{\overline{N}}^\alpha, P^\phi, t_n) \subset (\overline{N}^\beta, P^\phi, t_n)$.

Proof. Let $t_n \leqslant M_1, \forall n \in \mathbb{N}$ and $\limsup_{n\to\infty} \frac{n}{T_n^\beta} = M_2$. For any $\epsilon, \delta > 0$ setting $H = \{k \leqslant T_n : t_k\phi(P(|X_k - X| \geqslant \epsilon)) \geqslant \delta\}$ and $H^c = \{k \leqslant T_n : t_k\phi(P(|X_k - X| \geqslant \epsilon)) < \delta\}$. Then,

$$\begin{aligned} &\frac{1}{T_n^\beta}\sum_{k=1}^n t_k\phi(P(|X_k - X| \geqslant \epsilon)) = \frac{1}{T_n^\beta}\sum_{\substack{k=1 \\ k\in H}}^n t_k\phi(P(|X_k - X| \geqslant \epsilon)) \\ &+ \frac{1}{T_n^\beta}\sum_{\substack{k=1 \\ k\in H^c}}^n t_k\phi(P(|X_k - X| \geqslant \epsilon)) \\ &\leqslant \frac{M_1M_3}{T_n^\alpha}|\{k \leqslant T_n : t_k\phi(P(|X_k - X| \geqslant \epsilon)) \geqslant \delta\}| + (M_2 + 1)\delta, \end{aligned}$$

where M_3 is a positive constant. Since δ is arbitrary, so the result follows. □

Note 4.1. It is known that in [31] "If $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\mathcal{I}$-lacunary statistical convergence is equivalent to $N_\theta(\mathcal{I})$-convergence." However, for order α, this is not clear and it is an open problem in Remark 2.18 [3]. Theorem 4.6 is a partial answer of the open problem Remark 2.18 [3], if $\phi(x) = x$ forall $x \in [0, \infty)$.

Theorem 4.7. *Let* $\{t_n\}_{n\in\mathbb{N}}$ *be a bounded sequence and* $\liminf_{n\to\infty} t_n > 0$. *Then* $(S_{\overline{N}}^1, P^\phi, t_n) \subset (\overline{N}^1, P^\phi, t_n)$.

Proof is straight forward, so omitted.

The following example shows that the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is $(\overline{N}^\alpha, P^\phi, t_n)$ to X but it is not $(S_{\overline{N}}^\alpha, P^\phi, t_n)$ to X.

Example 4.5. We consider a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 0\} \text{ with p.m.f } P(X_n = -1) = \frac{1}{n^2}, \\ P(X_n = 0) = 1 - \frac{1}{n^2} \text{ if } n = m^2 \quad \text{where } \ m \in \mathbb{N}, \\ \{0, 1\} \text{ with p.m.f, } \ P(X_n = 0) = 1 - \frac{1}{n^8}, \\ P(X_n = 1) = \frac{1}{n^8}, \text{ if } n \neq m^2 \quad \text{where } \ m \in \mathbb{N}. \end{cases}$$

Let $0 < \epsilon, \delta < 1, t_n = n, \forall n \in \mathbb{N}, \phi(x) = \sqrt{x}, \forall x \in [0, \infty)$.

Then $\frac{1}{\sqrt{T_n}}\sum_{k=1}^n t_k\phi(P(|X_k - 0| \geqslant \epsilon)) \leqslant 2\left\{\frac{\sqrt{n+1}}{n} + \frac{1}{n}\left(\frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{n^3}\right)\right\}$.

Next $\frac{1}{\sqrt{T_n}}|\{k \leqslant T_n : t_k\phi(P(|X_k - 0| \geqslant \epsilon)) \geqslant \delta\}| \geqslant \frac{\sqrt{T_n} - 1}{\sqrt{T_n}} \geqslant \frac{1}{2}$.

So $X_n \xrightarrow{\left(\overline{N}^{\frac{1}{2}}, P^\phi, t_n\right)} 0$ but $\{X_n\}_{n\in\mathbb{N}} \notin \left(S_{\overline{N}}^{\frac{1}{2}}, P^\phi, t_n\right)$.

Acknowledgments

Thankful to respected Editor and Referees for his/her careful reading of the paper and several valuable suggestions which has improved the presentation of the paper.

References

[1] V. Karakaya, T.A. Chishti, Weighted statistical convergence, Iranian J. Sci. Technol. Trans. A 33 (A3) (2009) 219–223.

[2] M. Mursaleen, V. Karakaya, M. Erturk, F. Gursoy, Weighted statistical convergence and its application to Korovkin type approximation theorem, Appl. Math. Comput. 218 (2012) 9132–9137.

[3] P. Das, E. Savas, On $\mathcal{I}$-statistical and $\mathcal{I}$-lacunary statistical convergence of order α, Bull. Irani. Math. Soc. 40 (2) (2014) 459–472.

[4] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244, In French.
[5] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.
[6] I.J. Schonenberg, The integrability of certain functions and related summability methods, Am. Math. Month. 66 (1959) 361–375.
[7] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980) 139–150.
[8] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.
[9] M. Gürdal, On ideal convergent sequences in 2-normed spaces, Thai J. Math. 4 (1) (2006) 85–91.
[10] M. Gürdal, A. Sahiner, Ideal convergence in n-normed spaces and some new sequence spaces via n-norm, J. Fundament. Sci. 4 (1) (2008) 233–244.
[11] R. Çolak, Statistical Convergence of Order α, Modern Methods in Analysis and Its Applications, Anamaya Pub, New Delhi, India, 2010, pp. 121–129.
[12] S. Bhunia, P. Das, S. Pal, Restricting statistical convergence, Acta Math. Hungar. 134 (1–2) (2012) 153–161.
[13] R. Çolak, C.A. Bektaş, λ-statistical convergence of order α, Acta Math. Sci. 31B (3) (2011) 953–959.
[14] M. Et, M. Cinar, M. Karakas, On λ-statistical convergence of order α of sequences of function, J. Inequal. Appl. 204 (2013) 8.
[15] H. Sengül, M. Et, On lacunary statistical convergence of order α, Acta Math. Sci. 34B (2) (2014) 473–482.
[16] M. Cinar, M. Karakas, M. Et, On pointwise and uniform statistical convergence of order α for sequences of functions, Fixed Point Theory Appl. 33 (2013) 11.
[17] M. Gürdal, H. Sari, Extremal A-statistical limit points via ideals, J. Egypt. Math. Soc. 22 (2014) 55–58.
[18] M. Gürdal, M.B. Huban, On $\mathcal{I}$-convergence of double sequences in the topology induced by random 2-norms, Matematicki Vesnik 66 (1) (2014) 73–83.
[19] B.C. Tripathy, B. Hazarika, B. Choudhary, Lacunary $\mathcal{I}$-convergent sequences, Kyungpook Math. J. 52 (2012) 473–482.
[20] S. Ghosal, S_λ-statistical convergence of a sequence of random variables, J. Egypt. Math. Soc. 23 (1) (2015) 85–89.
[21] J.S. Connor, The statistical and strong p-Cesaro convergence sequence, Analysis 8 (1988) 47–63.
[22] I.J. Maddox, Elements of Functional Analysis, Cambridge University Press, 1970.
[23] I.J. Maddox, On Kuttner theorem, J. London Math. Soc. 100 (1986) 161–166.
[24] S. Ghosal, Statistical convergence of a sequence of random variables and limit theorems, Appl. Math. 4 (58) (2013) 423–437.
[25] P. Das, S. Ghosal, S. Som, Statistical convergence of order α in probability. Arab. J. Math. Sci., doi: 10.1016/j.ajmsc.2014.06.002.
[26] S. Ghosal, $\mathcal{I}$-statistical convergence of a sequence of random variables in probability, Afrika Matematica 25 (3) (2014) 681–692.
[27] I.J. Maddox, Sequence spaces define by a modulus, Math. Proc. Comb. Phil. Soc. 100 (1986) 161–166.
[28] W.H. Ruckle, FK space in which the sequence of co-ordinate vectors in bounded, Canad. J. Math. 25 (1973) 973–978.
[29] B.C. Tripathy, B. Sarma, Double sequence spaces of fuzzy numbers defined by orlicz function, Acta Math. Sci. 31 (1) (2011) 134–140.
[30] E. Savaş, R.F. Patterson, Some double lacunary sequence spaces defined by orlicz functions, Southeast Asian Bull. Math. 35 (1) (2011) 103–110.
[31] P. Das, E. Savas, S. Ghosal, On generalizations of certain summability methods using ideals, Appl. Math. Lett. 24 (2011) 1509–1514.