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Global dynamics of some systems of rational difference equations☆

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Abstract In this paper, we study the qualitative behavior of some systems of second-order rational difference equations. More precisely, we study the equilibrium points, local asymptotic stability of equilibrium point, unstability of equilibrium points, global character of equilibrium point, periodicity behavior of positive solutions and rate of convergence of positive solutions of these systems. Some numerical examples are given to verify our theoretical results.

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1. Introduction and preliminaries

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For a systematic study of rational difference equations we refer [1–15] and references therein. In Refs. [16–19] qualitative behavior of some biological models is discussed. Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations.

Bajo and Liz [5] investigated the global behavior of difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n},$$

for all values of real parameters a, b.

Aloqeili [6] discussed the stability properties and semi-cycle behavior of the solutions of the difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a - x_{n-1}x_n}, \quad n = 0, 1, \ldots,$$

with real initial conditions and positive real number a.

Motivated by the above studies, our aim in this paper is to investigate the qualitative behavior of following systems of second-order rational difference equations:

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}}, \quad y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

and

$$x_{n+1} = \frac{a y_{n-1}}{b - c x_n x_{n-1}}, \quad y_{n+1} = \frac{a_1 x_{n-1}}{b_1 - c_1 y_n y_{n-1}}, \quad n = 0, 1, \ldots, \tag{2}$$

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where the parameters α, β, γ, α_1, β_1, γ_1, a, b, c, a_1, b_1, c_1 and initial conditions x_0, x_{-1}, y_0, y_{-1} are positive real numbers.

Let us consider four-dimensional discrete dynamical system of the form

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \qquad (3)$$
$$y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \quad n = 0, 1, \ldots,$$

where $f : I^2 \times J^2 \to I$ and $g : I^2 \times J^2 \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (3) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-1, 0\}$. Along with system (3) we consider the corresponding vector map $F = (f, x_n, x_{n-1}, g, y_n, y_{n-1})$. An equilibrium point of (3) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}),$$
$$\bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y}).$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (3).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-1, 0\} \left\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \right\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\|\cdot\|$ is the usual Euclidian norm in $\mathbb{R}^2$.
(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.
(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\left\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \right\| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.
(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.
(v) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map

$$F = (f, x_n, x_{n-1}, g, y_n, y_{n-1}),$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (3) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and F_J is the Jacobian matrix of the system (3) about the equilibrium point $(\bar{x}, \bar{y})$.

Lemma 1 [2]. *For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$ of difference equations such let $\overline{X}$ be a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\overline{X}$ lie inside an open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has norm greater than one, then $\overline{X}$ is unstable.*

Lemma 2 [3]. *Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$ is a system of difference equations and $\overline{X}$ is the equilibrium point of this system. The characteristic polynomial of this system about the equilibrium point $\overline{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lies inside the open unit disk $|\lambda|$ if and only if $\Delta_k > 0$ for $k = 0, 1, \ldots$, where Δ_k is the principal minor of order k of the $n \times n$ matrix*

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}. \qquad (4)$$

The following result gives the rate of convergence of solution of a system of difference equations

$$X_{n+1} = (A + B(n))X_n, \qquad (5)$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \qquad (6)$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Proposition 1 (*Perron's Theorem [20]*). *Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n} \qquad (7)$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

Proposition 2 [20]. *Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \qquad (8)$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

2. On the system $x_{n+1} = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}}$, $y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}}$

In this section, we shall investigate the qualitative behavior of the system (1). Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1), then for $\beta > \alpha$ and $\beta_1 > \alpha_1$ system (1) has following two equilibrium points $P_0 = (0, 0)$, $P_1 = \left(\sqrt{\frac{\beta_1 - \alpha_1}{\gamma_1}}, \sqrt{\frac{\beta - \alpha}{\gamma}}\right)$.

To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \mapsto (f, f_1, g, g_1), \qquad (9)$$

where $f = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}}$, $f_1 = x_n$, $g = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}}$, $g_1 = y_n$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (14) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \delta_1 & \delta_2 & \delta_2 \\ 1 & 0 & 0 & 0 \\ \delta_3 & \delta_3 & 0 & \delta_4 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $\delta_1 = \frac{\alpha}{\beta-\gamma\bar{y}^2}$, $\delta_2 = \frac{\alpha\gamma\bar{x}\bar{y}}{(\beta-\gamma\bar{y}^2)^2}$, $\delta_3 = \frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1-\gamma_1\bar{x}^2)^2}$, $\delta_4 = \frac{\alpha_1}{\beta_1-\gamma_1\bar{x}^2}$.

2.1. Main results

Theorem 1. *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then every solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (1) is bounded.*

Proof. It is easy to verify that

$$0 \leqslant x_n \leqslant \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-1}, \quad \text{if } n = 2m+1,$$
$$0 \leqslant x_n \leqslant \left(\frac{\alpha}{\beta}\right)^{m+1} x_0, \quad \text{if } n = 2m+2,$$
$$0 \leqslant y_n \leqslant \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-1}, \quad \text{if } n = 2m+1,$$
$$0 \leqslant y_n \leqslant \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_0, \quad \text{if } n = 2m+2.$$

Taking $\delta_1 = \max\{x_{-1}, x_0\}$ and $\delta_2 = \max\{y_{-1}, y_0\}$. Then, $0 \leqslant x_n < \delta_1$ and $0 \leqslant y_n < \delta_2$ for all $n = 0, 1, 2, \ldots$ □

Theorem 2. *If $\alpha < \beta$ and $\alpha_1 < \beta_1$, then equilibrium point $P_0 = (0, 0)$ is locally asymptotically stable.*

Proof. The linearized system of (1) about the equilibrium point $P_0 = (0, 0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$, and $F_J(0,0) = \begin{pmatrix} 0 & \frac{\alpha}{\beta} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0,0)$ is given by

$$P(\lambda) = \lambda^4 - \left(\frac{\alpha}{\beta} + \frac{\alpha_1}{\beta_1}\right)\lambda^2 + \frac{\alpha\alpha_1}{\beta\beta_1}. \quad (10)$$

The roots of $P(\lambda)$ are $\lambda = \pm\sqrt{\frac{\alpha}{\beta}}$, $\lambda = \pm\sqrt{\frac{\alpha_1}{\beta_1}}$. Therefore, by Lemma 1 the equilibrium point $P_0 = (0, 0)$ is locally asymptotically stable if $\alpha < \beta$ and $\alpha_1 < \beta_1$. □

Theorem 3. *The positive equilibrium point $(\bar{x}, \bar{y}) = \left(\sqrt{\frac{\beta_1-\alpha_1}{\gamma_1}}, \sqrt{\frac{\beta-\alpha}{\gamma}}\right)$ of the system (1) is unstable.*

Proof. The linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y}) = \left(\sqrt{\frac{\beta_1-\alpha_1}{\gamma_1}}, \sqrt{\frac{\beta-\alpha}{\gamma}}\right)$ is given by

$$X_{n+1} = F_J(P_1)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and $F_J(P_1) = \begin{pmatrix} 0 & 1 & \mu_1 & \mu_1 \\ 1 & 0 & 0 & 0 \\ \mu_2 & \mu_2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(P_1)$ is given by

$$P(\lambda) = \lambda^4 - (2 + \mu_1\mu_2)\lambda^2 - 2\mu_1\mu_2\lambda + 1 - \mu_1\mu_2, \quad (11)$$

where $\mu_1 = \frac{1}{\alpha}\sqrt{\frac{\gamma(\alpha-\beta)(\alpha_1-\beta_1)}{\gamma_1}}$ and $\mu_2 = \frac{1}{\alpha_1}\sqrt{\frac{\gamma_1(\alpha-\beta)(\alpha_1-\beta_1)}{\gamma}}$. It is clear that not all of $\Delta_k > 0$ for $k = 1, 2, 3, 4$. Hence by Lemma 2, the positive equilibrium point $(\bar{x}, \bar{y}) = \left(\sqrt{\frac{\alpha_1-\beta_1}{\gamma_1}}, \sqrt{\frac{\alpha-\beta}{\gamma}}\right)$ is unstable. □

2.2. Global character

Theorem 4. *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then the equilibrium point $P_0 = (0, 0)$ of system (1) is global attractor.*

Proof. From Theorem 1, every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (1) is bounded. Now, it is sufficient to prove that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is decreasing. From system (1) one has

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}} \leqslant \frac{\alpha x_{n-1}}{\beta} < x_{n-1}.$$

This implies that $x_{2n+1} < x_{2n-1}$ and $x_{2n+3} < x_{2n+1}$. Hence, the subsequences $\{x_{2n+1}\}$, $\{x_{2n+2}\}$ are decreasing, i.e., the sequence $\{x_n\}$ is decreasing. Similarly, one has

$$y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}} \leqslant \frac{\alpha_1 y_{n-1}}{\beta_1} < y_{n-1}.$$

This implies that $y_{2n+1} < y_{2n-1}$ and $y_{2n+3} < y_{2n+1}$. Hence, the subsequences $\{y_{2n+1}\}$, $\{y_{2n+2}\}$ are decreasing, i.e., the sequence $\{y_n\}$ is decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. □

Lemma 3. *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then the equilibrium point $P_0 = (0, 0)$ of system (1) is globally asymptotically stable.*

Proof. The proof follows from Theorems 2 and 4. □

Theorem 5. *The system (1) has no prime period-two solutions.*

Proof. Assume that $(p_1, q_1), (p_2, q_2), (p_1, q_1), \ldots$ be prime period-two solution of system (1) such that p_i, $q_i \neq 0$ and $p_i \neq q_i$ for $i = 1, 2$. Then, from system (1) one has:

$$p_1 = \frac{\alpha p_1}{\beta - \gamma q_2 q_1}, \quad p_2 = \frac{\alpha p_2}{\beta - \gamma q_1 q_2}, \quad (12)$$

and

$$q_1 = \frac{\alpha_1 q_1}{\beta_1 - \gamma_1 p_2 p_1}, \quad q_2 = \frac{\alpha_1 q_2}{\beta_1 - \gamma_1 p_1 p_2}. \quad (13)$$

From (12) and (13), one has p_i, $q_i = 0$ for $i = 1, 2$. Which is a contradiction. Hence, system (1) has no prime period-two solutions. □

2.3. Rate of convergence

We investigate the rate of convergence of a solution that converges to the equilibrium point $(0, 0)$ of the system (1).

Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be any solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms, one has from the system (1)

$$x_{n+1} - \bar{x} = \sum_{i=0}^{1} A_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{1} B_i(y_{n-i} - \bar{y}),$$

$$y_{n+1} - \bar{y} = \sum_{i=0}^{1} C_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{1} D_i(y_{n-i} - \bar{y}).$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, one has

$$e^1_{n+1} = \sum_{i=0}^{1} A_i e^1_{n-i} + \sum_{i=0}^{1} B_i e^2_{n-i},$$

$$e^2_{n+1} = \sum_{i=0}^{1} C_i e^1_{n-i} + \sum_{i=0}^{1} D_i e^2_{n-i},$$

where $A_0 = 0$, $A_1 = \frac{\alpha}{\beta-\gamma y_n y_{n-1}}$, $B_0 = \frac{\alpha\gamma\bar{x}y_{n-1}}{(\beta-\gamma y_n y_{n-1})(\beta-\gamma\bar{y}^2)}$, $B_1 = \frac{\alpha\gamma\bar{x}\bar{y}}{(\beta-\gamma y_n y_{n-1})(\beta-\gamma\bar{y}^2)}$, $C_0 = \frac{\alpha_1\gamma_1\bar{y}x_{n-1}}{(\beta_1-\gamma_1 x_n x_{n-1})(\beta_1-\gamma_1\bar{x}^2)}$, $C_1 = \frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1-\gamma_1 x_n x_{n-1})(\beta_1-\gamma_1\bar{x}^2)}$, $D_0 = 0$, $D_1 = \frac{\alpha_1}{\beta_1-\gamma_1 x_n x_{n-1}}$.

Taking the limits, we obtain $\lim_{n\to\infty} A_0 = 0$, $\lim_{n\to\infty} A_1 = \frac{\alpha}{\beta-\gamma\bar{y}^2}$, $\lim_{n\to\infty} B_0 = \lim_{n\to\infty} B_1 = \frac{\alpha\gamma\bar{x}\bar{y}}{(\beta-\gamma\bar{y}^2)^2}$, $\lim_{n\to\infty} C_0 = \lim_{n\to\infty} C_1 = \frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1-\gamma_1\bar{x}^2)^2}$, $\lim_{n\to\infty} D_0 = 0$, $\lim_{n\to\infty} D_1 = \frac{\alpha_1}{\beta_1-\gamma_1\bar{x}^2}$. So, the limiting system of error terms can be written as

$$E_{n+1} = KE_n,$$

where $E_n = \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ e^2_n \\ e^2_{n-1} \end{pmatrix}$ and $K = \begin{pmatrix} 0 & \frac{\alpha}{\beta-\gamma\bar{y}^2} & \frac{\alpha\gamma\bar{x}\bar{y}}{(\beta-\gamma\bar{y}^2)^2} & \frac{\alpha\gamma\bar{x}\bar{y}}{(\beta-\gamma\bar{y}^2)^2} \\ 1 & 0 & 0 & 0 \\ \frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1-\gamma_1\bar{x}^2)^2} & \frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1-\gamma_1\bar{x}^2)^2} & 0 & \frac{\alpha_1}{\beta_1-\gamma_1\bar{x}^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$, which is similar to linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$.

Using proposition (1), one has following result.

Theorem 6. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *be a positive solution of the system (1) such that* $\lim_{n\to\infty} x_n = \bar{x}$, *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where* $(\bar{x}, \bar{y}) = (0, 0)$. *Then, the error vector* E_n *of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty} (\|E_n\|)^{\frac{1}{n}} = |\lambda F_J(\bar{x}, \bar{y})|, \quad \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_J(\bar{x}, \bar{y})|,$$

where $\lambda F_J(\bar{x}, \bar{y})$ *are the characteristic roots of the Jacobian matrix* $F_J(\bar{x}, \bar{y})$ *about* $(0, 0)$.

3. On the system $x_{n+1} = \frac{a y_{n-1}}{b - c x_n x_{n-1}}$, $y_{n+1} = \frac{a_1 x_{n-1}}{b_1 - c_1 y_n y_{n-1}}$

In this section, we shall investigate the qualitative behavior of the system (2). Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2), then system (2) has a unique equilibrium point $P_0 = (0, 0)$. To construct corresponding linearized form of the system (2) we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \mapsto (f, f_1, g, g_1), \tag{14}$$

where $f = \frac{a y_{n-1}}{b - c x_n x_{n-1}}$, $f_1 = x_n$, $g = \frac{a_1 x_{n-1}}{b_1 - c_1 y_n y_{n-1}}$, $g_1 = y_n$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (14) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} \zeta_1 & \zeta_1 & 0 & \zeta_2 \\ 1 & 0 & 0 & 0 \\ 0 & \zeta_4 & \zeta_3 & \zeta_4 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $\zeta_1 = \frac{ac\bar{x}\bar{y}}{(b-c\bar{x}^2)^2}$, $\zeta_2 = \frac{a}{b-c\bar{x}^2}$, $\zeta_3 = \frac{a_1}{b_1-c_1\bar{y}^2}$, $\zeta_4 = \frac{a_1 c_1 \bar{x}\bar{y}}{(b_1-c_1\bar{y}^2)^2}$.

3.1. Main results

Theorem 7. *Let* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *be positive solution of system (2), then for every* $m \geqslant 1$ *the following results hold.*

$$(i) \quad 0 \leqslant x_n \leqslant \begin{cases} \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^{m} y_{-1}, & \text{if } n = 4m+1, \\ \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^{m} y_0, & \text{if } n = 4m+2, \\ \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^{m} x_{-1}, & \text{if } n = 4m+3, \\ \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^{m} x_0, & \text{if } n = 4m+4. \end{cases}$$

$$(ii) \quad 0 \leqslant y_n \leqslant \begin{cases} \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} x_{-1}, & \text{if } n = 4m+1, \\ \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} x_0, & \text{if } n = 4m+2, \\ \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} y_{-1}, & \text{if } n = 4m+3, \\ \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} y_0, & \text{if } n = 4m+4. \end{cases}$$

Theorem 8. *For the equilibrium point* $P_0 = (0, 0)$ *of the system (2) following results hold true:*

(i) If $a < b$ and $a_1 < b_1$, then equilibrium point $P_0 = (0, 0)$ is locally asymptotically stable.
(ii) If $a > b$ or $a_1 > b_1$, then equilibrium point $P_0 = (0, 0)$ is unstable.

Proof.

(i) The linearized system of (2) about the equilibrium point $P_0 = (0, 0)$ is given by

$$X_{n+1} = F_J(0, 0) X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$, and $F_J(0, 0) = \begin{pmatrix} 0 & 0 & 0 & \frac{a}{b} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{a_1}{b_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0, 0)$ is given by

$$P(\lambda) = \lambda^4 - \frac{aa_1}{bb_1}. \tag{15}$$

The roots of $P(\lambda)$ are $\lambda = \pm(\frac{aa_1}{bb_1})^{\frac{1}{4}}$, $\lambda = \pm\iota\left(\frac{aa_1}{bb_1}\right)^{\frac{1}{4}}$. Therefore, by Lemma 1 the equilibrium point $P_0 = (0, 0)$ is locally asymptotically stable if $a < b$ and $a_1 < b_1$.

(ii) It is easy to see that if $a > b$ or $a_1 > b_1$, then there exists at least one root λ of Eq. (15) such that $|\lambda| > 1$. Hence, by Lemma 1 if $a > b$ or $a_1 > b_1$, then $P_0 = (0, 0)$ is unstable. □

3.2. Global character

Theorem 9. *Let* $a < b$ *and* $a_1 < b_1$, *then the equilibrium point* $P_0 = (0, 0)$ *of system (2) is global attractor.*

Proof. From Theorem 7, it is easy to show that if $a < b$ and $a_1 < b_1$ then every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (2) is bounded. It is sufficient to prove that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is decreasing. From system (2) one has

$$x_{n+1} = \frac{ay_{n-1}}{b - cx_nx_{n-1}} \leqslant \frac{ay_{n-1}}{b} < y_{n-1}.$$

This implies that $x_{4n+1} < y_{4n-1}$ and $x_{4n+5} < y_{4n+3}$. Also

$$y_{n+1} = \frac{a_1x_{n-1}}{b_1 - c_1y_ny_{n-1}} \leqslant \frac{ax_{n-1}}{b} < x_{n-1}.$$

This implies that $y_{4n+1} < x_{4n-1}$ and $y_{4n+5} < x_{4n+3}$. So, $x_{4n+5} < y_{4n+3} < x_{4n+1}$ and $y_{4n+5} < x_{4n+3} < y_{4n+1}$. Hence the subsequences $\{x_{4n+1}\}, \{x_{4n+2}\}$, $\{x_{4n+3}\}$, $\{x_{4n+4}\}$ and $\{y_{4n+1}\}$, $\{y_{4n+2}\}$, $\{y_{4n+3}\}$, $\{y_{4n+4}\}$ are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence, $\lim_{n\to\infty}x_n = \lim_{n\to\infty}y_n = 0$. □

Lemma 4. *If $a < b$ and $a_1 < b_1$ then the equilibrium point $P_0 = (0,0)$ of system (2) is globally asymptotically stable.*

Proof. The proof follows from Theorems 8 and 9. □

Theorem 10. *The system (2) has no prime period-two solutions.*

Proof. Assume that $(u_1, v_1), (u_2, v_2), (u_1, v_1), \ldots$ be prime period-two solution of system (2) such that $u_i,\ v_i \neq 0$ and $u_i \neq v_i$ for $i = 1, 2$. Then from system (2) one has:

$$u_1 = \frac{av_1}{b - cu_2u_1}, \quad u_2 = \frac{av_2}{b - cu_1u_2}, \tag{16}$$

and

$$v_1 = \frac{a_1u_1}{b_1 - c_1v_2v_1}, \quad v_2 = \frac{a_1u_2}{b_1 - c_1v_1v_2}. \tag{17}$$

From (16) and (17), one has $u_i,\ v_i = 0$ for $i = 1, 2$. Which is a contradiction. Hence, system (2) has no prime period-two solutions. □

3.3. Rate of convergence

We investigate the rate of convergence of a solution that converges to the equilibrium point $P_0 = (0,0)$ of the system (2).

Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be any solution of the system (2) such that $\lim_{n\to\infty}x_n = \bar{x}$, and $\lim_{n\to\infty}y_n = \bar{y}$. To find the error terms, one has from the system (2)

$$x_{n+1} - \bar{x} = \sum_{i=0}^{1} A_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{1} B_i(y_{n-i} - \bar{y}),$$

$$y_{n+1} - \bar{y} = \sum_{i=0}^{1} C_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{1} D_i(y_{n-i} - \bar{y}).$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, one has

$$e_{n+1}^1 = \sum_{i=0}^{1} A_ie_{n-i}^1 + \sum_{i=0}^{1} B_ie_{n-i}^2,$$

$$e_{n+1}^2 = \sum_{i=0}^{1} C_ie_{n-i}^1 + \sum_{i=0}^{1} D_ie_{n-i}^2,$$

where $A_0 = \frac{acx_{n-1}\bar{y}}{(b-cx_nx_{n-1})(b-c\bar{x}^2)}$, $A_1 = \frac{ac\bar{x}\bar{y}}{(b-cx_nx_{n-1})(b-c\bar{x}^2)}$, $B_0 = 0$, $B_1 = \frac{a}{b-cx_nx_{n-1}}$, $C_0 = 0$, $C_1 = \frac{a_1}{b_1-c_1y_ny_{n-1}}$, $D_0 = \frac{a_1c_1\bar{x}y_{n-1}}{(b_1-c_1y_ny_{n-1})(b_1-c_1\bar{y}^2)}$, $D_1 = \frac{a_1c_1\bar{x}\bar{y}}{(b_1-c_1y_ny_{n-1})(b_1-c_1\bar{y}^2)}$.

Taking the limits, we obtain $\lim_{n\to\infty}A_0 = \lim_{n\to\infty}A_1 = \frac{ac\bar{x}\bar{y}}{(b-c\bar{x}^2)^2}$, $\lim_{n\to\infty}B_0 = 0$, $\lim_{n\to\infty}B_1 = \frac{a}{b-c\bar{x}^2}$, $\lim_{n\to\infty}C_0 = 0$, $\lim_{n\to\infty}C_1 = \frac{a_1}{b_1-c_1\bar{y}^2}$, $\lim_{n\to\infty}D_0 = \lim_{n\to\infty}D_1 = \frac{a_1c_1\bar{x}\bar{y}}{(b_1-c_1\bar{y}^2)^2}$. So, the limiting system of error terms can be written as

$$G_{n+1} = MG_n,$$

where $G_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix}$ and $M = \begin{pmatrix} \frac{ac\bar{x}\bar{y}}{(b-c\bar{x}^2)^2} & \frac{ac\bar{x}\bar{y}}{(b-c\bar{x}^2)^2} & 0 & \frac{a}{b-c\bar{x}^2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{a_1}{b_1-c_1\bar{y}^2} & \frac{a_1c_1\bar{x}\bar{y}}{(b_1-c_1\bar{y}^2)^2} & \frac{a_1c_1\bar{x}\bar{y}}{(b_1-c_1\bar{y}^2)^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$, which is similar to linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$.

Using proposition (1), one has following result.

Theorem 11. *Assume that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a positive solution of the system (1) such that $\lim_{n\to\infty}x_n = \bar{x}$, and $\lim_{n\to\infty}y_n = \bar{y}$, where $(\bar{x}, \bar{y}) = (0,0)$. Then, the error vector E_n of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty}(\|E_n\|)^{\frac{1}{n}} = |\lambda F_J(\bar{x}, \bar{y})|, \quad \lim_{n\to\infty}\frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_J(\bar{x}, \bar{y})|,$$

where $\lambda F_J(\bar{x}, \bar{y})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(0,0)$.

4. Examples

In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples show that the equilibrium point $(0,0)$ of both systems (1) and (2) is globally asymptotically stable.

Example 1. Consider the system (1) with initial conditions $x_{-1} = 0.7,\ x_0 = 1.5,\ y_{-1} = 1.9,\ y_0 = 1.2$. Moreover, choosing the parameters $\alpha = 120, \beta = 129, \gamma = 3.3, \alpha_1 = 125, \beta_1 = 135, \gamma_1 = 10$. Then, the system (1) can be written as:

$$x_{n+1} = \frac{120x_{n-1}}{129 - 3.3y_ny_{n-1}}, \quad y_{n+1} = \frac{125y_{n-1}}{135 - 10x_nx_{n-1}}, \quad n = 0, 1, \ldots, \tag{18}$$

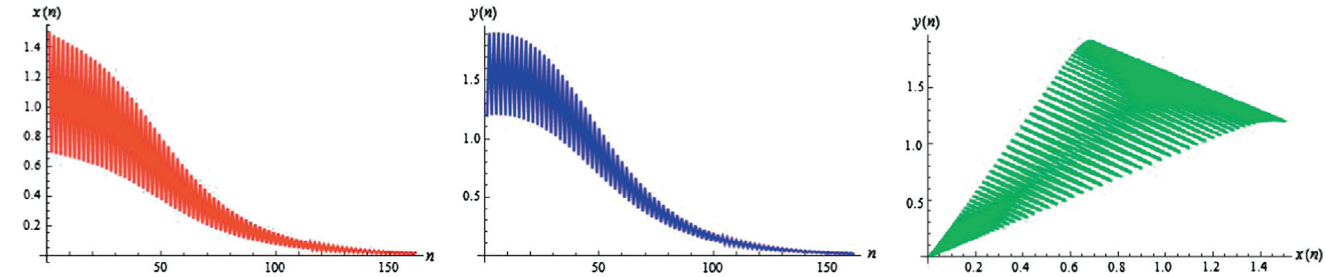
and with initial conditions $x_{-1} = 0.7,\ x_0 = 1.5,\ y_{-1} = 1.9,\ y_0 = 1.2$.

Moreover, in Fig. 1 the plot of x_n is shown in Fig. 1a, the plot of y_n is shown in Fig. 1b, and an attractor of the system (18) is shown in Fig. 1c.

Example 2. Consider the system (1) with initial conditions $x_{-1} = 2.9,\ x_0 = 1.8,\ y_{-1} = 0.07,\ y_0 = 0.1$. Moreover, choosing the parameters $\alpha = 538,\ \beta = 550,\ \gamma = 14,\ \alpha_1 = 600,\ \beta_1 = 625,\ \gamma_1 = 16$. Then, the system (1) can be written as:

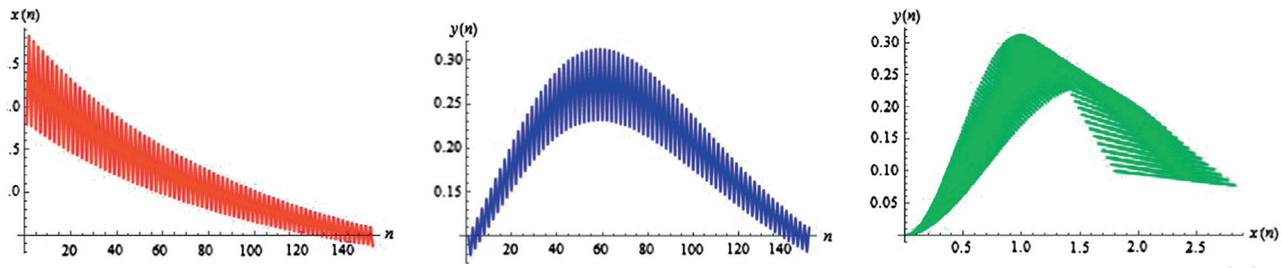
$$x_{n+1} = \frac{538x_{n-1}}{550 - 14y_ny_{n-1}}, \quad y_{n+1} = \frac{600y_{n-1}}{625 - 16x_nx_{n-1}}, \quad n = 0, 1, \ldots, \tag{19}$$

and with initial conditions $x_{-1} = 2.9,\ x_0 = 1.8,\ y_{-1} = 0.07,\ y_0 = 0.1$.



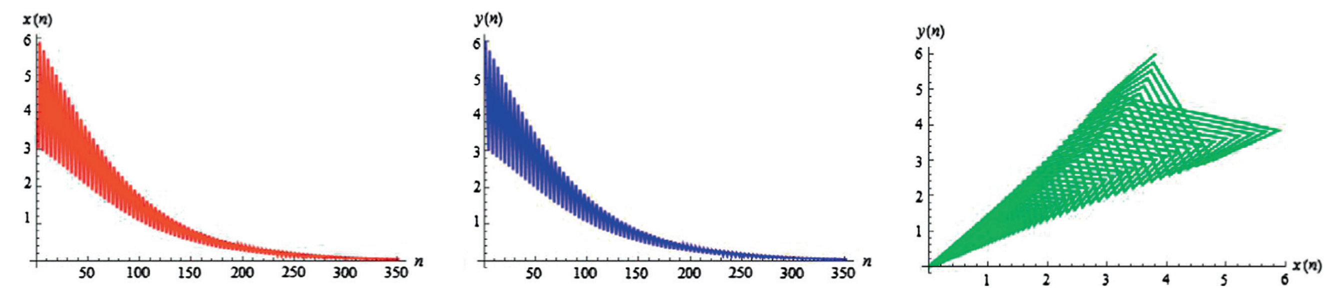
(a) Plot of x_n for the system (18) (b) Plot of y_n for the system (18) (c) An attractor of the system (18)

Figure 1 Plots for the system (18).



(a) Plot of x_n for the system (19) (b) Plot of y_n for the system (19) (c) An attractor of the system (19)

Figure 2 Plots for the system (19).



(a) Plot of x_n for the system (20) (b) Plot of y_n for the system (20) (c) An attractor of the system (20)

Figure 3 Plots for the system (20).

Moreover, in Fig. 2 the plot of x_n is shown in Fig. 2a, the plot of y_n is shown in Fig. 2b, and an attractor of the system (19) is shown in Fig. 2c.

Moreover, in Fig. 3 the plot of x_n is shown in Fig. 3a, the plot of y_n is shown in Fig. 3b, and an attractor of the system (20) is shown in Fig. 3c.

Example 3. Consider the system (2) with initial conditions $x_{-1} = 4.9,\ x_0 = 3.8,\ y_{-1} = 3.08,\ y_0 = 5.99$. Moreover, choosing the parameters $a = 1105,\ b = 1126,\ c = 0.008,\ a_1 = 1100,\ b_1 = 1146,\ c_1 = 2.01$. Then, the system (2) can be written as:

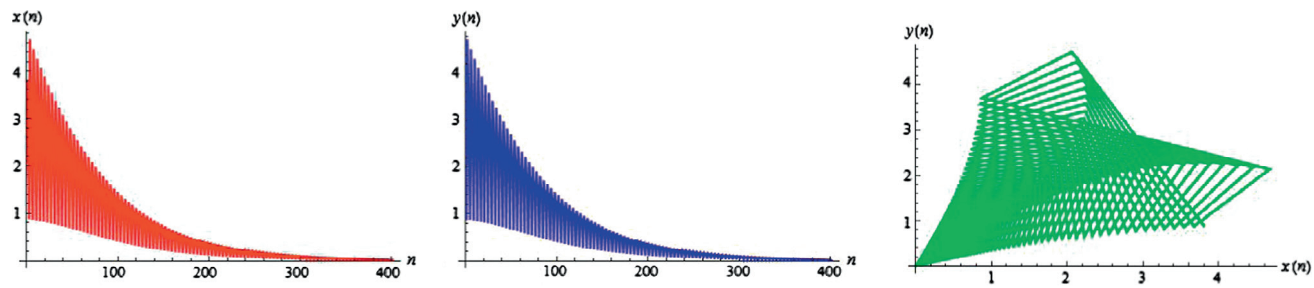
$$x_{n+1} = \frac{1105y_{n-1}}{1126 - 0.008x_nx_{n-1}},$$
$$y_{n+1} = \frac{1100x_{n-1}}{1146 - 2.01y_ny_{n-1}}, \quad n = 0, 1, \ldots, \tag{20}$$

and with initial conditions $x_{-1} = 4.9,\ x_0 = 3.8,\ y_{-1} = 3.08,\ y_0 = 5.99$.

Example 4. Consider the system (2) with initial conditions $x_{-1} = 4.89,\ x_0 = 3.8,\ y_{-1} = 2.08,\ y_0 = 0.89$. Moreover, choosing the parameters $a = 1101,\ b = 1116,\ c = 0.3,\ a_1 = 1090,\ b_1 = 1136,\ c_1 = 5.01$. Then, the system (2) can be written as:

$$x_{n+1} = \frac{1101y_{n-1}}{1116 - 0.3x_nx_{n-1}},$$
$$y_{n+1} = \frac{1090x_{n-1}}{1136 - 5.01y_ny_{n-1}}, \quad n = 0, 1, \ldots, \tag{21}$$

and with initial conditions $x_{-1} = 4.89,\ x_0 = 3.8,\ y_{-1} = 2.08,\ y_0 = 0.89$.



(a) Plot of x_n for the system (21) (b) Plot of y_n for the system (21) (c) An attractor of the system (21)

Figure 4 Plots for the system (21).

Moreover, in Fig. 4 the plot of x_n is shown in Fig. 4a, the plot of y_n is shown in Fig. 4b, and an attractor of the system (21) is shown in Fig. 4c.

5. Conclusion

This work is a natural extension of [6]. In this paper we have investigated the qualitative behavior of some systems of second-order rational difference equations. Each system has only one equilibrium point which is stable under some restriction to parameters. The most important finding here is that the unique equilibrium point $(0, 0)$ is a globally asymptotically stable for the systems (1) and (2). Moreover, we have determined the periodicity behavior of positive solutions and the rate of convergence of a solution that converges to the equilibrium point $(0, 0)$ of the systems (1) and (2). Some numerical examples are provided to support our theoretical results. These examples are experimental verifications of theoretical discussions.

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