Journal of the Egyptian Mathematical Society (2016) **24**, 1–4

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Cleanness of overrings of polynomial rings

Hani A. Khashan *, **Worood Burhan**

Department of Mathematics, Al al-Bayt University, Mafraq, Jordan

Received 22 February 2014; revised 17 June 2014; accepted 2 August 2014
Available online 11 February 2015

KEYWORDS

Clean rings;
Almost clean rings;
The rings $R(x)$ and $R\langle x\rangle$

Abstract Let R be a commutative ring with identity. R is called clean (respectively, almost clean) if every element in R is a sum of an idempotent and a unit (respectively, a regular element). In this paper, we clarify conditions under which the two overrings $R(x)$ and $R\langle x\rangle$ of the ring of polynomials $R[x]$ are (almost) clean rings.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 13A15; 13A99; 13B99

1. Introduction

Throughout this paper a ring R will be commutative with identity, $U(R)$ denotes the set of all units of R, $reg(R)$ the set of regular elements of R, $Z(R)$ the set of all zero divisors of R and $Id(R)$ the set of all idempotent elements of R. If $f \in R[x]$, the ring of all polynomials over R, then $C(f)$ denotes the ideal of R generated by the coefficients of f. Let $S = \{f \in R[x] : C(f) = R\}$ and $W = \{f \in R[x] : f \text{ is monic}\}$. Then S and W are regular multiplicatively closed subsets of $R[x]$ and the rings of fractions $S^{-1}R[x]$ and $W^{-1}R[x]$ are denoted by $R(x)$ and $R\langle x\rangle$ respectively. As clearly $W \subseteq S$, then $R(x)$ is itself a ring of fractions of $R\langle x\rangle$, see Theorem 3.16 in [1]. Some basic properties and related theorems of $R(x)$ and $R\langle x\rangle$ can be found in [2–5] where we can see that the rings $R, R(x)$ and $R\langle x\rangle$ share many properties. In the following lemma, we survey some properties of the rings $R[x], R(x)$ and $R\langle x\rangle$ that will be needed in this paper.

Lemma 1.1. *Let R be a ring. Then*

(1) *If f and g are two nonzero polynomials in* $R[x]$ *with* $deg(f) = k$, *then* $c(g)^{k+1}c(f) = c(g)^k c(fg)$.
(2) *If R is a finite dimensional ring, then* $dim(R(x)) = dim(R\langle x\rangle) = dim(R[x]) - 1$.
(3) $Id(R(x)) = Id(R\langle x\rangle) = Id(R)$.
(4) $U(R(x)) = \left\{\frac{f}{g} \in R(x) : c(f) = c(g) = R\right\}$.
(5) *There is a one to one correspondence between the maximal (minimal) ideals of R and the maximal (minimal) ideals of* $R(x)$ *(or* $R\langle x\rangle$*) given by,* $P \leftrightarrow PR(x)$ *(or* $P \leftrightarrow PR\langle x\rangle$*).*
(6) *If I is an ideal of R, then* $IR(x) \cap R = IR\langle x\rangle \cap R = I$.
(7) *If* $R = R_1 \times R_2 \times \cdots \times R_n$ *is a direct product of rings, then* $R(x) = R_1(x) \times R_2(x) \times \cdots \times R_n(x)$ *and* $R\langle x\rangle = R_1\langle x\rangle \times R_2\langle x\rangle \times \cdots \times R_n\langle x\rangle$.
(8) *If R is an indecomposable ring, then* $R(x)$ *and* $R\langle x\rangle$ *are also indecomposable.*
(9) *If R is a Noetherian ring, then* $R(x)$ *and* $R\langle x\rangle$ *are also Noetherian.*

* Corresponding author.
E-mail address: hakhashan@aabu.edu.jo (H.A. Khashan).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2014.08.003
1110-256X

Proof. See [2,5,4]. □

Let R be a ring. An element $r \in R$ is called clean (respectively, almost clean) if $r = u + e$ where $u \in U(R)$ (respectively, $u \in reg(R)$) and $e \in Id(R)$. If every element in R is clean (respectively, almost clean), then R is called a clean (respectively, almost clean) ring. Since clearly every unit in a ring R is regular, then the class of almost clean rings is a generalization of the class of clean rings. This generalization is proper since any non quasi-local integral domain is almost clean which is not clean, see [6]. The class of clean rings (not necessarily commutative) was first introduced by Nicholson [7] as early as 1977 in his study of lifting idempotents and exchange rings. In fact, he proved that the class of commutative clean rings coincide with the class of commutative exchange rings. Since then, a great deal is known about clean rings and their generalizations. In particular in 2002, Anderson and Camillo [6] studied commutative clean rings and gave a characterization for a Noetherian clean rings. Then Ahn and Anderson [8] in 2006, defined and studied weakly clean and almost clean rings as two important generalizations of clean rings. In 2012, Anderson and Badawi [9] gave a complete classification of the clean elements of the ring $R[x]$. More results about clean rings and some of their generalizations can be found also in [10,11].

In this paper, we clarify conditions under which the following statements are equivalent for a ring R:

(1) R is clean (almost clean).
(2) $R\langle x\rangle$ is clean (almost clean).
(3) $R(x)$ is clean (almost clean).

2. When $R(x)$ and $R\langle x\rangle$ are (almost) clean rings

In this section, we clarify conditions on a ring R under which the ring $R(x)$ and $R\langle x\rangle$ are clean rings or almost clean rings.

Theorem 2.1. *Let R be a ring and consider the following statements*

(1) *R is clean.*
(2) *$R\langle x\rangle$ is clean.*
(3) *$R(x)$ is clean.*

then we have $(3) \Rightarrow (1)$ *and* $(2) \Rightarrow (1)$.

Proof. $(3) \Rightarrow (1)$: Suppose that $R(x)$ is a clean ring and let $r \in R \subseteq R(x)$. Since $R(x)$ is clean, then there exists $\frac{f}{g} \in U(R(x))$ and $\frac{f'}{g'} \in Id(R(x))$ such that $r = \frac{f}{g} + \frac{f'}{g'}$. Hence by Lemma 1.1, we have $c(g) = c(f) = R$ and $\frac{f'}{g'} = e \in Id(R)$. Therefore, $\frac{f}{g} = r - e \in U(R(x)) \cap R$. It is enough now to prove that $U(R(x)) \cap R = U(R)$. Let $\frac{f}{g} = r \in U(R(x)) \cap R$, then $f = rg$ where $c(g) = c(f) = R$ and so again by Lemma 1.1, $c(g)^1 c(r) = c(g)^0 c(rg) = c(rg)$. Hence, $(r) = c(r) = c(rg) = c(f) = R$ and so $\frac{f}{g} = r \in U(R)$. The other containment is obvious. Therefore, R is a clean ring.

$(2) \Rightarrow (1)$: As $R \subseteq R\langle x\rangle \subseteq R(x)$, we have $U(R) \subseteq U(R\langle x\rangle) \cap R \subseteq U(R(x)) \cap R \subseteq U(R)$ and so $U(R\langle x\rangle) \cap R = U(R)$. Since also $Id(R) = Id(R\langle x\rangle)$, then similar to the proof of $(3) \Rightarrow (1)$, we see that R is a clean ring. □

Let R be a ring and let $\widehat{P}$ be a prime ideal of $R(x)$. Then $\widehat{P} = S^{-1}Q$ where Q is a prime ideal of $R[x]$ and so $\widehat{P} \cap R[x] = S^{-1}Q \cap R[x] = Q$. Therefore, $\widehat{P} \cap R = Q \cap R$ is a prime ideal of R. Similarly, $\breve{P} \cap R$ is a prime ideal of R for any prime ideal $\breve{P}$ of $R\langle x\rangle$.

We recall that a ring R is called a pm-ring if each prime ideal of R is contained in a unique maximal ideal. In the following lemma, we clarify a condition under which $R(x)$ and $R\langle x\rangle$ are pm-rings.

Lemma 2.2. *Let R be a ring. Then the following are equivalent*

(1) *R is a pm-ring.*
(2) *$R(x)$ is a pm-ring.*
(3) *$R\langle x\rangle$ is a pm-ring.*

Proof. $(1) \Rightarrow (2)$: Suppose that R is a pm-ring and let $\widehat{P}$ be a prime ideal of $R(x)$ and suppose that there are two distinct maximal ideals $\widehat{M}_1$ and $\widehat{M}_2$ of $R(x)$ such that $\widehat{P} \subseteq \widehat{M}_1$ and $\widehat{P} \subseteq \widehat{M}_2$. By Lemma 1.1, $\widehat{M}_1 = M_1R(x)$ and $\widehat{M}_2 = M_2R(x)$ for some distinct maximal ideals M_1 and M_2 of R. Thus, $\widehat{P} \cap R \subseteq M_1R(x) \cap R = M_1$ and $\widehat{P} \cap R \subseteq M_2R(x) \cap R = M_2$. But, $\widehat{P} \cap R$ is a prime ideal of R which contradicts that R is a pm-ring. Therefore, any prime ideal of $R(x)$ is contained in a unique maximal ideal and so $R(x)$ is a pm-ring.

$(2) \Rightarrow (3)$: Suppose $R(x)$ is a pm-ring and let $\breve{P}$ be a prime ideal of $R\langle x\rangle$. As $R(x) = T^{-1}R\langle x\rangle$ for some multiplicatively closed subset T of $R\langle x\rangle$, then $T^{-1}\breve{P}$ is a prime ideal of $R(x)$. Thus, there is a unique maximal ideal $\widehat{M}$ of $R(x)$ such that $T^{-1}\breve{P} \subseteq \widehat{M}$. Write $\widehat{M} = T^{-1}Q$ where Q is a prime ideal of $R\langle x\rangle$. Then it follows clearly that $\widehat{M} \cap R\langle x\rangle = T^{-1}Q \cap R\langle x\rangle = Q$ is the unique maximal ideal of $R\langle x\rangle$ containing $\breve{P}$. Therefore, $R\langle x\rangle$ is a pm-ring.

$(3) \Rightarrow (1)$: Suppose $R\langle x\rangle$ is a pm-ring. Let P be a prime ideal of R and suppose that there are two distinct maximal ideals M_1 and M_2 of R such that $P \subseteq M_1$ and $P \subseteq M_2$. Then $PR\langle x\rangle$ is a prime ideal of $R\langle x\rangle$ with $PR\langle x\rangle \subseteq M_1R\langle x\rangle$ and $PR\langle x\rangle \subseteq M_2R\langle x\rangle$. But, $M_1R\langle x\rangle$ and $M_2R\langle x\rangle$ are distinct maximal ideals of $R\langle x\rangle$ which is a contradiction. Thus, R is a pm-ring. □

In [6], it has been proved that every clean ring is a pm-ring. The converse is true under a certain condition as we can see in the following Lemma.

Lemma 2.3 [6]. *Let R be a ring with a finite number of minimal prime ideals. Then the following are equivalent:*

(1) *R is a finite direct product of quasi-local rings.*
(2) *R is a clean ring.*
(3) *R is a pm-ring.*

Now, we use the above two lemmas to prove the following main theorem in this paper.

Theorem 2.4. *Let R be a ring with a finite number of minimal prime ideals. The following are equivalent.*

(1) *R is a clean ring.*
(2) *$R(x)$ is a clean ring.*
(3) *$R\langle x\rangle$ is a clean ring.*

Proof. (1) $\Rightarrow$ (2): If R is a clean ring, then it is a pm-ring by Corollary 4 in [6] and so $R(x)$ is a pm-ring by Lemma 2.2. By Lemma 1.1, there is a one to one correspondence between minimal prime ideals of R and those of $R(x)$. It follows that $R(x)$ also has a finite number of minimal prime ideals. Therefore, $R(x)$ is a clean ring by Lemma 2.3.

(2) $\Rightarrow$ (3): As $R(x)$ is a clean ring, it is a pm-ring. Again we use Lemma 2.2 to conclude that $R\langle x\rangle$ is a pm-ring. Since also, $R\langle x\rangle$ has a finite number of minimal prime ideals by Lemma 1.1, then $R\langle x\rangle$ is a clean ring.

(3) $\Rightarrow$ (1): This is followed by Theorem 2.1. (in this direction we do not need the condition that R has a finite number of minimal prime ideals). □

In the following Theorem we give other conditions under which $R(x)$ and $R\langle x\rangle$ are clean rings.

Theorem 2.5. *If R is a zero dimensional ring or a finite direct product of quasi-local rings, then $R(x)$ and $R\langle x\rangle$ are clean rings.*

Proof. Suppose that $\dim(R)=0$, then by Lemma 1.1, we get $\dim(R[x])=1$. But, then by Lemma 1.1 again, we conclude that $\dim(R(x))=\dim(R\langle x\rangle)=\dim(R[x])-1=0$. The result follows now by Corollary (11) in [6].

Now, suppose $R=R_1\times R_2\times\cdots\times R_n$ is a direct product of quasi-local rings. Then $R(x)=R_1(x)\times R_2(x)\times\cdots\times R_n(x)$ is also a direct product of quasi-local rings by Lemma 1.1. Now, the result follows by Proposition 2 in [6]. The proof for $R\langle x\rangle$ is similar. □

Now, for a ring R, we study conditions under which $R(x)$ and $R\langle x\rangle$ are almost clean rings. First we note that If R is an integral domain, then one can easily prove that both $R(x)$ and $R\langle x\rangle$ are integral domains. Hence $R(x)$ and $R\langle x\rangle$ are almost clean rings. In the following two lemmas, we can see characterizations for indecomposable almost clean and Noetherian almost clean rings.

Lemma 2.6 [8]. *A ring R is an indecomposable almost clean rings if and only if for prime ideals P and Q of R where P, $Q\subseteq Z(R)$, we have $P+Q\neq R$.*

Lemma 2.7 [8]. *A Noetherian ring R is almost clean if and only if for prime ideals P and Q of R where $P,Q\subseteq Z(R)$ and $P+Q=R$, there exists an idempotent e in R with $e\in P$ and $1-e\in Q$.*

Theorem 2.8. *Let R be a ring. If $R(x)$ or $R\langle x\rangle$ is an almost clean ring, then R is an almost clean ring.*

Proof. Suppose $R(x)$ is almost clean and let $r\in R\subseteq R(x)$. As $R(x)$ is almost clean and $Id(R(x))=Id(R)$, then $r=\frac{f}{g}+e$ where $\frac{f}{g}\in reg(R(x))$ and $e\in Id(R)$. Thus $\frac{f}{g}=r-e\in reg(R(x))\cap R$. It is enough to prove that $reg(R(x))\cap R=reg(R)$. Let $t\in reg(R(x))\cap R$ and suppose that there exists $0\neq a\in R$ such that $ta=0$. Then $\frac{0}{1}\neq\frac{a}{1}\in R(x)$ with $t\frac{a}{1}=\frac{ta}{1}=\frac{0}{1}$ which is a contradiction. Therefore, $t\in reg(R)$. The other containment is clear. Hence, $reg(R(x))\cap R=reg(R)$ and R is an almost clean ring. Similarly, one can prove that $reg(R\langle x\rangle)\cap R=reg(R)$ and so $R\langle x\rangle$ is an almost clean ring implies that R is almost clean.

In general, we do not know whether the converse of Theorem 2.8 is always true or not. However, in the following Theorem, we prove that this is true in a special case. First, we give the following definition

Definition 2.9. A ring R is said to satisfy the property $(*)$ if for any multiplicatively closed subset S of R, we have $S^{-1}(P+Q)\cap R=P+Q$ for any prime ideals P and Q of R.

If a ring R is a zero dimensional ring (such as $\mathbb{Z}_n$), then every prime ideal of R is maximal. Thus, for any distinct prime ideals P and Q of R, we have $S^{-1}(P+Q)\cap R=S^{-1}R\cap R=R=P+Q$. Hence, R satisfies the property $(*)$. Similarly, one dimensional integral domains satisfy the property $(*)$. Also, if the set of all prime ideals of a ring R form a chain, say $P_0\subseteq P_1\subseteq\cdots\subseteq P_n$, then $P_i+P_j=P_k$ where $k=\max\{i,j\}$. Therefore, $S^{-1}(P_i+P_j)\cap R=P_i+P_j$ and also R satisfy the property $(*)$. In general, if I and J are any two ideals of a valuation ring, then either $I\subseteq J$ or $J\subseteq I$. Hence, one can easily see that any valuation ring satisfies the property $(*)$.

Theorem 2.10. *Let R be a ring in which $R[x]$ satisfies the property $(*)$. If R is an indecomposable almost clean ring, then, $R(x)$ and $R\langle x\rangle$ are also indecomposable almost clean.*

Proof. Let $\widehat{P_1}$ and $\widehat{P_2}$ be prime ideals of $R(x)$ with $\widehat{P_1},\widehat{P_2}\subseteq Z(R(x))$. Then $\widehat{P_1}=S^{-1}Q_1$ and $\widehat{P_2}=S^{-1}Q_2$ where Q_1 and Q_2 are prime ideals of $R[x]$. Suppose that $\widehat{P_1}+\widehat{P_2}=S^{-1}Q_1+S^{-1}Q_2=S^{-1}(Q_1+Q_2)=R(x)$. Then as $R(x)$ satisfies the property $(*)$, we have $(\widehat{P_1}+\widehat{P_2})\cap R[x]=S^{-1}(Q_1+Q_2)\cap R[x]=Q_1+Q_2$. Thus, $(Q_1\cap R)+(Q_2\cap R)=(Q_1+Q_2)\cap R=\left(\widehat{P_1}+\widehat{P_2}\right)\cap R=R(x)\cap R=R$. Moreover, $(Q_1\cap R),(Q_2\cap R)\subseteq Z(R)$. Indeed, if $r\in Q_1\cap R$, then $\frac{r}{1}\in\widehat{P_1}\subseteq Z(R(x))$ and so there exists $\frac{0}{1}\neq\frac{f}{g}\in R(x)$ such that $\frac{rf}{g}=\frac{0}{1}$. Therefore, $rf=0$. If $f=a_0+a_1x+\cdots+a_nx^n$, then $ra_i=0$ for some nonzero coefficient a_i of f. Thus, $r\in Z(R)$ and $Q_1\cap R\subseteq Z(R)$. Similarly, $Q_2\cap R\subseteq Z(R)$. But, this contradicts Lemma 2.6 and so $\widehat{P_1}+\widehat{P_2}\neq R(x)$. Hence, again by using Lemma 2.6, we see that $R(x)$ is almost clean. Since R is indecomposable, then $R(x)$ is also indecomposable by Lemma 1.1. Similarly, $R\langle x\rangle$ is indecomposable almost clean ring. □

Theorem 2.11. *Let R be a ring in which $R[x]$ satisfies the property $(*)$. If R is a Noetherian almost clean ring, then $R(x)$ and $R\langle x\rangle$ are also almost clean.*

Proof. First, we note that $R(x)$ and $R\langle x\rangle$ are Noetherian rings by Lemma 1.1. Let $\widehat{P}_1$ and $\widehat{P}_2$ be prime ideals of $R(x)$ with $\widehat{P}_1, \widehat{P}_2 \subseteq Z(R(x))$ and $\widehat{P}_1 + \widehat{P}_2 = R(x)$. Then as in the proof of Theorem 2.10, we have $(Q_1 \cap R) + (Q_2 \cap R) = R$ where Q_1 and Q_2 are prime ideals of $R[x]$. Since R is a Noetherian almost clean ring, then by Lemma 2.7, there is an element $e \in Id(R)$ such that $e \in (Q_1 \cap R)$ and $1 - e \in (Q_2 \cap R)$. Thus, by Lemma 1.1, $e \in Id(R(x))$ with $e \in \widehat{P}_1$ and $1 - e \in \widehat{P}$. Therefore, $R(x)$ is almost clean again by Lemma 2.7. Similarly, $R\langle x\rangle$ is an almost clean ring. □

Acknowledgments

The authors would like to thank referees for their valuable comments and suggestions, which have improved the quality of the paper.

References

[1] M. Larsen, P. McCarthy, Multiplicative Theory of Ideals, Academic Press, New York and London, 1971.
[2] D.D. Anderson, D.F. Anderson, R. Markanda, The rings $R(x)$ and $R\langle x\rangle$, J. Algebra 95 (1985) 96–115.
[3] S. Glaz, Hereditary localizations of polynomial rings, J. Algebra 140 (1991) 101–112.
[4] L.R. Riche, The ring $R\langle x\rangle$, J. Algebra 67 (1980) 327–341.
[5] J.A. Huckaba, Commutative rings with zero divisors, Monographs and Textbooks in Pure and Applied Mathematics, vol. 117, Marcel Dekker, Inc., New York, 1988.
[6] D.D. Anderson, V.P. Camillo, Commutative rings whose elements are a sum of a unit and an idempotent, Commun. Algebra 30 (7) (2002) 3327–3336.
[7] W.K. Nicholson, Lifting idempotents and exchange rings, Trans. Am. Math. Soc. 229 (1977) 278–279.
[8] M. Ahn, D.D. Anderson, Weakly clean rings and almost clean rings, Rocky Mt. J. Math. 36 (3) (2006) 783–798.
[9] D.F. Anderson, A. Badawi, Von Neumann regular and related elements in commutative rings, Algebra Colloq. 19 (1) (2012) 1017–1040.
[10] W.D. Burgess, R. Raphael, On commutative clean rings and pm rings, Contemp. Math. 480 (2009) 35–55.
[11] W.D. Burgess, R. Raphael, Clean classical rings of quotients of commutative rings with applications to C(X), J. Algebra Appl. 7 (2) (2008) 195–209.