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Poisson–logarithmic half-logistic distribution with inference under a progressive-stress model based on adaptive type-II progressive hybrid censoring

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Abstract

The researchers, engineers, and physical experimenters may face difficulty to get a distribution that fits the failure data arising from certain systems. So, in this paper, a new distribution is introduced, named Poisson–logarithmic half-logistic distribution, based on a parallel–series system's failure times. Specific statistical properties are investigated for the introduced distribution. Also, two real data sets are considered to compare the introduced distribution with some other distributions. The progressive-stress accelerated life test is applied using an increasing exponential function of time to units whose lifetimes are expected to follow the new distribution at normal stress conditions. Different estimation methods, such as maximum likelihood, percentile, least squares, and weighted least square methods, are considered on the basis of adaptive type-II progressive hybrid censoring. To assess the efficiency of the estimation methods, a simulation study is conducted, as well as numerical calculations.

Keywords: Compounding of distributions, Parallel–series system, Poisson, logarithmic, and half-logistic distributions, Progressive-stress accelerated life test, Adaptive type-II progressive hybrid censoring, Maximum likelihood, percentile, least squares, and weighted least squares estimations

Mathematics Subject Classification: 62E15, 62F10, 62N01, 62N05

Introduction

Describing the failures of certain systems may be of great importance for researchers, engineers, and physical experimenters, especially when the units constituting each system are connected in a parallel–series system. Well-known classical distributions may fail to give adequate fit to such failures. Therefore, it is needful to construct (generate) a distribution for such cases taking into account the parallel–series system combination and the number of units in each sub-system which may be a random variable. Compounding of distributions is considered as one of the methods that could be implemented to construct (generate) new distributions. The new distributions include more parameters, and hence, they possess different shapes of the hazard rate function (HRF); this feature makes them more flexible to fit the failure data.

Kuş [1] and Tahmasbi and Rezaei [2] proposed the exponential–Poisson and exponential–logarithmic distributions, respectively, which have decreasing hazard rates. Louzada et al. [3] discussed the statistical properties of the complementary exponential–geometric distribution, which has an increasing hazard rate. Abdel-Hamid [4] introduced the Poisson-half-logistic distribution (PHLD), which has an increasing-constant hazard rate and investigated its properties. Rezaei et al. [5], Ristić and Nadarajah [6] and Nadarajah et al. [7] discussed the statistical properties of the exponentiated exponential–geometric, exponentiated exponential–Poisson, and geometric–exponential–Poisson distributions, which have increasing, decreasing and unimodal hazard rates, respectively. Abdel-Hamid and Hashem [8, 9] and Nadarajah et al. [10] introduced the doubly Poisson–exponential and two exponential–Poisson–geometric and geometric–Poisson–Rayleigh distributions, respectively, which have monotone increasing, decreasing, bathtub-shaped, unimodal, and increasing–decreasing–increasing hazard rates and investigated their properties.

With the permanent development in manufacturing technology, modern products are designed and manufactured to run without failure for a long interval of time under normal functioning conditions. Hence, when applying traditional life testing experiments, manufacturers find it difficult to get enough information about the failure times for their products in a suitable time. For this reason, accelerated life tests (ALTs) are applied to obtain the required information about the product failure time in a short time interval and also to derive the relationship between the external stress variables and product lifetimes. In ALTs, products are experimented in such tests under stresses that are more severe than those under regular conditions. The information collected from the experiment under accelerated conditions is used to predict product performance in regular conditions. The stress applied in ALTs can be widely used in different methods, such as constant, step, and progressive stresses. See, for example, Nelson [11], AL-Hussaini and Abdel-Hamid [12, 13], Abdel-Hamid and AL-Hussaini [14–16], Yin and Sheng [17], Abdel-Hamid and Abushal [18], AL-Hussaini et al. [19] and Nadarajah et al. [10], for more details on ALTs.

In life testing and reliability experiments, in which items are removed or lost from testing before failure due to occasional breakage or an item being tested drops out, the experimenter may be unable to get all information about failure time for each experimental item. Data obtained from such tests are named censored data. One of the major advantages of censoring may appear in decreasing the total experiment time and the associated cost. A censoring scheme (CS), which can balance among the number of items used in the experiment, the total experiment time, and the efficiency of statistical inference based on the experimental data, is desirable.

The majority in applying censoring are type-I and type-II. The mixing of these two types of CSs constitutes a new censoring called hybrid CS. These types of CSs do not possess the elasticity of allowing removal of items from the experiment at different points other than the end point of the experiment. Hence, to overcome this problem, a new generalization of existing plans of censoring named progressive type-I hybrid CS is proposed, see Kundu and Joarder [20] and Childs et al. [21]. It can be applied as follows:

Assume that the effective sample size $m(< n)$ and the experimental time $\mathcal{T}$ are fixed before starting the experiment with progressive CS $(R_1, R_2, \ldots, R_m)$.

R_1 functioning units are randomly excluded from the experiment at the first failure time z_1. R_2 functioning units are randomly excluded from the experiment at the second failure time z_2. The experiment continues in the same way until the m-th failure z_m or time $\mathcal{T}$ whichever occurs first. In the case of the m-th failure time, z_m occurs before time $\mathcal{T}$, and the remaining functioning units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ will be excluded from the experiment, thereby finishing the experiment at z_m. But, if the experimental time $\mathcal{T}$ is reached before occurring the m-th failure time z_m and only D failures occur before fixed time $\mathcal{T}$, $D < m$, then at the time $\mathcal{T}$ exclude all the remaining functioning units $R_D^* = n - D - \sum_{i=1}^{D} R_i$ from the experiment, thereby finishing the experiment at $\mathcal{T}$.

One of the drawbacks of the progressive type-I hybrid CS is that the effective sample size is random and may be quite small (up to zero). Therefore, statistical inference procedures may not be applicable or will be less efficient. For increasing the desired efficiency of statistical analysis, Ng et al. [22] and Lin et al. [23] proposed an adaptive type-II progressive hybrid CS, in which the effective sample size m was constant. In this CS, we may also allow the experiment to run over time $\mathcal{T}$, which is considered fixed before the experiment, and modify the CS adaptively through the experiment. The main objective of the current scheme is to accelerate the test as much as possible when the test period exceeds a predetermined time $\mathcal{T}$. For this scheme, in the case of the m-th failure time z_m occurs before time $\mathcal{T}$, and the remaining functioning units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ will be excluded from the experiment, thereby finishing the experiment at z_m. But, if the experimental time $\mathcal{T}$ is reached before occurring the m-th failure time z_m and only D failures occur before time $\mathcal{T}$, $D < m$, then we will not remove any functioning unit from the experiment immediately following the $(D + 1)$-th ...$(m - 1)$-th failure time and at the m-th failure time z_m, we remove the remaining functioning units $R_m^* = n - m - \sum_{i=1}^{D} R_i$ from the experiment, thereby finishing the experiment at z_m. Here, D is a discrete random variable representing the number of observed failure times up to time $\mathcal{T}$.

The last CS provides the experimenter a guarantee to acquire m observed failure times for the efficacy of statistical inference and also to control the total test time to be near to the proposed time $\mathcal{T}$. Moreover, the value of experimental time $\mathcal{T}$ may have a role in determining the censoring values R_i. This value of $\mathcal{T}$ enables the experimenter flexibility between stopping the experiment in a short period of time and a higher opportunity to detect some large failure times, see Ng et al. [22].

In this paper, we propose a new distribution, named Poisson–logarithmic half-logistic distribution (PLHLD), based on a parallel–series system's failure times and study some of its important properties. The new proposed distribution can be obtained by compounding zero-truncated Poisson and logarithmic distributions with half-logistic distribution (HLD). The progressive-stress ALT is applied using an increasing exponential function of time to units whose lifetimes are supposed to have the PLHLD at normal stress conditions. Different methods of estimation, based on adaptive type-II progressive hybrid censoring, are used to estimate the parameters involved in the PLHLD under progressive-stress ALT.

The remaining sections of the article are structured according to the following: In the "The Poisson–logarithmic half-logistic distribution" section, we propose the PLHLD and study some of its important properties, in addition to the application to two real data sets. In the “PLHLD under progressive-stress model” section, we consider the PLHLD under progressive-stress ALT. Some estimation methods are applied in the “Different methods of estimation” section. To assess the performance of the estimation methods, some simulation studies are presented in the “Simulation study” section followed by the significant results with their discussion in the “Results and discussion” section. Finally, some important remarks are given in the “Concluding remarks” section.

The Poisson–logarithmic half-logistic distribution

Compounding of distributions based on failure times of a parallel–series system is considered as one of the physical motivations to introduce the new distribution. According to the parallel–series system structured in Fig. 1 and as shown in Nadarajah et al. [10], suppose a is a realization of a random variable (RV) A. The distribution of A is assumed to be a zero-truncated Poisson distribution with probability mass function (PMF)

$$P(A = a; \theta) = \frac{e^{-\theta}\theta^{a}}{a!(1 - e^{-\theta})}, \quad a = 1, 2, \ldots, \quad (\theta > 0). \tag{1}$$

Suppose that the l-th system, $l = 1, 2, \ldots, A$, has B_l series items, where B_l is a RV having the logarithmic distribution with PMF

$$P(B_l = b_l; p) = \frac{(1 - p)^{b_l}}{-b_l \ln(p)}, \quad b_l = 1, 2, \ldots, \quad (0 < p < 1). \tag{2}$$

The motivation for assuming A and B_l as RVs comes from a practical point of view in which the failure system sometimes occurs due to existence of an unknown number of initial defective items in the system.

The failure times of the b_l series items, say $Y_{1l}, Y_{2l}, \ldots, Y_{b_l l}$, are assumed to be independent and identically distributed (iid) RVs. The lifetime of the l-th series system is given by the minimum lifetime of its items, i.e., $T_l = \min(Y_{1l}, Y_{2l}, \ldots, Y_{B_l l}), l = 1, \ldots, A$. Since the a systems constitute a parallel system, the entire system structured in Fig. 1 is in operation when at least one series system operates. In other words, the entire system terminates if all of the series systems fail. Therefore, the lifetime of the parallel–series system is given by

$$Z \equiv Z_A = \max_{1 \le l \le A} T_l = \max_{1 \le l \le A} \min_{1 \le i \le B_l} Y_{il}, \tag{3}$$

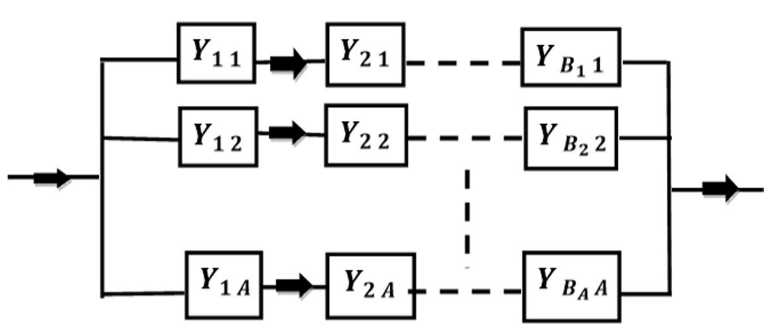

Fig. 1 Description of a parallel–series system

where $Y_{il}, i = 1, \ldots, B_l, l = 1, \ldots, A$ are iid RVs.

In the following theorem, the probability density function (PDF) and cumulative distribution function (CDF) of the RV Z are given.

Theorem 1 *For $i = 1, \ldots, B_j, l = 1, \ldots, A$, suppose that Y_{il} are iid RVs with common PDF $g_Y(y; \boldsymbol{\lambda})$ and CDF $G_Y(y; \boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_\xi\}$ is a vector of parameters of dimension ξ. Suppose also that A and B_j are two independent zero-truncated Poisson and logarithmic RVs with PMFs (1) and (2), respectively. Then, the PDF and CDF of Z, given by (3), are given, respectively, by*

$$g(z) \equiv g(z; \theta, p, \boldsymbol{\lambda}) = \frac{\theta(1-p)p^{\frac{\theta}{\ln(p)}}}{(1-e^\theta)\ln(p)} \frac{g_Y(z; \boldsymbol{\lambda})}{\left(1-(1-p)[1-G_Y(z; \boldsymbol{\lambda})]\right)^{\frac{\theta}{\ln(p)}+1}}, \tag{4}$$

$$G(z) \equiv G(z; \theta, p, \boldsymbol{\lambda}) = \frac{1}{e^\theta - 1}\left[\left(\frac{p}{1-(1-p)[1-G_Y(z; \boldsymbol{\lambda})]}\right)^{\frac{\theta}{\ln(p)}} - 1\right]. \tag{5}$$

Proof

Since $T_l = \min_{1 \le i \le B_l} Y_{il}$, $l = 1, \ldots, A$, the conditional PDF of T_l, given $B_l = b_l$, is given by

$$g(t_l|b_l; \boldsymbol{\lambda}) = b_l[1 - G_Y(t_l; \boldsymbol{\lambda})]^{b_l - 1} g_Y(t_l; \boldsymbol{\lambda}).$$

Thus, the unconditional PDF of T_l takes the form

$$\begin{aligned} g_T(t_l; p, \boldsymbol{\lambda}) &= \sum_{b_l=1}^{\infty} g(t_l|b_l; \boldsymbol{\lambda}) P(B_l = b_l; p) \\ &= \frac{-(1-p)}{\ln(p)} \frac{g_Y(t_l; \boldsymbol{\lambda})}{1-(1-p)[1-G_Y(t_l; \boldsymbol{\lambda})]}. \end{aligned} \tag{6}$$

The corresponding CDF of $T_l, l = 1, \ldots, A$, is given by

$$G_T(t_l; p, \boldsymbol{\lambda}) = \ln\left(\frac{p}{1-(1-p)[1-G_Y(t_l; \boldsymbol{\lambda})]}\right)^{\frac{1}{\ln(p)}}, \quad t_l \ge 0. \tag{7}$$

Since $Z = \max_{1 \le l \le A} \min_{1 \le i \le B_l} Y_{il} = \max_{1 \le l \le A} T_l$, the conditional PDF of Z given $A = a$ is given by

$$g(z|a; p, \boldsymbol{\lambda}) = a[G_T(z; p, \boldsymbol{\lambda})]^{a-1} g_T(z; p, \boldsymbol{\lambda}).$$

Therefore, the unconditional PDF of Z takes the form

$$\begin{aligned} g_Z(z; \theta, p, \boldsymbol{\lambda}) &= \sum_{a=1}^{\infty} g(z|a; p, \boldsymbol{\lambda}) P(A = a; \theta) \\ &= \frac{\theta}{e^\theta - 1} g_T(z; p, \boldsymbol{\lambda})\, e^{\theta G_T(z; p, \boldsymbol{\lambda})}. \end{aligned}$$

The corresponding CDF of Z is given by

$$G_Z(z;\theta,p,\boldsymbol{\lambda}) = \frac{e^{\theta G_T(z;p,\boldsymbol{\lambda})}-1}{e^{\theta}-1}, \quad z>0,$$

where $G_T(z;p,\boldsymbol{\lambda})$ is given by (7). From Equations (6) and (7), the PDF (4) and the corresponding CDF (5) of Z hold. □

The new class of distributions (5) includes several lifetime distributions as special cases by assuming different forms of the CDF $G(.)$. Many real-life systems do not have constant hazard rates. So, we consider here the HLD as a distribution with increasing hazard rate. There, we suppose that the lifetimes of the items presented in the l-th system have HLD. Assume, in Equations (4) and (5), that the lifetimes of the items included in the l-th system are independent and identical half-logistic RVs with PDF $g(y;\lambda)=\dfrac{2\lambda e^{-\lambda y}}{(1+e^{-\lambda y})^2}$ and CDF $G(y;\lambda)=\dfrac{1-e^{-\lambda y}}{1+e^{-\lambda y}}$. Thus, the lifetime Z of the parallel–series system has a PLHLD with PDF and CDF given, respectively, by

$$g(z)\equiv g(z;\theta,p,\lambda)=\mathcal{C}(\theta,p,\lambda)\ \frac{e^{-\lambda z}}{(1+e^{-\lambda z})^2}\left[\eta(z)\right]^{\frac{\theta}{\ln(1/p)}-1}, \quad z,\theta,p,\lambda>0, p\neq 1, \tag{8}$$

$$G(z)\equiv G(z;\theta,p,\lambda)=\frac{1}{e^{\theta}-1}\left[\left(\frac{1}{p}\ \eta(z)\right)^{\frac{\theta}{\ln(1/p)}}-1\right], \tag{9}$$

where

$$\mathcal{C}(\theta,p,\lambda)=\frac{2\theta\lambda(1-p)p^{\frac{\theta}{\ln(p)}}}{(1-e^{\theta})\ln(p)}, \tag{10}$$

$$\eta(z)\equiv\eta(z;p,\lambda)=1-\frac{2(1-p)}{1+e^{\lambda z}}. \tag{11}$$

We have mentioned before that $0<p<1$, but it can be noticed here that function (8) is still a PDF for $p>1$. Clearly, λ is a scale parameter of PLHLD.

The HRF of the PLHLD is given by

$$h(z)\equiv h(z;\theta,p,\lambda)=\mathcal{C}(\theta,p,\lambda)\ \frac{(1-e^{\theta})e^{-\lambda z}}{(1+e^{-\lambda z})^2}\ \frac{\left[\eta(z)\right]^{\frac{\theta}{\ln(1/p)}-1}}{\left(\frac{1}{p}\ \eta(z)\right)^{\frac{\theta}{\ln(1/p)}}-e^{\theta}}, \quad z,\theta,p,\lambda>0, p\neq 1. \tag{12}$$

Notice that the CDF (9) of the PLHLD reduces to CDF of the

1 PHLD, proposed by Abdel-Hamid [4], as $p\to 0^+$,
2 logarithmic half-logistic distribution (LHLD) as $\theta\to 0^+$,
3 HLD as $p\to 0^+$ and $\theta\to 0^+$.

Statistical properties of PLHLD

Here, we discuss some important statistical properties of the PLHLD, such as its PDF, shapes of the HRF, q-th quantile, moments, mean, variance, kurtosis, skewness, mean residual lifetime, PDF and CDF of the i-th order statistic, Lorenz and Bonferroni curves, and Shannon's and Rényi entropies.

Modality and quantiles of PLHLD

In Fig. 2, the PDF (8) of PLHLD is drawn for different values of θ and p with $\lambda = 1.5$. It can be noticed, from Figure 2, that the PDF (8) may be decreasing or increasing–decreasing (unimodal). This can be concluded as follows:

The first derivative of $\Im(z) = \ln(g(t))$ takes the form

$$\Im'(t) = \frac{\lambda}{1+e^{-\lambda z}}\left[e^{-\lambda z} + \frac{2(1-p)\left(\frac{\theta}{\ln(1/p)}-1\right)}{e^{\lambda z}+2p-1} - 1\right].$$

Then, $\Im'(t) = 0 \Rightarrow \left(e^{\lambda z} + \frac{\theta(p-1)}{\ln(1/p)}\right)^2 = \left(\frac{\theta(p-1)}{\ln(1/p)}\right)^2 + 2p - 1.$

This yields the unique solution

$$z^* = \frac{1}{\lambda}\ln\left(\frac{-\theta(p-1)}{\ln(1/p)} + \sqrt{\left(\frac{\theta(p-1)}{\ln(1/p)}\right)^2 + 2p - 1}\right).$$

Then, $z^* > 0 \Rightarrow 2p - 1 > 1 + \frac{2\theta(p-1)}{\ln(1/p)} \Rightarrow pe^{\theta} > 1$. So, the PDF (8) is unimodal, with mode satisfied at $z = z^*$, if $pe^{\theta} > 1$ and is decreasing if $pe^{\theta} < 1$.

In Fig. 3, the HRF (12) of PLHLD is drawn for different values of θ and p with $\lambda = 1.5$. Different shapes of the HRF can be noticed in Fig. 3, such as decreasing-constant, increasing-constant, and v-shaped. It can be noticed also that, at the end of the constant failure rate, sudden fluctuations are exhibited. Usually, these fluctuations indicate that product performance has deteriorated over time. Such a phenomenon can be observed in non-stationary data; thus, the PLHLD can kindly represent such data. The non-stationary nature of the failure times may be useful to the experimenter in predicting the environmental behavior of some products. The different shapes of HRF (12) supply another motivation for selecting the PLHLD as a likely candidate for data analysis.

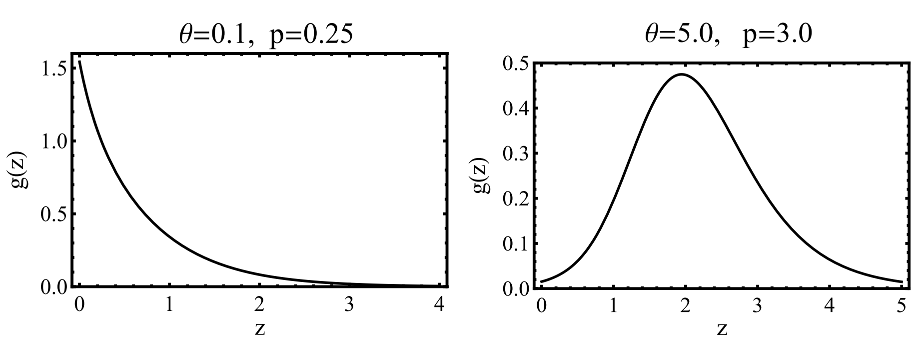

Fig. 2 PDF of the PLHLD for different values of θ and p with $\lambda = 1.5$

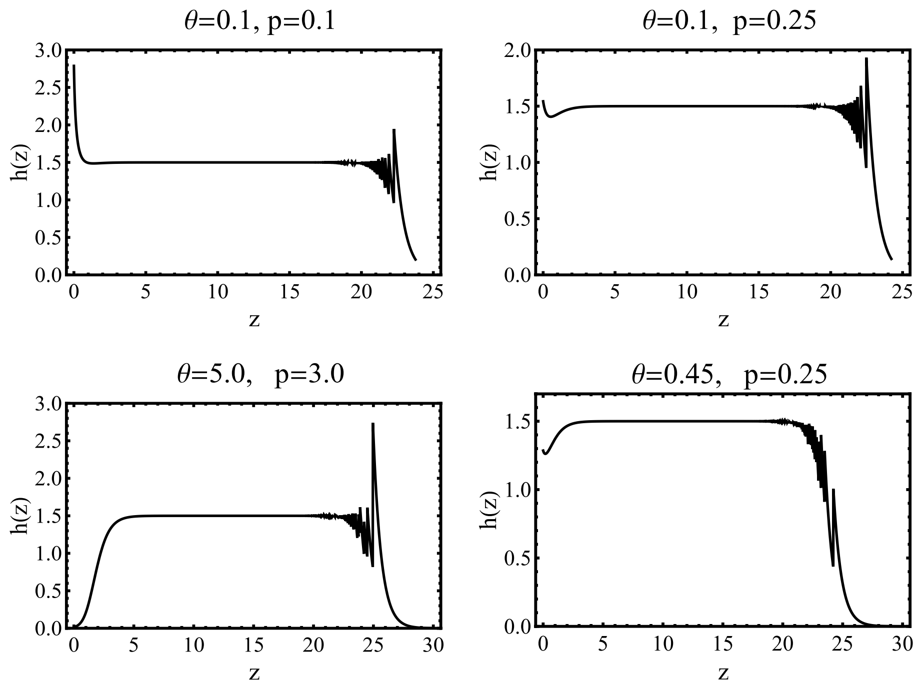

Fig. 3 HRF of the PLHLD for different values of θ and p with $\lambda = 1.5$

The q-th quantile z_q of the PLHLD with CDF (9) can be obtained by solving the equation $G(z_q) - q = 0$, which is given by

$$z_q = \frac{1}{\lambda} \ln\left(\frac{2(1-p)}{1 - p[q(e^{\theta} - 1) + 1]^{\frac{\ln(1/p)}{\theta}}} - 1 \right), \qquad 0 < q < 1. \tag{13}$$

Particularly, the median of the PLHLD with CDF (9) can be obtained from Eq. (13) by putting $q = 0.5$ as

$$z_{0.5} = \frac{1}{\lambda} \ln\left(\frac{2(1-p)}{1 - p[0.5(e^{\theta} + 1)]^{\frac{\ln(1/p)}{\theta}}} - 1 \right).$$

Moments and mean residual lifetime

Let the RV Z have the PLHLD with PDF (8). Then, the rth moment of Z can be readily obtained by applying the Legendre–Gauss quadrature formula (LGQF), see Canuto et al. [24].

$$\begin{aligned}
\mu_r &= \int_0^{\infty} z^r g(z)\, \mathrm{d}\, z \\
&= \mathcal{C}(\theta, p, \lambda) \int_{-1}^{1} \frac{2}{(1-x)^2} \quad \Psi_r\left(\frac{1+x}{1-x}\right) \mathrm{d}\, x \\
&= 2\,\mathcal{C}(\theta, p, \lambda) \sum_{j=0}^{M} \frac{\pi_j}{(1-x_j)^2} \quad \Psi_r\left(\frac{1+x_j}{1-x_j}\right),
\end{aligned} \tag{14}$$

where $\mathcal{C}(\theta, p, \lambda)$ is given by (10) and

$$\Psi_r(z) = \frac{z^r e^{-\lambda z}}{(1+e^{-\lambda z})^2} \left[\eta(z)\right]^{\frac{\theta}{\ln(1/p)}-1}, \tag{15}$$

where $\eta(.)$ is given by (11) and x_j and π_j are the zeros and corresponding Christoffel numbers of the LGQF on the interval $(-1, 1)$,

$$\pi_j = \frac{2}{(1-x_j^2)[L'_{M+1}(x_j)]^2} \quad \text{and} \quad L'_{M+1}(x_j) = \frac{\mathrm{d}\, L_{M+1}(x)}{\mathrm{d}\, x} \quad \text{at} \quad x = x_j, \tag{16}$$

and L_M denotes the Legendre polynomial of degree M. The relationship between the degree M of Legendre polynomial and mean of the PLHLD is drawn in Fig. 4 in which one can select the value of M required to obtain stable results for true mean.

Equation (14) could be used to calculate the mean, variance, kurtosis, and skewness of the PLHLD. Those four quantities against θ are drawn in Fig. 5 for different values of p with $\lambda = 1.5$. From Figure 5, it can be noticed that the mean for fixed values of p is increasing, while, for fixed values of θ, higher values of the mean can be obtained as p increases. The variance is unimodal and tends to be increasing-constant as p decreases. Also, the kurtosis and skewness are decreasing–increasing and tend to be decreasing-constant as p decreases.

The mean residual lifetime is very important in the study of survival analysis. It is defined as the expected residual lifetime given that the system has survived to time z. It can be obtained for the PLHLD using LGQF as follows:

$$\begin{aligned} m(z_0) &= \mathrm{E}[Z - z_0 \mid Z > z_0] \\ &= \frac{1}{S(z_0)} \int_{z_0}^{\infty} S(z)\, \mathrm{d}\, z \\ &= \frac{2z_0}{e^{\theta} - \left(\frac{1}{p}\, \eta(z_0)\right)^{\frac{\theta}{\ln(1/p)}}} \int_{-1}^{1} \frac{1}{(1-x)^2} \left[e^{\theta} - \left(\frac{1}{p}\, \eta\left(\frac{2z_0}{1-x}\right)\right)^{\frac{\theta}{\ln(1/p)}}\right] \mathrm{d}\, x \\ &= \frac{2z_0}{e^{\theta} - \left(\frac{1}{p}\, \eta(z_0)\right)^{\frac{\theta}{\ln(1/p)}}} \sum_{j=0}^{M} \frac{\pi_j}{(1-x_j)^2} \left[e^{\theta} - \left(\frac{1}{p}\, \eta\left(\frac{2z_0}{1-x_j}\right)\right)^{\frac{\theta}{\ln(1/p)}}\right], \end{aligned}$$

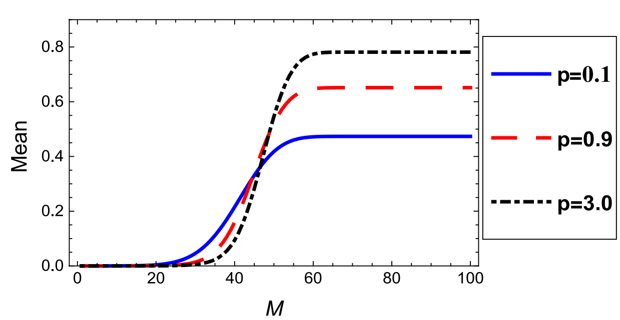

Fig. 4 The relationship between the degree M of Legendre polynomial and mean of the PLHLD for $\theta = 6.0$ and $\lambda = 4.5$ and different values of p

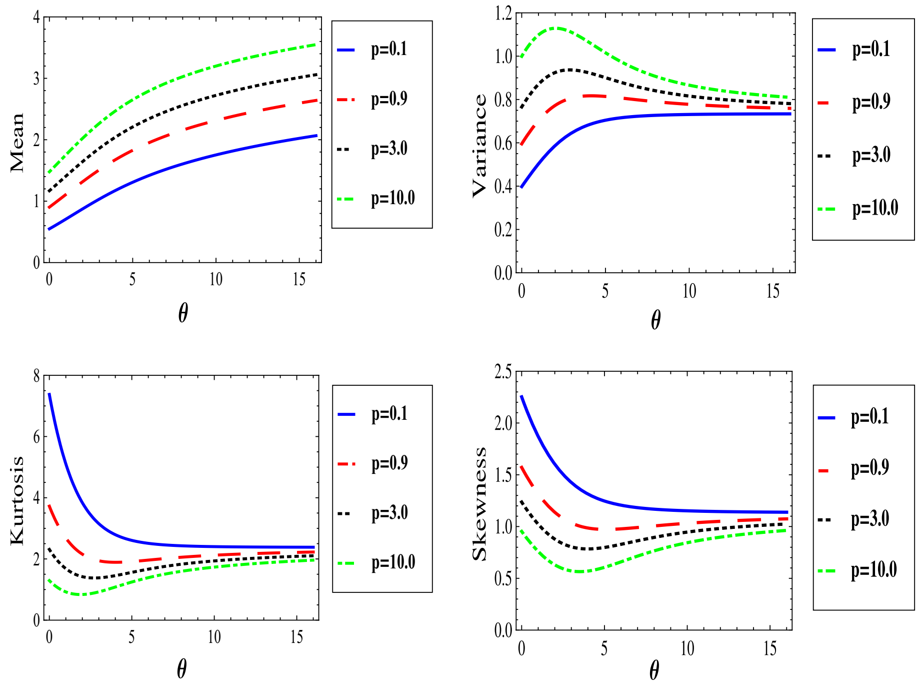

Fig. 5 The mean, variance, kurtosis, and skewness of the PLHLD against θ for different values of p with $\lambda = 1.5$

where $S(.)$ is the survival function of PLHLD, and x_j and π_j are defined as in (16).

Order statistics

Order statistics play an important role in different fields of statistical applications. They have a major role in quality control and reliability, in which an experimenter wants to predict with the failure of future units based on the times of past failures. Now, let $Z_1, \ldots, Z_n$ be a random sample of size n from the PLHLD with PDF (8) and CDF (9). The PDF of the ith order statistic, say $Z_{i:n}$, is given by, see, for example, Arnold et al. [25],

$$\begin{aligned} g_{i:n}(z) &= i\binom{n}{i} g(z)[G(z)]^{i-1}[1-G(z)]^{n-i} \\ &= i\binom{n}{i} \frac{2\theta\lambda(1-p)p^{\frac{\theta}{\ln(p)}}}{\ln(p)} \frac{e^{-\lambda z}}{(1+e^{-\lambda z})^2} \sum_{j_1=0}^{n-i}\sum_{j_2=0}^{i+j_1-1}\left[(-1)^{j_1+j_2}\binom{n-i}{j_1}\right. \\ &\quad \left. \times \binom{i+j_1-1}{j_2} \frac{p^{\frac{j_2\theta}{\ln(p)}}}{(1-e^{\theta})^{i+j_1}} \left(\eta(z)\right)^{\frac{\theta(j_2+1)}{\ln(1/p)}-1}\right]. \end{aligned}$$

The corresponding CDF, $G_{i:n}$, is given by

$$\begin{aligned} G_{i:n}(z) &= \sum_{j_3=i}^{n}\binom{n}{j_3}[G(z)]^{j_3}[1-G(z)]^{n-j_3} \\ &= \sum_{j_3=i}^{n}\sum_{j_4=0}^{n-j_3}\sum_{j_5=0}^{j_3+j_4}(-1)^{j_4+j_5}\binom{n}{j_3}\binom{n-j_3}{j_4}\binom{j_3+j_4}{j_5} \frac{\left(\frac{1}{p}\,\eta(z)\right)^{\frac{\theta j_5}{\ln(1/p)}}}{(1-e^{\theta})^{j_3+j_4}}, \end{aligned}$$

where $\eta(.)$ is given by (11).

Lorenz and Bonferroni curves

The Lorenz and Bonferroni curves have important meanings in economics to study income and poverty. They also have meanings in reliability, demography, insurance, and medicine. Let the RV Z have the PLHLD with PDF (8). Then, the Lorenz curve is given by

$$\begin{aligned}\mathfrak{L}(\epsilon) &= \frac{1}{\mu_1}\int_0^{q_\epsilon} zg(z)\,\mathrm{d}\,z\\ &= \frac{\mathcal{C}(\theta,p,\lambda)}{\mu_1}\int_0^{q_v}\Psi_1(z)\,\mathrm{d}\,z\\ &= \frac{q_\epsilon\,\mathcal{C}(\theta,p,\lambda)}{2\mu_1}\int_{-1}^{1}\Psi_1\Big(\frac{q_\epsilon}{2}(x+1)\Big)\,\mathrm{d}\,x\\ &= \frac{q_\epsilon\,\mathcal{C}(\theta,p,\lambda)}{2\mu_1}\sum_{j=0}^{M}\pi_j\,\Psi_1\Big(\frac{q_\epsilon}{2}(x_j+1)\Big),\end{aligned}$$

where $\mathcal{C}(\theta,p,\lambda)$ is given by (10), $q_\epsilon = G^{-1}(\epsilon)$, μ_1 is the mean of PLHLD, Ψ_1 is given, at $r = 1$, by (15), and x_j and π_j are defined as in (16) (Fig. 6).

The Bonferroni curve of the PLHLD is given by

$$\begin{aligned}\mathfrak{B}(\epsilon) &= \frac{1}{\epsilon\mu_1}\int_0^{q_\epsilon} zg(z)\,\mathrm{d}\,z\\ &= \frac{q_\epsilon\,\mathcal{C}(\theta,p,\lambda)}{2\epsilon\mu_1}\sum_{j=0}^{M}\pi_j\,\Psi_1\Big(\frac{q_\epsilon}{2}(x_j+1)\Big).\end{aligned}$$

Shannon's and Rényi entropies

Entropy plays an essential role in the field of information theory. It can be used to measure the randomness or uncertainty of dynamical systems, and it is widely used in science and engineering. Two commonly entropy measures are known as the Shannon's and Rényi entropies, see Shannon [26] and Rényi [27]. Now, let the RV Z have the PLHLD with PDF (8). Then, Shannon's entropy of Z is given by

$$\begin{aligned}\Upsilon &= \mathrm{E}[-\ln\big(g(z)\big)]\\ &= -\ln\left(\mathcal{C}(\theta,p,\lambda)\right) + 2\mathcal{C}(\theta,p,\lambda)\sum_{j=0}^{M}\frac{\pi_j}{(1-x_j)^2}\,\Omega\left(\frac{1+x_j}{1-x_j}\right)\Psi_0\left(\frac{1+x_j}{1-x_j}\right),\end{aligned}$$

where $\mathcal{C}(\theta,p,\lambda)$ is given by (10), $\Psi_0(.)$ is given, at $r = 0$, by (15), and

$$\Omega(z) = \lambda z + 2\ln(1+e^{-\lambda z}) - \left(\frac{\theta}{\ln(1/p)} - 1\right)\ln(\eta(z)).$$

Based on PDF (8), the Rényi entropy of Z is given by

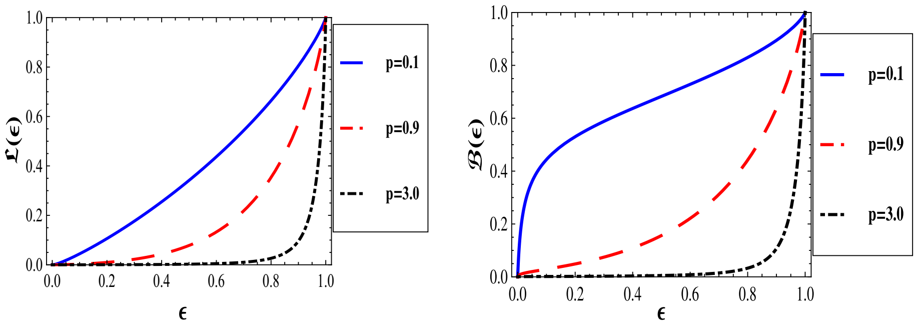

Fig. 6 Left (Right) panel: The Lorenz (Bonferroni) curve of the PLHLD for $\theta = 6.0, \lambda = 4.5$ and different values of p

$$\mathcal{J}(k) = \frac{1}{1-k} \ln \left(\int_0^\infty g^k(z)\, \mathrm{d}\, z \right)$$
$$= \frac{1}{1-k} \left[k \ln \left(\mathcal{C}(\theta, p, \lambda) \right) + \ln \left(\sum_{j=0}^{M} \frac{2\pi_j}{(1-x_j)^2} \quad \Delta_k \left(\frac{1+x_j}{1-x_j} \right) \right) \right],$$

where $k > 0, k \neq 1, \mathcal{C}(\theta, p, \lambda)$ is given by (10), x_j and π_j are defined as in (16), and

$$\Delta_k(z) = \frac{e^{-k\lambda z}}{(1+e^{-\lambda z})^{2k}} \left[\eta(z) \right]^{k \left(\frac{\theta}{\ln(1/p)} - 1 \right)}.$$

Application of PLHLD to real data

Five distributions are used to fit two real data sets. The distributions are PLHLD, PHLD, half-logistic generated Weibull distribution (HLGWD) (suggested by AL-Hussaini and Abdel-Hamid [28]), HLD, and Weibull distribution (WD). The CDFs of the last four distributions are given, respectively, by

$$G_{PHLD} = \frac{\exp\left\{ p_1 \left(\frac{1-\exp(-p_2 z)}{1+\exp(-p_2 z)} \right) \right\} - 1}{\exp(p_1) - 1}, \quad G_{HLGWD} = \frac{2\{1-\exp(-p_1 z^{p_3})\}^{p_2}}{1+\{1-\exp(-p_1 z^{p_3})\}^{p_2}},$$
$$G_{HLD} = \frac{1-\exp(-p_1 z)}{1+\exp(-p_1 z)}, \qquad G_{WD} = 1 - \exp\left(-p_2 z^{p_1}\right).$$

Four numerical methods such as Nelder–Mead (NM), BFGS (it is a quasi-Newton method introduced by **B**royden, **F**letcher, **G**oldfarb, and **S**hanno, simultaneously), conjugate gradients (CG), and L-BFGS-B (it is a modification of BFGS with box constraints) are used to obtain estimates of the distribution's parameters. These methods can be applied by "optim" function in R. In the tables, p_1, p_2 and p_3 stand for θ, λ and p, respectively.

The maximum likelihood estimates (MLEs) of parameters are attained by numerical methods to maximize the log-likelihood function. 5000 initial values are uniformly generated from a subset of parameter space. The four numerical methods are performed with these initials values of parameters to maximize the likelihood function.

The likelihood values for all estimates are ordered from large to small. The estimates that give the largest likelihood value are treated as MLEs of parameters.

The PLHLD, PHLD, HLGWD, HLD, and WD are now fitted to two real data sets. The first real data set is taken from Aarset [29]. It represents the failure times of 50 devices that have been put to life test at time 0. The data are given as follows: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92.

The second real data set is taken from Choulakian and Stephen [30]. It consists of 72 exceedances for the years from 1958 to 1984, rounded to one decimal place. The data are given as follows: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0.

For the first and second data sets, the MLEs of parameters with their corresponding standard errors under PLHLD, PHLD, HLGWD, HLD, and WD are given in Tables 1 and 2, respectively, in which several comparison criteria are also shared. Calculations of log-likelihood ℓ, -2ℓ, the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent AIC (CAIC), Hannan–Quinn information criterion (HQIC), Kolmogorov–Smirnov statistic (K-S), Anderson–Darling statistic (A-D), Cramér–von Mises statistic (CvM) with their associated p values, and the MLE $\widehat{p}_i$ $(i = 1, 2, 3)$ of parameter p_i with standard error $SE(\widehat{p}_i)$ are presented also in Tables 1 and 2. In that tables, initial parameters and the related numerical method are given to get MLEs for all models included in the analysis. It can be noticed, from Tables 1 and 2, that the PLHLD has the smallest values of -2ℓ, AIC, CAIC, HQIC, K-S, A-D, and CvM for the all data sets. Furthermore, the goodness-of-fit tests K-S, A-D, and CvM confirm the PLHLD model validity (p values>0.05). It is concluded that the PLHLD is better to fit the given data than the other distributions in terms of almost all criteria. Figures 7 and 8 present the fitted CDFs with the empirical CDF. From Figs. 7 and 8, it is concluded that fitted CDF of PLHLD exhibits better than the others.

PLHLD under progressive-stress model

Several authors, such as Abdel-Hamid and AL-Hussaini [16], Yin and Sheng [17], Abdel-Hamid and Abushal [18], and AL-Hussaini et al. [19], studied progressive-stress ALTs assuming that the applied stress is expressed as a linear increasing function of time, $V(z) = kz$, $k > 0$. Nadarajah et al. [10] proposed progressive-stress ALTs supposing that the applied stress is expressed as a nonlinear increasing function of time, $V(z) = kz^a$, $k, a > 0$. The main goal of this section is to consider an exponentially increasing stress with time, $V(z) = e^{kz}$, $k > 0$. Here are some basic assumptions for applying the progressive-stress ALT.

Assumptions:

1. Under design stress, the lifetime of a unit follows the PLHLD with CDF (9).
2. The inverse power law controls the relationship between the scale parameter λ in CDF (9) and the applied stress V, i.e.,

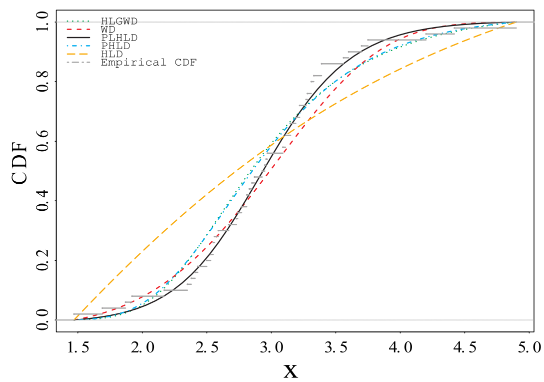

Fig. 7 Plots of fitted and empirical CDFs for the first data set

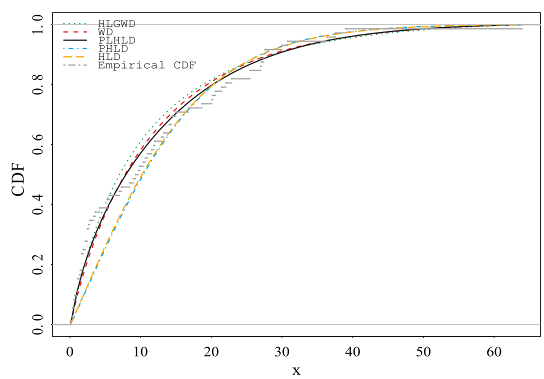

Fig. 8 Plots of fitted and empirical CDFs for the second data set

$$\lambda[z] = \lambda[V(z)] = \frac{1}{\mu\,[V(z)]^{\delta}},$$

where μ and δ are two positive parameters should be estimated.

3. The stress $V(z)$ is an increasing exponential function of time z with the form

$$V(z) = e^{kz}, \quad k > 0.$$

4. The testing process starts with dividing the N tested units into $\hbar(>1)$ groups each of them containing n_j units operating under progressive-stress ALT. Thus,

$$V_j(z) = e^{k_j z}, \quad j = 1, \ldots, \hbar, \quad k_1 < k_2 < \cdots < k_{\hbar}.$$

5. The lifetimes of units, $Z_{1j}, \ldots, Z_{n_j j}$ with realizations $z_{1j}, \ldots, z_{n_j j}$, $j = 1, \ldots, \hbar$, are iid RVs.
6. Under any stress level, the failure mechanisms of the units do not change.

Based on Assumptions 2 and 4, the cumulative exposure model, see Nelson [11], is given by

Table 1 Data analysis results for the first real data set

	PLHLD	PHLD	HLGWD	HLD	WD
ℓ	− 46.5507	− 50.0825	− 51.2945	− 95.0418	− 50.0752
-2ℓ	93.1014	100.1650	102.5890	190.0836	100.1503
AIC	99.1014	104.1650	108.5890	192.0836	104.1503
BIC	104.8375	107.9890	114.3251	193.9956	107.9743
CAIC	99.6232	104.4203	109.1108	192.1669	104.4056
HQIC	101.2858	105.6212	110.7734	192.8117	105.6065
K-S	0.0712	0.1141	0.1200	0.4432	0.1299
A-D	0.2208	1.0201	1.1767	13.8602	0.9453
CvM	0.0273	0.1609	0.1767	2.9365	0.1453
p-value (K-S)	0.9618	0.5336	0.4677	0.0000	0.3675
p-value (A-D)	0.9836	0.3465	0.2764	0.0000	0.3868
p-value (CvM)	0.9849	0.3589	0.3182	0.0000	0.4053
$\widehat{p}_1$	8.6078	42.1315	1.3734	0.5184	4.7833
$\widehat{p}_2$	2.7786	1.6689	55.6645	–	0.0038
$\widehat{p}_3$	1355.9139	–	0.9999	–	–
$SE(\widehat{p}_1)$	2.3069	16.5562	1.3374	0.0569	0.2501
$SE(\widehat{p}_2)$	0.5230	0.1597	99.4232	–	0.0011
$SE(\widehat{p}_3)$	2874.8663	–	0.4989	–	–
Numerical Methods	L-BFGS-B	L-BFGS-B	BFGS	CG	NM
Initial value for $\widehat{p}_1$	96.2858	12.0576	60.6673	46.2549	1.0779
Initial value for $\widehat{p}_2$	4.5543	0.2161	87.5668	–	3.6835
Initial value for $\widehat{p}_3$	79.4243	–	0.9679	–	–

$$\varpi_j(z) = \int_0^z \frac{1}{\lambda[V_j(u)]}\,\mathrm{d}u = \frac{\mu}{k_j\delta}\,(e^{k_j\delta z} - 1), \qquad j = 1,\ldots,\hbar. \tag{17}$$

Therefore, CDF (9) of PLHLD under progressive-stress ALT becomes

$$F_j(z) = G(\varpi_j(z)) = \frac{1}{e^{\theta}-1}\left[\left(\frac{1}{p}\;\mathbb{Q}_j(z)\right)^{\frac{\theta}{\ln(1/p)}} - 1\right], \tag{18}$$

where $G(.)$ is the supposed CDF with scale parameter value 1 and

$$\mathbb{Q}_j(z) = 1 - \frac{2(1-p)}{1+e^{\varpi_j(z)}}. \tag{19}$$

The corresponding PDF is given by

$$f_j(z) = \mathcal{C}(\theta,p,\mu)\;\frac{e^{k_j\delta z+\varpi_j(z)}}{(1+e^{\varpi_j(z)})^2}\;\left[\mathbb{Q}_j(z)\right]^{\frac{\theta}{\ln(1/p)}-1}, \tag{20}$$

where $\mathcal{C}(\theta,p,\mu)$, $\varpi_j(z)$ and $\mathbb{Q}_j(z)$ are given, respectively, by (10), (17) and (19).

Different methods of estimation

Under progressive-stress ALT, the adaptive type-II progressive hybrid censoring may be applied under the following assumptions:

Table 2 Data analysis results for the second real data set

	PLHLD	PHLD	HLGWD	HLD	WD
ℓ	− 250.1338	− 255.7655	− 252.0314	− 255.1664	− 251.4986
-2ℓ	500.2676	511.5310	504.0628	510.3328	502.9972
AIC	506.2675	515.5528	510.0627	512.3328	506.9973
BIC	513.0975	520.1062	516.8927	514.6095	511.5506
CAIC	506.6205	515.7267	510.4157	512.3900	507.1712
HQIC	508.9866	517.3655	512.7818	513.2392	508.8100
K-S	0.0870	0.2030	0.1019	0.1969	0.1054
A-D	0.5465	3.8577	0.9028	3.4854	0.8450
CvM	0.0944	0.5319	0.1683	0.4856	0.1488
p-value (K-S)	0.6464	0.0053	0.4427	0.0075	0.4004
p-value (A-D)	0.6994	0.0103	0.4121	0.0157	0.4492
p-value (CvM)	0.6146	0.0328	0.3389	0.0431	0.3939
$\widehat{p}_1$	1.6567	0.1681	0.0556	0.1082	0.9011
$\widehat{p}_2$	0.0796	0.1116	0.9727	–	0.1096
$\widehat{p}_3$	0.0261	–	0.9985	–	–
$SE(\widehat{p}_1)$	1.5216	0.5792	0.0678	0.0109	0.0855
$SE(\widehat{p}_2)$	0.0142	0.0156	0.4156	–	0.0301
$SE(\widehat{p}_3)$	0.0407	–	0.3020	–	–
Numerical Method	NM	CG	NM	CG	BFGS
Initial value for $\widehat{p}_1$	52.0855	0.8000	14.0013	2.6950	0.6077
Initial value for $\widehat{p}_2$	0.5705	0.8000	16.1198	–	37.2225
Initial value for $\widehat{p}_3$	42.8379	–	0.6379	–	–

1. Assume that n_j units are placed on a life testing experiment, $j = 1, ..., \hbar$.
2. Assume that the effective sample size $m_j (< n_j)$ is fixed before the experiment with progressive CS $(R_{1j}, R_{2j}, ..., R_{m_j j})$, $j = 1, ..., \hbar$.
3. Assume that the experimental time $\mathcal{T}_j$ is fixed before the experiment but we may permit the experiment to run over time $\mathcal{T}_j$. So, some of the R_{ij} values may be changed through the experiment, $j = 1, ..., \hbar$.
4. In group j, R_{1j} functioning units are randomly excluded from the experiment at the first failure time z_{1j}. R_{2j} functioning units are randomly excluded from the experiment at the second failure time z_{2j}. The experiment continues in the same way until the m_j-th failure $z_{m_j j}$ or time $\mathcal{T}_j$ whichever occurs first.
5. Case I: In group j, if the m_j-th failure time $z_{m_j j}$ occurs before time $\mathcal{T}_j$, then all the remaining functioning units $R_{m_j j} = n_j - m_j - \sum_{i=1}^{m_j - 1} R_{ij}$ are excluded from the experiment, thereby finishing the experiment at $z_{m_j j}$, see Fig. 9.
6. Case II: In group j, if the experimental time $\mathcal{T}_j$ is reached before occurring the m_j-th failure time $z_{m_j j}$ and only D_j failures occur before time $\mathcal{T}_j$. Then, we will not exclude any functioning unit from the experiment immediately following the $(D_j + 1)$-th ...$(m_j - 1)$-th failure time and exclude the remaining functioning units $R^*_{m_j j} = n_j - m_j - \sum_{i=1}^{D_j} R_{ij}$ from the experiment, thereby finishing the experiment at $z_{m_j j}$. That is, $R_{D_j+1 j} = \cdots = R_{m_j - 1 j} = 0$, see Fig. 9.

The data obtained from adaptive type-II progressive hybrid censoring are presented in the following two cases:

- Case I: $(z_{1j}; R_{1j}), \ldots, (z_{m_j j}; R_{m_j j})$, if $z_{m_j j} < \mathcal{T}_j$.
- Case II: $(z_{1j}; R_{1j}), \ldots, (z_{D_j j}; R_{D_j j}), (z_{D_j+1j}; 0), \ldots, (z_{m_j-1j}; 0), (z_{m_j j}; R^*_{m_j j})$, if $z_{m_j j} > \mathcal{T}_j$,

where $z_{1j} < \cdots < z_{m_j j}$ denote the m_j ordered observed failure times in group j and $R_{1j}, \ldots, R_{m_j j}$ (or $R_{1j}, \ldots, R_{D_j j}, 0, \ldots, 0, R^*_{m_j j}$) denote the number of units excluded from the experiment at failure times $z_{1j}, \ldots, z_{m_j j}$ (or $z_{1j}, \ldots, z_{D_j j}, \ldots, z_{m_j j}$). Notice that the adaptive type-II progressive hybrid censoring reduces to the progressive type-II censoring, as $\mathcal{T}_j \to \infty$, and reduces to type-II censoring, as $\mathcal{T}_j = 0$.

In the following subsections, based on adaptive type-II progressive hybrid censoring, we discuss four methods of estimation to estimate the parameters θ, p, μ and δ.

Maximum likelihood estimation

For $D_j = d_j$, $j = 1, \ldots, \hbar$, and based on adaptive type-II progressive hybrid censoring under progressive-stress ALT, the likelihood function is then given by

$$L(\theta, p, \mu, \delta; \mathbf{z}) \propto \prod_{j=1}^{\hbar} \left(\left[\prod_{i=1}^{m_j} f(z_{ij}) \right] \left[\prod_{i=1}^{d_j} [1 - F(z_{ij})]^{R_{ij}} \right] [1 - F(z_{m_j j})]^{R^*_{m_j j}} \right),$$

where $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_\hbar)$, $\mathbf{z}_j = (z_{1j}, \ldots, z_{m_j j})$, $R^*_{m_j j} = n_j - m_j - \sum_{i=1}^{d_j} R_{ij}$ and $d_j = 0, 1, \ldots, m_j$.

Using Eqs. (18) and (20), the log-likelihood function takes the form

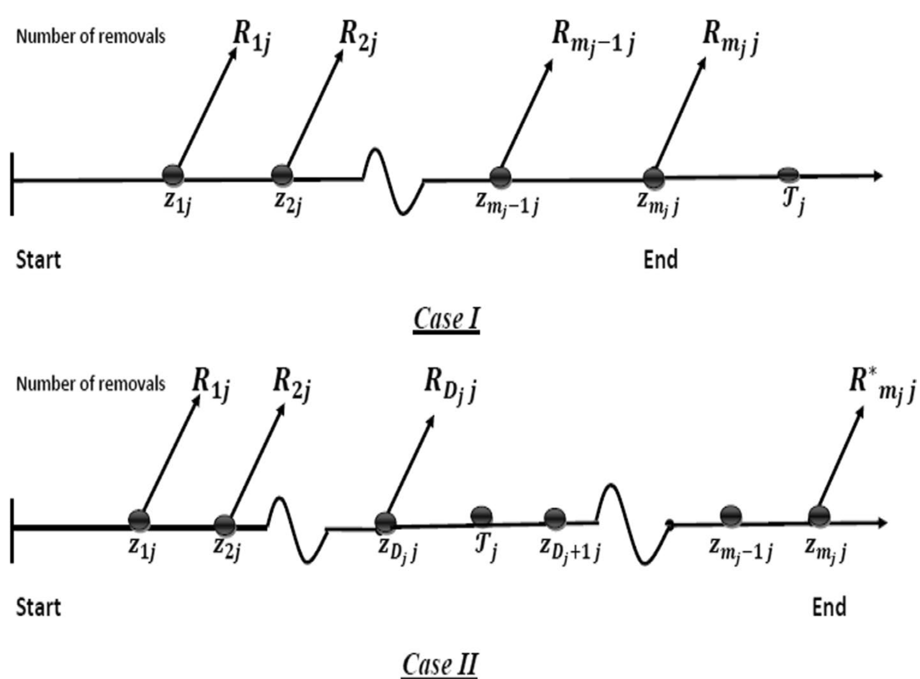

Fig. 9 Generation process of order statistics under adaptive type-II progressive hybrid censoring

$$
\begin{aligned}
£ &= \ln(L(\theta, p, \mu, \delta; \mathbf{z})) \\
&\propto \left(\sum_{j=1}^{\hbar} m_j\right) \ln\left(\mathcal{C}(\theta, p, \mu)\right) + \sum_{j=1}^{\hbar} \sum_{i=1}^{m_j} \left[k_j \delta z_{ij} + \varpi_j(z_{ij}) - 2\ln\left(1 + e^{\varpi_j(z_{ij})}\right)\right. \\
&\quad \left. + \left(\frac{\theta}{\ln(1/p)} - 1\right) \ln\left(\mathbb{Q}_j(z_{ij})\right)\right] + \sum_{j=1}^{\hbar} \sum_{i=1}^{d_j} R_{ij} \ln\left(e^{\theta} - \left(\frac{1}{p}\left[\mathbb{Q}_j(z_{ij})\right]\right)^{\frac{\theta}{\ln(1/p)}}\right) \\
&\quad + \sum_{j=1}^{\hbar} R^*_{m_j j} \ln\left(e^{\theta} - \left(\frac{1}{p}\left[\mathbb{Q}_j(z_{m_j j})\right]\right)^{\frac{\theta}{\ln(1/p)}}\right) - \ln\left(e^{\theta} - 1\right) \sum_{j=1}^{\hbar} (n_j - m_j).
\end{aligned}
$$

The MLEs $\hat{\theta}, \hat{p}, \hat{\mu}$ and $\hat{\delta}$ of θ, p, μ and δ could be obtained by solving simultaneously the likelihood equations, $\frac{\partial £}{\partial \alpha_r} = 0$, $r = 1, 2, 3, 4$, with respect to α_r, where $(\alpha_1 = \theta, \alpha_2 = p, \alpha_3 = \mu, \alpha_4 = \delta)$. These MLEs cannot be obtained in closed forms, so a numerical iteration technique to solve the likelihood equations should be applied.

Percentile estimation

Kao [31] introduced the percentile estimation to estimate the unknown parameters. If the data are obtained from a closed form of a CDF, then it is just normal to estimate the parameters by fitting a straight line to the theoretical points obtained by the CDF and the sample percentile points. The empirical CDF used in this method may be written as

$$
\widehat{F}_j(z_{ij}) = 1 - \prod_{s=1}^{i} (1 - \widehat{\mathbf{q}}_{sj}), \qquad i = 1, \ldots, m_j, \quad j = 1, \ldots, \hbar,
$$

where

$$
\widehat{\mathbf{q}}_{sj} = \frac{1}{n_j - \left[\sum_{k=1}^{\min(s-1, d_j)} R_{kj}\right] - s + 1}, \qquad s = 1, \ldots, m_j, \quad j = 1, \ldots, \hbar.
$$

The percentile estimates (PEs) $\check{\theta}, \check{p}, \check{\mu}$ and $\check{\delta}$ of θ, p, μ and δ can be obtained by minimizing the following quantity with respect to θ, p, μ and δ

$$
\mathbf{\Omega} \equiv \mathbf{\Omega}(\boldsymbol{\alpha}; \mathbf{z}) = \sum_{j=1}^{\hbar} \sum_{i=1}^{m_j} \left[z_{ij} - \frac{1}{k_j \delta} \ln\left(\frac{k_j \delta}{\mu} \ln\left(\frac{2(1-p)}{1 - p[(e^{\theta} - 1)\varrho_{ij} + 1]^{\frac{\ln(1/p)}{\theta}}} - 1\right) + 1\right)\right]^2,
$$

where

$$
\varrho_{ij} = \frac{\widehat{F}_j(z_{i-1j}) + \widehat{F}_j(z_{ij})}{2}.
$$

Minimization of the quantity $\mathbf{\Omega}$ could be obtained by solving the equations $\frac{\partial \mathbf{\Omega}}{\partial \alpha_r} = 0$ with respect to α_r, $r = 1, 2, 3, 4$.

Least squares and weighted least squares estimations

The least squares and weighted least squares estimation methods are considered by Swain et al. [32] to estimate the unknown parameters of Beta distribution. Based on progressive type-II censoring, Abdel-Hamid and Hashem [8] used these two methods to estimate the parameters included in the doubly Poisson–exponential distribution.

Aggarwala and Balakrishnan [33] obtained the expectation and variance of the empirical CDF $\widehat{F}(.)$ under progressive type-II censoring. Their procedure may be modified to adaptive type-II progressive hybrid censoring as follows: Let $(Z_{1j},\ldots,Z_{m_jj})$, $j=1,\ldots,\hbar$, be the ordered sample of size m_j from the PLHLD with CDF (18) under adaptive type-II progressive hybrid censoring. Thus, the expectation and variance of the empirical CDF $\widehat{F}_j(.)$, $j=1,\ldots,\hbar$, are given, respectively, by

$$E[\widehat{F}_j(z_{ij})]=1-\prod_{l=m_j-i+1}^{m_j}W_{lj},\qquad i=1,\ldots,m_j,\quad j=1,\ldots,\hbar,$$
$$V[\widehat{F}_j(z_{ij})]=\left(\prod_{l=m_j-i+1}^{m_j}W_{lj}\right)\left(\prod_{l=m_j-i+1}^{m_j}H_{lj}-\prod_{l=m_j-i+1}^{m_j}W_{lj}\right),\quad i=1,\ldots,m_j,\quad j=1,\ldots,\hbar,$$

where

$$W_{lj}=\frac{U_{lj}}{1+U_{lj}},\qquad l=1,\ldots,m_j,\quad j=1,\ldots,\hbar,$$
$$O_{lj}=\frac{1}{(U_{lj}+1)(U_{lj}+2)},\qquad l=1,\ldots,m_j,\quad j=1,\ldots,\hbar,$$
$$U_{lj}=\begin{cases} l+\sum_{s=m_j-l+1}^{m_j}R_{sj}, & m_j<d_j,\\ l+R^*_{m_jj}+\sum_{s=m_j-l+1}^{d_j}R_{sj}, & m_j-l+1<d_j<m_j,\\ l+R^*_{m_jj}, & d_j<m_j-l+1<m_j,\end{cases}$$
$$R^*_{m_jj}=n_j-m_j-\sum_{i=1}^{d_j}R_{ij},\qquad j=1,\ldots,\hbar,$$
$$H_{lj}=W_{lj}+O_{lj},\qquad l=1,\ldots,m_j,\quad j=1,\ldots,\hbar.$$

The least squares estimates (LSEs) $\tilde{\theta},\tilde{p},\tilde{\mu}$ and $\tilde{\delta}$ of θ,p,μ and δ could be determined by minimizing the next quantity with respect to θ,p,μ and δ

$$\mathbb{S}^*\equiv\mathbb{S}^*(\boldsymbol{\alpha};\mathbf{z})=\sum_{j=1}^{\hbar}\sum_{i=1}^{m_j}\left[F_j(z_{ij})-E[\widehat{F}_j(z_{ij})]\right]^2.$$

The weighted LSEs (WLSEs) $\ddot{\theta},\ddot{p},\ddot{\mu}$ and $\ddot{\delta}$ of θ,p,μ and δ could be determined by minimizing the next quantity with respect to θ,p,μ and δ

$$\mathbb{S}^{**}\equiv\mathbb{S}^{**}(\boldsymbol{\alpha};\mathbf{z})=\sum_{j=1}^{\hbar}\sum_{i=1}^{m_j}\varpi_j\left(F_j(z_{ij})-E[\widehat{F}_j(z_{ij})]\right)^2,$$

where ϖ_j is the weight factor given by

$$\varpi_j=\frac{1}{V[\widehat{F}_j(z_{ij})]}.$$

Minimizations of the two quantities $\mathbb{S}^*$ and $\mathbb{S}^{**}$ could be determined by solving the equations $\frac{\partial \mathbb{S}^*}{\partial \alpha_r} = 0$ and $\frac{\partial \mathbb{S}^{**}}{\partial \alpha_r} = 0$ with respect to $\alpha_r, r = 1, 2, 3, 4$.

Simulation study

The performance of the four estimation methods cannot be compared theoretically. Consequently, a Monte Carlo simulation study is applied to compare the proposed methods. In this section, the MLE, PE, LSE, and WLSE of the parameters θ, p, μ and δ are calculated in order to compare the performance of these methods via Monte Carlo simulation according to the following algorithm:

Algorithm:

1. Assign the values of $n_j, m_j (1 < m_j < n_j), \mathcal{T}_j$ and $(R_{1j}, \ldots, R_{m_j j}), j = 1, \ldots, \hbar$.
2. For given values of the parameters (θ, p, μ, δ) and values of the stress rates $k_j, j = 1, \ldots, \hbar$, generate an adaptive type-II progressively hybrid censored sample of size m_j $(z_{1j}, \ldots, z_{m_j j})$ from PLHLD with CDF (18) based on the method introduced in Ng et al. [22].
3. The MLE, PE, LSE, and WLSE of the parameters θ, p, μ and δ are calculated as shown in the "Different methods of estimation" section.
4. Repeat the above steps $\mathcal{K} (= 1,000)$ times.
5. Evaluate the average of estimates, relative absolute biases (RABs) and mean squared errors (MSEs) of $\hat{\xi}$ over $\mathcal{K}$ samples as follows:

$$\bar{\hat{\xi}} = \frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} \hat{\xi}_i, \quad \text{RAB}(\hat{\xi}) = \frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} \frac{|\hat{\xi}_i - \xi|}{\xi}, \quad \text{MSE}(\hat{\xi}) = \frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} (\hat{\xi}_i - \xi)^2,$$

   where $\hat{\xi}$ is an estimate of ξ.
6. Evaluate the average of estimates of the parameters θ, p, μ and δ with their RABs and MSEs as shown in Step 5.
7. Evaluate the average of the RABs (ARAB) and the average of the MSEs (AMSE).

The following four CSs are considered in the generation of samples:

- CS1: For $j = 1, \ldots, \hbar$

$$\begin{aligned} R_{ij} &= 1, \qquad i = 1, \ldots, n_j - m_j, \\ R_{ij} &= 0, \qquad \text{otherwise.} \end{aligned}$$

- CS2: For $j = 1, \ldots, \hbar$

$$\begin{aligned} R_{ij} &= 1, \qquad i = 2m_j - n_j + 1, \ldots, m_j, \\ R_{ij} &= 0, \qquad \text{otherwise.} \end{aligned}$$

- CS3: For $j = 1, \ldots, \hbar$

$$
\begin{aligned}
&R_{ij} = \left[\frac{n_j - m_j}{2}\right], \qquad i = 1,\\
&R_{ij} = n - m - \left[\frac{n_j - m_j}{2}\right], \qquad i = \frac{m_j}{2} \quad \text{if } m_j \text{ is even} \quad \text{or} \quad i = \frac{m_j + 1}{2} \quad \text{if } m_j \text{ is odd},\\
&R_{ij} = 0, \qquad \text{otherwise},
\end{aligned}
$$

where $[\nu]$ indicates the greatest integer value less than or equal ν.

- CS4: For $j = 1, \dots, \hbar$

$$
\begin{aligned}
&R_{ij} = n_j - m_j \qquad i = 1,\\
&R_{ij} = 0, \qquad \text{otherwise}.
\end{aligned}
$$

Results and discussion

The results due to the simulation study are shown in Tables 3 and 4 considering the following values for the population parameters: $\theta = 1.5$, $p = 0.9$, $\mu = 0.7$ and $\delta = 0.3$. For comparison among the MLEs, PEs, LSEs, and WLSEs, the following values have been taken into consideration:

$$
\begin{aligned}
&\hbar = 2, \quad \text{and} \qquad n_1 = n_2 = N/2,\\
&m_1 = m_2 = 60\%,\ 80\% \text{ and } 100\% \text{ of the sample size},\\
&k_1 = 0.3 \text{ and } k_2 = 0.5,\\
&(\mathcal{T}_1 = 2.0 \text{ and } \mathcal{T}_2 = 1.5) \ \text{ or } \ (\mathcal{T}_1 = 4.0 \text{ and } \mathcal{T}_2 = 3.5).
\end{aligned}
$$

From Tables 3 and 4, we observe:

1. The LSEs are the best estimates through the AMSEs and ARABs.
2. The WLSEs are better than the MLEs and PEs through the AMSEs and ARABs.
3. The MLEs are better than the PEs through the AMSEs and ARABs.
4. For fixed $\mathcal{T}_j$ and n_j, by increasing m_j, $j = 1, \dots, \hbar$, the MSEs and RABs decrease.
5. For fixed $\mathcal{T}_j$ and m_j (= 60%, 80% and 100% of the sample size), by increasing n_j, $j = 1, \dots, \hbar$, the MSEs and RABs decrease.
6. For fixed n_j and m_j by increasing $\mathcal{T}_j, j = 1, \dots, \hbar$, the MSEs and RABs decrease for the MLEs and PEs, while the MSEs and RABs increase for the LSEs and WLSEs.
7. For fixed n_j and m_j by increasing $\mathcal{T}_j, j = 1, \dots, \hbar$, the $\overline{d_j}$ increases, where $\overline{d_j}$ is the average number of observed failure up to time $\mathcal{T}_j$.

The above results are true except for some rare states, and this may be due to data fluctuation.

Concluding remarks

In this article, we have proposed a new lifetime distribution, named PLHLD, which has been derived by compounding zero-truncated Poisson and logarithmic distributions with HLD based on parallel–series system's failures. We have discussed some statistical properties of the PLHLD, including the PDF, shapes of the HRF, q-th quantile, moments, mean, variance, kurtosis, skewness, mean residual lifetime, PDF and CDF of the i-th

Table 3 MLEs and PEs of θ, p, μ and δ with their MSEs, RABs, AMSE, and ARAB based on 1000 simulations. Population parameter values are $\theta = 1.5$, $p = 0.9$, $\mu = 0.7$ and $\delta = 0.3$

						MLE				**PE**				
						$\bar{\hat{\theta}}$	**MSE**($\hat{\theta}$)	**RAB**($\hat{\theta}$)		$\bar{\check{\theta}}$	**MSE**($\check{\theta}$)	**RAB**($\check{\theta}$)		
						$\bar{\hat{p}}$	**MSE**($\hat{p}$)	**RAB**($\hat{p}$)		$\bar{\check{p}}$	**MSE**($\check{p}$)	**RAB**($\check{p}$)		
		n_1	m_1	τ_1		$\bar{\hat{\mu}}$	**MSE**($\hat{\mu}$)	**RAB**($\hat{\mu}$)	**AMSE**	$\bar{\check{\mu}}$	**MSE**($\check{\mu}$)	**RAB**($\check{\mu}$)	**AMSE**	$\overline{d_1}$
$\hbar$	***N***	n_2	m_2	τ_2	**CS**	$\bar{\hat{\delta}}$	**MSE**($\hat{\delta}$)	**RAB**($\hat{\delta}$)	**ARAB**	$\bar{\check{\delta}}$	**MSE**($\check{\delta}$)	**RAB**($\check{\delta}$)	**ARAB**	$\overline{d_2}$
2	40	20	12	2.0	1	1.6730	1.3543	0.6052	0.8181	1.9700	1.9392	0.7162	0.8777	4
		20	12	1.5		1.2687	1.6291	0.9871	0.6452	1.2806	1.1538	0.8549	0.6884	3
						0.9020	0.2620	0.5551		0.8620	0.3790	0.6633		
						0.2882	0.0272	0.4333		0.3258	0.0390	0.5194		
					2	1.6311	1.4646	0.6173	0.8457	1.9318	1.8072	0.7000	0.8168	5
						1.2757	1.5808	0.9732	0.6624	1.2724	0.9941	0.7553	0.6709	4
						0.9426	0.3096	0.6122		0.8665	0.4271	0.7028		
						0.2836	0.0277	0.4467		0.3310	0.0389	0.5254		
					3	1.6712	1.4729	0.6305	0.8978	1.9398	1.9019	0.7180	0.8884	4
						1.2952	1.8535	1.0302	0.6584	1.3529	1.3099	0.8947	0.6856	3
						0.8792	0.2380	0.5396		0.8182	0.3014	0.5965		
						0.2931	0.0268	0.4334		0.3390	0.0405	0.5332		
					4	1.7065	1.7162	0.6601	0.8627	1.9678	2.0097	0.7290	1.0619	4
						1.1531	1.5287	0.9602	0.6203	1.4402	1.9704	1.1555	0.7415	3
						0.8705	0.1812	0.4505		0.8002	0.2200	0.5208		
						0.2803	0.0248	0.4102		0.3393	0.0474	0.5608		
				4.0	1	1.6928	1.5273	0.6463	0.8407	1.8039	1.8936	0.7118	1.0184	9
				3.5		1.2119	1.5853	0.9712	0.6414	1.4490	1.9375	1.1282	0.7130	10
						0.8849	0.2200	0.5039		0.7927	0.2014	0.4925		
						0.2899	0.0302	0.4440		0.3300	0.0413	0.5195		
					2	1.6372	1.4802	0.6265	0.8358	1.8762	1.6516	0.6602	0.7989	10
						1.2821	1.6117	0.9817	0.6434	1.3216	1.2642	0.8760	0.6426	11
						0.8955	0.2238	0.5304		0.7598	0.2426	0.5389		
						0.2881	0.0275	0.4350		0.3390	0.0373	0.4955		
					3	1.7198	1.7149	0.6779	0.8360	1.8362	1.6565	0.6687	0.9691	9
						1.1298	1.3961	0.9223	0.6290	1.4399	1.9561	1.1345	0.7129	9
						0.8799	0.2049	0.4838		0.8258	0.2219	0.5198		
						0.2869	0.0280	0.4318		0.3272	0.0420	0.5285		
					4	1.7301	1.7304	0.6755	0.8908	1.9738	2.0424	0.7448	1.0624	8
						1.1796	1.6190	0.9880	0.6380	1.4244	1.9400	1.1306	0.7359	8
						0.8658	0.1864	0.4581		0.8202	0.2202	0.5164		
						0.2870	0.0274	0.4304		0.3356	0.0471	0.5516		

Table 3 (continued)

						MLE				PE				
						$\overline{\hat{\theta}}$	MSE($\hat{\theta}$)	RAB($\hat{\theta}$)		$\overline{\check{\theta}}$	MSE($\check{\theta}$)	RAB($\check{\theta}$)		
						$\overline{\hat{p}}$	MSE($\hat{p}$)	RAB($\hat{p}$)		$\overline{\check{p}}$	MSE($\check{p}$)	RAB($\check{p}$)		
		n_1	m_1	τ_1		$\overline{\hat{\mu}}$	MSE($\hat{\mu}$)	RAB($\hat{\mu}$)	AMSE	$\overline{\check{\mu}}$	MSE($\check{\mu}$)	RAB($\check{\mu}$)	AMSE	$\overline{d_1}$
$\hbar$	**N**	n_2	m_2	τ_2	CS	$\overline{\hat{\delta}}$	MSE($\hat{\delta}$)	RAB($\hat{\delta}$)	ARAB	$\overline{\check{\delta}}$	MSE($\check{\delta}$)	RAB($\check{\delta}$)	ARAB	$\overline{d_2}$
			16	2.0	1	1.7740	1.7492	0.6753	0.8599	1.9366	2.0243	0.7324	0.9353	4
			16	1.5		1.1741	1.5386	0.9603	0.6105	1.3195	1.4664	0.9845	0.6890	3
						0.8283	0.1289	0.4028		0.7822	0.2095	0.5107		
						0.2914	0.0230	0.4035		0.3378	0.0408	0.5283		
					2	1.7122	1.4990	0.6293	0.8688	2.0237	2.0485	0.7289	0.8426	5
						1.2894	1.7734	1.0304	0.6401	1.2392	1.0784	0.8190	0.6398	4
						0.8381	0.1753	0.4704		0.7388	0.2053	0.5075		
						0.2981	0.0274	0.4305		0.3496	0.0383	0.5039		
					3	1.7405	1.7201	0.6719	0.8748	1.9591	1.9860	0.7076	0.8724	4
						1.2320	1.5721	0.9839	0.6365	1.3258	1.2692	0.9021	0.6542	4
						0.8519	0.1810	0.4657		0.7372	0.1940	0.4924		
						0.2930	0.0260	0.4245		0.3516	0.0403	0.5145		
					4	1.7411	1.7510	0.6781	0.8804	1.9956	1.9497	0.7186	1.0211	4
						1.1861	1.5967	0.9806	0.6223	1.4485	1.9413	1.1363	0.7061	3
						0.8401	0.1489	0.4153		0.7410	0.1502	0.4378		
						0.2895	0.0250	0.4150		0.3543	0.0431	0.5318		
				4.0	1	1.6991	1.6230	0.6567	0.8130	1.8916	1.9040	0.7084	0.9759	10
				3.5		1.1513	1.4709	0.9615	0.6023	1.4230	1.7970	1.0925	0.6865	11
						0.8449	0.1364	0.4048		0.7798	0.1643	0.4473		
						0.2815	0.0216	0.3863		0.3351	0.0384	0.4978		
					2	1.7754	1.7039	0.6628	0.9496	1.9611	2.0128	0.7325	0.8514	12
						1.3530	1.9060	1.0952	0.6616	1.2904	1.1442	0.8519	0.6527	13
						0.8120	0.1598	0.4491		0.7649	0.2099	0.5166		
						0.3097	0.0288	0.4393		0.3427	0.0389	0.5097		
					3	1.7685	1.5754	0.6348	0.8235	1.9091	1.9200	0.7227	0.9759	11
						1.2176	1.5609	0.9753	0.6024	1.4201	1.7721	1.0651	0.6933	11
						0.8096	0.1342	0.3961		0.7712	0.1684	0.4565		
						0.2996	0.0236	0.4035		0.3438	0.0433	0.5288		

Table 3 (continued)

						MLE				**PE**				
						$\overline{\hat{\theta}}$	**MSE($\hat{\theta}$)**	**RAB($\hat{\theta}$)**		$\overline{\check{\theta}}$	**MSE($\check{\theta}$)**	**RAB($\check{\theta}$)**		
						$\overline{\hat{p}}$	**MSE($\hat{p}$)**	**RAB($\hat{p}$)**		$\overline{\check{p}}$	**MSE($\check{p}$)**	**RAB($\check{p}$)**		
		n_1	m_1	τ_1		$\overline{\hat{\mu}}$	**MSE($\hat{\mu}$)**	**RAB($\hat{\mu}$)**	**AMSE**	$\overline{\check{\mu}}$	**MSE($\check{\mu}$)**	**RAB($\check{\mu}$)**	**AMSE**	$\overline{d_1}$
$\hbar$	**N**	n_2	m_2	τ_2	**CS**	$\overline{\hat{\delta}}$	**MSE($\hat{\delta}$)**	**RAB($\hat{\delta}$)**	**ARAB**	$\overline{\check{\delta}}$	**MSE($\check{\delta}$)**	**RAB($\check{\delta}$)**	**ARAB**	$\overline{d_2}$
					4	1.8024	1.8721	0.6942	0.8770	1.9223	1.8233	0.7006	0.9365	10
						1.1156	1.4602	0.9356	0.6143	1.3787	1.7363	1.0638	0.6725	11
						0.8457	0.1513	0.4150		0.7579	0.1476	0.4264		
						0.2858	0.0244	0.4123		0.3400	0.0390	0.4993		
			20	–	–	1.7299	1.6335	0.6462	0.8400	1.8386	1.8867	0.7036	0.9090	–
			20			1.2046	1.5875	0.9791	0.5963	1.3425	1.5659	0.9996	0.6548	–
						0.8167	0.1175	0.3741		0.7699	0.1476	0.4262		
						0.2918	0.0215	0.3858		0.3323	0.0358	0.4899		
2	60	30	18	2.0	1	1.6728	1.3786	0.6020	0.8709	1.7983	1.4493	0.6256	0.6511	6
		30	18	1.5		1.3715	1.9030	1.0647	0.6416	1.2409	0.9017	0.7216	0.5879	5
						0.8187	0.1741	0.4675		0.7448	0.2192	0.5241		
						0.3086	0.0280	0.4324		0.3433	0.0340	0.4804		
					2	1.6123	1.2288	0.5665	0.8167	1.8619	1.4103	0.6276	0.6580	7
						1.4298	1.8067	1.0597	0.6421	1.2308	0.9283	0.7017	0.5865	5
						0.8341	0.2025	0.5068		0.7788	0.2607	0.5510		
						0.3122	0.0287	0.4354		0.3377	0.0328	0.4657		
					3	1.6468	1.4402	0.6099	0.8593	1.8520	1.6317	0.6668	0.7227	6
						1.3810	1.8079	1.0692	0.6380	1.2709	1.0052	0.7755	0.614	4
						0.8128	0.1641	0.4576		0.7512	0.2168	0.5194		
						0.3063	0.0251	0.4155		0.3437	0.0369	0.4941		
					4	1.6504	1.4739	0.6327	0.7964	1.9315	1.8738	0.7011	0.9253	5
						1.2069	1.5570	0.9844	0.5969	1.4008	1.6418	1.0536	0.6681	4
						0.8347	0.1334	0.3910		0.7475	0.1475	0.4167		
						0.2838	0.0214	0.3794		0.3449	0.0379	0.5011		
				4.0	1	1.6366	1.4533	0.6200	0.7896	1.8192	1.5981	0.6567	0.8564	13
				3.5		1.2730	1.5443	0.9683	0.5942	1.3970	1.6377	1.0218	0.6470	14
						0.8143	0.1381	0.4052		0.7608	0.1539	0.4328		
						0.3007	0.0225	0.3832		0.3358	0.0359	0.4768		

Table 3 (continued)

						MLE				PE				
						$\overline{\hat{\theta}}$	MSE($\hat{\theta}$)	RAB($\hat{\theta}$)		$\overline{\check{\theta}}$	MSE($\check{\theta}$)	RAB($\check{\theta}$)		
						$\overline{\hat{p}}$	MSE($\hat{p}$)	RAB($\hat{p}$)		$\overline{\check{p}}$	MSE($\check{p}$)	RAB($\check{p}$)		
		n_1	m_1	τ_1		$\overline{\hat{\mu}}$	MSE($\hat{\mu}$)	RAB($\hat{\mu}$)	AMSE	$\overline{\check{\mu}}$	MSE($\check{\mu}$)	RAB($\check{\mu}$)	AMSE	$\overline{d_1}$
$\hbar$	**N**	n_2	m_2	τ_2	**CS**	$\overline{\hat{\delta}}$	MSE($\hat{\delta}$)	RAB($\hat{\delta}$)	ARAB	$\overline{\check{\delta}}$	MSE($\check{\delta}$)	RAB($\check{\delta}$)	ARAB	$\overline{d_2}$
					2	1.6537	1.3527	0.6009	0.7755	1.8108	1.5108	0.6383	0.7006	15
						1.3281	1.5589	0.9806	0.6111	1.3009	1.0742	0.7997	0.5964	16
						0.8141	0.1650	0.4519		0.6963	0.1829	0.4734		
						0.3087	0.0256	0.4111		0.3519	0.0345	0.4741		
					3	1.6658	1.5330	0.6394	0.8148	1.8197	1.6630	0.6664	0.8904	13
						1.2249	1.5567	0.9816	0.6027	1.4373	1.7196	1.0615	0.6627	13
						0.8394	0.1481	0.4124		0.7447	0.1411	0.4215		
						0.2914	0.0212	0.3776		0.3438	0.0379	0.5013		
					4	1.7420	1.4577	0.6233	0.7935	2.0186	2.0407	0.7311	1.0060	11
						1.2581	1.5853	1.0158	0.5984	1.3924	1.7977	1.0800	0.6861	12
						0.7858	0.1071	0.3567		0.7473	0.1466	0.4258		
						0.3072	0.0238	0.3979		0.3470	0.0390	0.5076		
			24	2.0	1	1.6811	1.4593	0.6177	0.8396	1.8312	1.5713	0.6398	0.8140	6
			24	1.5		1.2960	1.7597	1.0449	0.6105	1.3955	1.5108	0.9905	0.6329	5
						0.7948	0.1153	0.3785		0.7145	0.1362	0.4124		
						0.3032	0.0240	0.4011		0.3534	0.0378	0.4888		
					2	1.6710	1.4174	0.6147	0.8031	1.8316	1.4141	0.6202	0.6162	7
						1.3185	1.6533	1.0194	0.6028	1.2030	0.8599	0.7410	0.5656	5
						0.7853	0.1190	0.3870		0.7197	0.1597	0.4483		
						0.3065	0.0228	0.3902		0.3427	0.0309	0.4530		
					3	1.6895	1.5036	0.6309	0.8307	1.8869	1.6566	0.6704	0.7610	6
						1.3103	1.6813	1.0314	0.6069	1.3089	1.1992	0.8704	0.6161	5
						0.7855	0.1161	0.3787		0.7170	0.1526	0.4427		
						0.3060	0.0220	0.3865		0.3514	0.0355	0.4811		
					4	1.7448	1.5816	0.6397	0.7986	1.8661	1.7059	0.6816	0.8623	6
						1.1782	1.4921	0.9625	0.5765	1.3616	1.5881	1.0120	0.6378	5
						0.7968	0.1018	0.3428		0.7430	0.1210	0.3873		
						0.2941	0.0190	0.3609		0.3383	0.0341	0.4701		

Table 3 (continued)

						MLE				PE				
						$\overline{\hat{\theta}}$	MSE($\hat{\theta}$)	RAB($\hat{\theta}$)		$\overline{\check{\theta}}$	MSE($\check{\theta}$)	RAB($\check{\theta}$)		
						$\overline{\hat{p}}$	MSE($\hat{p}$)	RAB($\hat{p}$)		$\overline{\check{p}}$	MSE($\check{p}$)	RAB($\check{p}$)		
		n_1	m_1	τ_1		$\overline{\hat{\mu}}$	MSE($\hat{\mu}$)	RAB($\hat{\mu}$)	AMSE	$\overline{\check{\mu}}$	MSE($\check{\mu}$)	RAB($\check{\mu}$)	AMSE	$\overline{d_1}$
$\hbar$	**N**	n_2	m_2	τ_2	CS	$\overline{\hat{\delta}}$	MSE($\hat{\delta}$)	RAB($\hat{\delta}$)	ARAB	$\overline{\check{\delta}}$	MSE($\check{\delta}$)	RAB($\check{\delta}$)	ARAB	$\overline{d_2}$
				4.0	1	1.7346	1.5235	0.6292	0.8181	1.8295	1.5036	0.6429	0.8263	15
				3.5		1.2895	1.6386	1.0045	0.5827	1.4068	1.6608	1.0560	0.6319	16
						0.7732	0.0908	0.3342		0.7265	0.1097	0.3752		
						0.3060	0.0195	0.3629		0.3415	0.0312	0.4534		
					2	1.7481	1.6436	0.6558	0.8600	1.8407	1.5361	0.6418	0.6987	18
						1.3444	1.6573	1.0315	0.6167	1.2780	1.0618	0.8052	0.5925	19
						0.7743	0.1144	0.3811		0.7111	0.1636	0.4532		
						0.3151	0.0245	0.3984		0.3490	0.0333	0.4697		
					3	1.8187	1.7826	0.6679	0.8806	1.8190	1.6746	0.6677	0.8485	16
						1.2442	1.6276	0.9909	0.5929	1.3730	1.5593	1.0010	0.6328	17
						0.7739	0.0921	0.3384		0.7482	0.1268	0.3988		
						0.3056	0.0200	0.3743		0.3354	0.0334	0.4638		
					4	1.7345	1.5845	0.6395	0.8099	1.8163	1.5654	0.6473	0.8374	15
						1.2444	1.5417	0.9765	0.5802	1.3890	1.6319	1.0282	0.6316	16
						0.7838	0.0933	0.3384		0.7303	0.1183	0.3838		
						0.3018	0.0201	0.3664		0.3432	0.0341	0.4671		
			30	–	–	1.7431	1.6372	0.6556	0.8263	1.8132	1.5076	0.6381	0.7707	–
			30			1.2965	1.5735	0.9866	0.5715	1.3232	1.4382	0.9650	0.6011	–
						0.7604	0.0775	0.3054		0.7347	0.1084	0.3652		
						0.3068	0.0168	0.3382		0.3348	0.0287	0.4363		

order statistic, Lorenz and Bonferroni curves, and Shannon's and Rényi entropies. Two real data sets have been considered to compare among PLHLD, PHLD, HLGWD, HLD, and WD. The comparison shows that the PLHLD is better to fit the considered data than the other four distributions. The progressive-stress ALT with an increasing exponential function of time has been applied when the lifetime of a unit under use stress follows the PLHLD. Based on adaptive type-II progressive hybrid censoring, some estimation methods, such as maximum likelihood, percentile, least squares, and weighted least squares estimations, have been discussed to estimate the parameters involved in the PLHLD under progressive-stress ALT. Based on four different progressive CSs, a simulation

Table 4 LSEs and WLSEs of θ, p, μ and δ with their MSEs, RABs, AMSE, and ARAB based on 1000 simulations. Population parameter values are $\theta = 1.5$, $p = 0.9$, $\mu = 0.7$ and $\delta = 0.3$

						LSE				**WLSE**			
						$\bar{\tilde{\theta}}$	**MSE**($\tilde{\theta}$)	**RAB**($\tilde{\theta}$)		$\bar{\ddot{\theta}}$	**MSE**($\ddot{\theta}$)	**RAB**($\ddot{\theta}$)	
						$\bar{\tilde{p}}$	**MSE**($\tilde{p}$)	**RAB**($\tilde{p}$)		$\bar{\ddot{p}}$	**MSE**($\ddot{p}$)	**RAB**($\ddot{p}$)	
		n_1	m_1	τ_1		$\bar{\tilde{\mu}}$	**MSE**($\tilde{\mu}$)	**RAB**($\tilde{\mu}$)	**AMSE**	$\bar{\ddot{\mu}}$	**MSE**($\ddot{\mu}$)	**RAB**($\ddot{\mu}$)	**AMSE**
$\hbar$	***N***	n_2	m_2	τ_2	**CS**	$\bar{\tilde{\delta}}$	**MSE**($\tilde{\delta}$)	**RAB**($\tilde{\delta}$)	**ARAB**	$\bar{\ddot{\delta}}$	**MSE**($\ddot{\delta}$)	**RAB**($\ddot{\delta}$)	**ARAB**
2	40	20	12	2.0	1	1.7476	1.5643	0.6488	0.5771	1.7429	1.4233	0.6136	0.5773
		20	12	1.5		0.7821	0.5388	0.5801	0.5383	0.8788	0.6592	0.6281	0.5466
						0.6917	0.1788	0.4880		0.7045	0.2022	0.5177	
						0.3068	0.0266	0.4365		0.3051	0.0245	0.4269	
					2	1.6528	1.2170	0.5766	0.5291	1.6206	1.2292	0.5766	0.5044
						0.8894	0.6853	0.6175	0.5289	0.8171	0.5731	0.5829	0.5109
						0.6604	0.1919	0.5165		0.7282	0.1960	0.5073	
						0.3098	0.0224	0.4050		0.2832	0.0192	0.3768	
					3	1.7242	1.2915	0.5668	0.4920	1.9930	1.8233	0.6570	0.6624
						0.7856	0.5071	0.5622	0.4899	0.8444	0.6413	0.6390	0.5425
						0.6562	0.1469	0.4386		0.6539	0.1587	0.4551	
						0.3147	0.0224	0.3919		0.3313	0.0262	0.4188	
					4	1.7651	1.5152	0.6400	0.5933	1.9459	1.8754	0.6922	0.7619
						0.8507	0.6383	0.6313	0.5661	0.9998	0.9535	0.7771	0.6131
						0.6679	0.1854	0.5012		0.7051	0.1824	0.4854	
						0.3308	0.0343	0.4920		0.3325	0.0363	0.4978	
				4.0	1	1.7739	1.4100	0.6148	0.5786	1.8638	1.6331	0.6546	0.7122
				3.5		0.8594	0.6942	0.6596	0.5574	0.9860	0.9724	0.7730	0.6025
						0.6732	0.1790	0.4855		0.7328	0.2098	0.5009	
						0.3238	0.0313	0.4697		0.3257	0.0337	0.4814	
					2	1.7279	1.4832	0.6302	0.5641	1.7624	1.5691	0.6286	0.6103
						0.8110	0.5749	0.6078	0.5363	0.8723	0.6610	0.6436	0.5450
						0.6588	0.1721	0.4815		0.7057	0.1875	0.4916	
						0.3158	0.0261	0.4259		0.3028	0.0235	0.4163	
					3	1.7636	1.2841	0.5804	0.5183	1.9637	1.6699	0.6390	0.6745
						0.8549	0.5821	0.6200	0.5290	0.9595	0.8231	0.6934	0.5579
						0.6731	0.1794	0.4790		0.6812	0.1777	0.4723	
						0.3280	0.0275	0.4365		0.3386	0.0275	0.4268	
					4	1.7543	1.5024	0.6309	0.6068	1.9761	1.9618	0.7047	0.7834
						0.8351	0.6990	0.6573	0.5666	0.9421	0.9514	0.7876	0.6139
						0.7114	0.1932	0.5017		0.7247	0.1874	0.4867	
						0.3154	0.0325	0.4768		0.3222	0.0330	0.4764	

study accompanied by numerical computations has been done to assess the performance of these methods. The numerical results indicate that the LSEs are the best estimates among the other estimates. In summary, the features of PLHLD can be summarized as follows:

1. The CDF of PLHLD has closed form.
2. The three parameters included in the CDF of PLHLD give it the ability to fit several data.
3. The CDF of PLHLD includes the CDFs of PHLD, LHLD, and HLD as special cases.

Table 4 (continued)

						LSE				WLSE			
						$\bar{\tilde{\theta}}$	**MSE($\tilde{\theta}$)**	**RAB($\tilde{\theta}$)**		$\bar{\ddot{\theta}}$	**MSE($\ddot{\theta}$)**	**RAB($\ddot{\theta}$)**	
						$\bar{\tilde{p}}$	**MSE($\tilde{p}$)**	**RAB($\tilde{p}$)**		$\bar{\ddot{p}}$	**MSE($\ddot{p}$)**	**RAB($\ddot{p}$)**	
		n_1	m_1	τ_1		$\bar{\tilde{\mu}}$	**MSE($\tilde{\mu}$)**	**RAB($\tilde{\mu}$)**	**AMSE**	$\bar{\ddot{\mu}}$	**MSE($\ddot{\mu}$)**	**RAB($\ddot{\mu}$)**	**AMSE**
$\hbar$	***N***	n_2	m_2	τ_2	**CS**	$\bar{\tilde{\delta}}$	**MSE($\tilde{\delta}$)**	**RAB($\tilde{\delta}$)**	**ARAB**	$\bar{\ddot{\delta}}$	**MSE($\ddot{\delta}$)**	**RAB($\ddot{\delta}$)**	**ARAB**
			16	2.0	1	1.8238	1.5582	0.6453	0.5936	1.9250	1.6210	0.6419	0.6518
			16	1.5		0.8371	0.6289	0.6232	0.5489	0.9175	0.7954	0.7021	0.5693
						0.6753	0.1557	0.4584		0.7033	0.1585	0.4594	
						0.3276	0.0316	0.4685		0.3300	0.0324	0.4739	
					2	1.8333	1.4968	0.6290	0.5560	1.8974	1.7667	0.6747	0.6420
						0.8027	0.5415	0.5719	0.5270	0.7988	0.5995	0.6124	0.5518
						0.6491	0.1580	0.4639		0.7178	0.1740	0.4811	
						0.3314	0.0276	0.4431		0.3142	0.0279	0.4390	
					3	1.8035	1.3400	0.6031	0.4966	1.9787	1.5802	0.6362	0.6237
						0.7910	0.4769	0.5515	0.5068	0.9199	0.7494	0.6798	0.5462
						0.6321	0.1422	0.4372		0.6285	0.1362	0.4271	
						0.3367	0.0272	0.4356		0.3510	0.0292	0.4416	
					4	1.7836	1.5947	0.6478	0.5893	1.9350	1.8131	0.6864	0.7090
						0.8229	0.5817	0.6132	0.5405	0.9666	0.8331	0.7122	0.5836
						0.6708	0.1528	0.4535		0.6936	0.1553	0.4507	
						0.3218	0.0280	0.4476		0.3364	0.0344	0.4851	
				4.0	1	1.8652	1.6027	0.6511	0.6191	1.9406	1.7167	0.6680	0.7036
				3.5		0.8753	0.6691	0.6470	0.5643	1.0255	0.8987	0.7596	0.5974
						0.6657	0.1711	0.4767		0.6866	0.1628	0.4651	
						0.3364	0.0337	0.4825		0.3443	0.0362	0.4970	
					2	1.7382	1.3469	0.6062	0.5441	1.8507	1.6026	0.6422	0.6484
						0.8014	0.6247	0.6006	0.5323	0.9015	0.7581	0.6558	0.5601
						0.6766	0.1765	0.4775		0.6788	0.2020	0.4861	
						0.3172	0.0281	0.4448		0.3300	0.0310	0.4563	
					3	1.8569	1.5358	0.6364	0.5812	1.9194	1.6035	0.6444	0.6573
						0.8475	0.5918	0.6106	0.5485	0.9481	0.8290	0.7185	0.5805
						0.6663	0.1646	0.4740		0.6834	0.1615	0.4673	
						0.3350	0.0326	0.4731		0.3359	0.0350	0.4918	
					4	1.8105	1.5793	0.6354	0.6093	1.9559	1.7478	0.6750	0.7322
						0.8786	0.6662	0.6399	0.5486	0.9942	0.9803	0.7647	0.6004
						0.6763	0.1621	0.4630		0.6912	0.1650	0.4676	
						0.3266	0.0297	0.4559		0.3401	0.0356	0.4942	

Table 4 (continued)

						LSE				WLSE			
						$\bar{\tilde{\theta}}$	**MSE($\tilde{\theta}$)**	**RAB($\tilde{\theta}$)**		$\bar{\ddot{\theta}}$	**MSE($\ddot{\theta}$)**	**RAB($\ddot{\theta}$)**	
						$\bar{\tilde{p}}$	**MSE($\tilde{p}$)**	**RAB($\tilde{p}$)**		$\bar{\ddot{p}}$	**MSE($\ddot{p}$)**	**RAB($\ddot{p}$)**	
		n_1	m_1	τ_1		$\bar{\tilde{\mu}}$	**MSE($\tilde{\mu}$)**	**RAB($\tilde{\mu}$)**	**AMSE**	$\bar{\ddot{\mu}}$	**MSE($\ddot{\mu}$)**	**RAB($\ddot{\mu}$)**	**AMSE**
$\hbar$	***N***	n_2	m_2	τ_2	**CS**	$\bar{\tilde{\delta}}$	**MSE($\tilde{\delta}$)**	**RAB($\tilde{\delta}$)**	**ARAB**	$\bar{\ddot{\delta}}$	**MSE($\ddot{\delta}$)**	**RAB($\ddot{\delta}$)**	**ARAB**
			20	–	–	1.8430	1.3954	0.6189	0.5286	2.0121	1.5934	0.6340	0.6314
			20			0.8430	0.5503	0.5813	0.5181	0.9993	0.7949	0.6939	0.5341
						0.6781	0.1407	0.4366		0.6509	0.1109	0.3859	
						0.3315	0.0280	0.4354		0.3511	0.0266	0.4227	
2	60	30	18	2.0	1	1.8088	1.4489	0.6264	0.5422	1.8590	1.6434	0.6531	0.6239
		30	18	1.5		0.7982	0.5207	0.5637	0.5274	0.8883	0.6660	0.6397	0.5436
						0.6821	0.1720	0.4818		0.6875	0.1616	0.4658	
						0.3173	0.0271	0.4376		0.3185	0.0244	0.4160	
					2	1.7355	1.2457	0.5717	0.5086	1.6656	1.2524	0.5853	0.5187
						0.8270	0.5941	0.5791	0.5153	0.8191	0.6097	0.6068	0.5169
						0.6625	0.1713	0.4962		0.7335	0.1943	0.4989	
						0.3146	0.0234	0.4141		0.2870	0.0185	0.3767	
					3	1.7526	1.3844	0.6036	0.5384	1.8925	1.6218	0.6545	0.5934
						0.7971	0.5768	0.5706	0.5194	0.8082	0.5667	0.5988	0.5339
						0.6870	0.1659	0.4687		0.6826	0.1594	0.4577	
						0.3158	0.0266	0.4348		0.3203	0.0257	0.4248	
					4	1.8456	1.5166	0.6345	0.6206	2.0217	1.8712	0.7045	0.7279
						0.9106	0.7609	0.6628	0.5656	0.9326	0.8366	0.7215	0.5954
						0.6792	0.1712	0.4755		0.6907	0.1686	0.4674	
						0.3380	0.0336	0.4895		0.3396	0.0350	0.4881	
				4.0	1	1.8596	1.4955	0.6249	0.5931	1.9388	1.5929	0.6496	0.6812
				3.5		0.8500	0.6795	0.6477	0.5510	1.0126	0.9343	0.7473	0.5823
						0.6981	0.1667	0.4709		0.6857	0.1651	0.4601	
						0.3245	0.0307	0.4605		0.3407	0.0323	0.4723	
					2	1.8757	1.5150	0.6276	0.5844	1.8301	1.3882	0.6042	0.5590
						0.8299	0.6353	0.6042	0.5352	0.8749	0.6640	0.6394	0.5276
						0.6455	0.1593	0.4652		0.6707	0.1607	0.4600	
						0.3301	0.0281	0.4439		0.3138	0.0232	0.4067	
					3	1.8612	1.4240	0.6101	0.5516	1.9589	1.6194	0.6331	0.6848
						0.8439	0.5827	0.6164	0.5443	0.9870	0.9215	0.7652	0.5873
						0.6934	0.1676	0.4742		0.6929	0.1637	0.4638	
						0.3307	0.0321	0.4765		0.3385	0.0347	0.4870	

Table 4 (continued)

						LSE				WLSE			
						$\overline{\tilde{\theta}}$	MSE($\tilde{\theta}$)	RAB($\tilde{\theta}$)		$\overline{\ddot{\theta}}$	MSE($\ddot{\theta}$)	RAB($\ddot{\theta}$)	
						$\overline{\tilde{p}}$	MSE($\tilde{p}$)	RAB($\tilde{p}$)		$\overline{\ddot{p}}$	MSE($\ddot{p}$)	RAB($\ddot{p}$)	
		n_1	m_1	τ_1		$\overline{\tilde{\mu}}$	MSE($\tilde{\mu}$)	RAB($\tilde{\mu}$)	AMSE	$\overline{\ddot{\mu}}$	MSE($\ddot{\mu}$)	RAB($\ddot{\mu}$)	AMSE
$\hbar$	N	n_2	m_2	τ_2	CS	$\overline{\tilde{\delta}}$	MSE($\tilde{\delta}$)	RAB($\tilde{\delta}$)	ARAB	$\overline{\ddot{\delta}}$	MSE($\ddot{\delta}$)	RAB($\ddot{\delta}$)	ARAB
					4	1.7979	1.5242	0.6381	0.6256	1.9212	1.6542	0.6509	0.6902
						0.9322	0.7836	0.6589	0.5509	1.0042	0.9179	0.7549	0.5821
						0.6626	0.1644	0.4515		0.7005	0.1565	0.4496	
						0.3352	0.0303	0.4552		0.3358	0.0322	0.4727	
			24	2.0	1	1.8412	1.4738	0.6282	0.5732	1.8890	1.5936	0.6540	0.6182
			24	1.5		0.8553	0.6413	0.6167	0.5357	0.9296	0.7080	0.6709	0.5540
						0.6877	0.1469	0.4390		0.6997	0.1401	0.4322	
						0.3288	0.0306	0.4589		0.3305	0.0311	0.4588	
					2	1.8718	1.4285	0.6220	0.5653	2.0369	1.7420	0.6764	0.6619
						0.8530	0.6547	0.6035	0.5298	0.8632	0.7354	0.6509	0.5506
						0.6621	0.1499	0.4517		0.6625	0.1414	0.4355	
						0.3333	0.0282	0.4419		0.3382	0.0288	0.4397	
					3	1.9248	1.6900	0.6624	0.5986	2.0166	1.8173	0.6867	0.6801
						0.7865	0.5470	0.6015	0.5264	0.8532	0.7359	0.6617	0.5517
						0.6725	0.1303	0.4135		0.6802	0.1372	0.4213	
						0.3274	0.0269	0.4281		0.3321	0.0298	0.4370	
					4	1.8486	1.5039	0.6348	0.6216	1.8923	1.5359	0.6414	0.6473
						0.9268	0.8073	0.6802	0.5538	0.9861	0.8774	0.7198	0.5693
						0.6733	0.1431	0.4360		0.6986	0.1420	0.4358	
						0.3356	0.0320	0.4642		0.3347	0.0337	0.4802	
				4.0	1	1.8830	1.5098	0.6307	0.5971	2.0012	1.7325	0.6838	0.6509
				3.5		0.9221	0.7109	0.6405	0.5384	0.9413	0.7187	0.6805	0.5547
						0.6609	0.1372	0.4248		0.6788	0.1226	0.4036	
						0.3401	0.0305	0.4575		0.3377	0.0297	0.4510	
					2	1.8442	1.4598	0.6168	0.5515	1.9614	1.5925	0.6500	0.5932
						0.8180	0.5777	0.5853	0.5144	0.8339	0.5984	0.6158	0.5343
						0.6652	0.1430	0.4378		0.6851	0.1557	0.4455	
						0.3269	0.0255	0.4177		0.3278	0.0264	0.4258	
					3	1.9149	1.5559	0.6474	0.6105	1.9565	1.6471	0.6542	0.6461
						0.8756	0.7152	0.6457	0.5460	0.9536	0.7882	0.7052	0.5477
						0.6712	0.1407	0.4349		0.6837	0.1214	0.3991	
						0.3355	0.0302	0.4560		0.3330	0.0274	0.4322	

Table 4 (continued)

						LSE				WLSE			
						$\bar{\tilde{\theta}}$	MSE($\tilde{\theta}$)	RAB($\tilde{\theta}$)		$\bar{\ddot{\theta}}$	MSE($\ddot{\theta}$)	RAB($\ddot{\theta}$)	
						$\bar{\tilde{p}}$	MSE($\tilde{p}$)	RAB($\tilde{p}$)		$\bar{\ddot{p}}$	MSE($\ddot{p}$)	RAB($\ddot{p}$)	
		n_1	m_1	τ_1		$\bar{\tilde{\mu}}$	MSE($\tilde{\mu}$)	RAB($\tilde{\mu}$)	AMSE	$\bar{\ddot{\mu}}$	MSE($\ddot{\mu}$)	RAB($\ddot{\mu}$)	AMSE
$\hbar$	***N***	n_2	m_2	τ_2	CS	$\bar{\tilde{\delta}}$	MSE($\tilde{\delta}$)	RAB($\tilde{\delta}$)	ARAB	$\bar{\ddot{\delta}}$	MSE($\ddot{\delta}$)	RAB($\ddot{\delta}$)	ARAB
					4	1.9539	1.6475	0.6665	0.6388	2.0221	1.7197	0.6650	0.6751
						0.8935	0.7104	0.6451	0.5642	0.9599	0.8241	0.7061	0.5588
						0.6754	0.1626	0.4633		0.6746	0.1243	0.4061	
						0.3420	0.0345	0.4821		0.3447	0.0325	0.4581	
			30	–	–	1.8967	1.5395	0.6380	0.5831	1.9899	1.5518	0.6327	0.6309
			30			0.8613	0.6485	0.6000	0.5178	0.9956	0.8520	0.6957	0.5246
						0.6737	0.1161	0.3978		0.6633	0.0925	0.3514	
						0.3353	0.0283	0.4354		0.3465	0.0273	0.4187	

4. The PLHLD can describe the failure times of parallel–series systems. This feature is very important for physical experimenters and engineers.
5. The HRF of PLHLD has various shapes such as decreasing-constant, increasing-constant, and v-shaped. This feature gives it more flexibility to fit and analyze several data.
6. The PLHLD can represent the non-stationary data. This feature may be useful for the experimenter to predict the environmental behavior of some products.
7. The PLHLD fits the data better than some other distributions, such as PHLD, HLGWD, HLD, and WD.

Abbreviations

AD Anderson–Darling statistic
AIC Akaike information criterion
ALT Accelerated life test
AMSE Average of mean squared errors
ARAB Average of relative absolute biases
BIC Bayesian information criterion
CAIC Consistent AIC
CDF Cumulative distribution function
CS Censoring scheme
CvM Cramér–von Mises statistic
HQIC Hannan–Quinn information criterion
HLD Half-logistic distribution
HLGWD Half-logistic generated Weibull distribution
HRF Hazard rate function
KS Kolmogorov–Smirnov
LGQF Legendre–Gauss quadrature formula
LSE Least squares estimate
MLE Maximum likelihood estimate
MSE Mean squared error
PDF Probability density function
PE Percentile estimate
PLHLD Poisson–logarithmic half-logistic distribution

PHLD	Poisson-half-logistic distribution
PMF	Probability mass function
RAB	Relative absolute bias
RV	Random variable
SE	Standard error
WD	Weibull distribution
WLSE	Weighted least squares estimate.

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Declarations

Competing interests
The authors declare that they have no competing interests.

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