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k-Zumkeller labeling of super subdivision of some graphs

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Abstract

A simple graph $G = (V, E)$ is said to be *k*-Zumkeller graph if there is an injective function *f* from the vertices of *G* to the natural numbers *N* such that when each edge $xy \in E$ is assigned the label *f*(*x*)*f*(*y*), the resulting edge labels are *k* distinct Zumkeller numbers. In this paper, we show that the super subdivision of path, cycle, comb, ladder, crown, circular ladder, planar grid and prism are *k*-Zumkeller graphs.

Keywords: Zumkeller number, *k*-Zumkeller labeling, Complete bipartite graph, Super subdivision

Mathematics Subject Classification: 05C, 05C78, 11Axx

Introduction

Graph labeling was first introduced by Alex Rosa in the mid-sixties [1]. It concerns the assignment of mathematical objects, such as integers, prime numbers, modular integers, element of group, etc. The properties of the mathematical object are used through an evaluating function that assigns the values to the edges and/or vertices of graph $G = (V, E)$ based upon certain criteria. There are enormous papers have been devoted to several kinds of labeling of graphs over the past three decades, which are updated by Gallian [2]. Labeled graphs are used in numerous areas like coding theory, X-ray crystallography, the design of good radar type codes, astronomy, circuit design, communication network addressing, data base management [2]. Graphs with labeled edges are frequently used to model networks, with restrictions on the network represented as restrictions on the labels of edges. For example, when modeling transportation networks, such labels can be used to indicate various factors, from cost to level of traffic stream. In generally Ahuja, Magnati and Orlin [3] point out different applications in statistical physics, particle physics, computer science, biology, economics, operations research and sociology. For all notations and terminology in graph theory we follow Harary [4].

Graph labeling is a strong relation between number theory and graph structures. The study of Zumkeller numbers [5] is a part of number theory which is one of the important branches of mathematics. Some parts of number theory play an important role in modern coding and cryptography. Balamurugan et al. [6] introduced Zumkeller labeling using Zumkeller numbers, which is defined as an injective function $f : V \to N$

such that the induced function $f^*: E \to N$ defined by $f^*(xy) = f(x)f(y)$ is a Zumkeller number for all $xy \in E, x, y \in V$. The concept of k-Zumkeller labeling of graphs has been introduced and investigated in the literature [7, 8].

Definition 1.1 [7] A function f is called k-Zumkeller labeling of the graph G if $f: V(G) \to N$ is injective and the induced function $f^*: E \to N$ defined by $f^*(uv) = f(u)f(v)$ is a Zumkeller number for all $uv \in E(G)$,$u, v \in V(G)$ and the resulting edge labels are k distinct Zumkeller numbers.

In 2001, Sethuraman and Selvaraju [9] have introduced a graph operation called super subdivision of graph, denoted $SSD(G)$ if $SSD(G)$ is obtained from G by replacing every edge xy of G by a complete bipartite graph $K_{2,t}$ by in such a way that the end vertices x, y of each edge are merged with the two vertices of 2-vertices part of $K_{2,t}$ after removing the edge xy from G (in the complete bipartite graph $K_{2,t}$ the part consisting of two vertices is referred as 2-vertices part of $K_{2,t}$ and the part consisting of t vertices is referred as t-vertices part of $K_{2,t}$). In graph theory, subdivision is a significant aspect that allows one to calculate properties of some complex graphs in terms of some easier graphs. Subdivision graphs are used to drive many mathematical and chemical properties of more complex graphs from more basic graphs and there are many results on these graphs, so it helps to study the physical, chemical properties of the object which is modeled by the graph [10].

Zumkeller numbers: definition and properties

In this section, we survey the notations of Zumkeller numbers and some properties of Zumkeller numbers. A positive integer n is known as a perfect number if the sum of its proper positive factors is equal to n. Generalizing the notion of perfect numbers, Zumkeller presented in Encyclopedia of Integer Sequences [5] A083207 a sequence of integers that their positive factors can be separated into two disjoint subsets with equal sum.

Definition 2.1 Let n be a positive number, if all the positive factors of n can be partitioned into two disjoint subsets such that the sums of the two subsets are equal. Then n is called a Zumkeller number.

This partition is called as a Zumkeller partition. For example, all the following numbers 6, 12, 20, 24, 28, 30 are Zumkeller numbers.

Properties of Zumkeller Numbers:

1. If the prime factorization of an even Zumkeller number n is $2^k p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$. Hence, at least one of k_i must be an odd number.
2. Let n be a Zumkeller number and p be a prime where $(n, p) = 1$, then np^ℓ is a Zumkeller number for any positive integer ℓ.
3. If $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$ is the prime factorization of the Zumkeller number n. Hence, for any positive integers $\ell_1, \ell_2, \ldots, \ell_m$. The number $p_1^{k_1+\ell_1(k_1+1)} p_2^{k_2+\ell_2(k_2+1)} \ldots p_m^{k_m+\ell_m(k_m+1)}$ is also Zumkeller number.

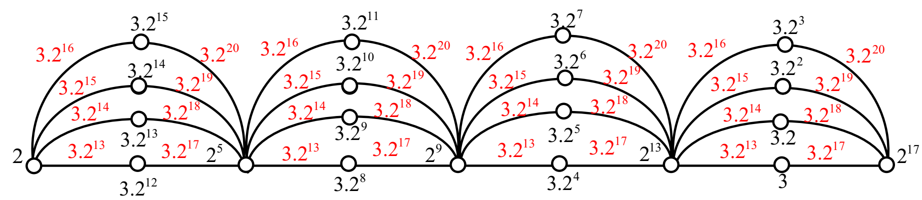

Fig. 1 8-Zumkeller labeling of $SSD(P_5)$

4. Let $p \neq 2$ be a prime number and let k be a positive integer with $p \leq 2^{k+1} - 1$. Then, $2^k p$ is a Zumkeller number.

Main results

In this section, we prove that the super subdivision of graphs such as path P_n, cycle C_n, comb $P_n \odot K_1$, crown $C_n \odot K_1$, ladder L_n, circular ladder CL_n, planer grid $P_m \times P_n$ and prism $C_m \times P_n$ are k-Zumkeller graphs.

Theorem 3.1 *$SSD(P_n)$ admits a $2t$-Zumkeller labeling for all $n \geq 2$.*

Proof Let P_n be a path with vertex set $V = \{u_i : 1 \leq i \leq n\}$ and edge set $E = \{e_i = u_i u_{i+1} : 1 \leq i \leq n-1\}$. Then, the super subdivision of the path P_n, $SSD(P_n)$ is obtained by replacing each edge e_i by a complete bipartite graph $K_{2,t}$ for some $t \in N$. Let $u^k_{i(i+1)} (1 \leq i \leq n-1, 1 \leq k \leq t)$ be the vertices of t-vertex part of the complete bipartite $K_{2,t}$. From the constructions, we observe that the graph $SSD(P_n)$ has $(n-1)t + n$ vertices and $2(n-1)t$ edges, where t is the number of vertices in the t-vertices part of $K_{2,t}$. We define $f : V(SSD(P_n)) \rightarrow N$ as follows:

For $1 \leq k \leq t$.

$$f(u_i) = 2^{ti+(1-t)}, 1 \leq i \leq n$$
$$f(u^k_{i(i+1)}) = p2^{t(n-1)-ti+(k-1)}, 1 \leq i \leq n-1.$$

where $p < 10, p \neq 2$ is a prime number. Hence the labels of the edges of $SSD(P_n)$ are given as follows:

$$f^*(u_i u^k_{i(i+1)}) = p2^{t(n-2)+k}, \tag{1}$$

$$f^*(u^k_{i(i+1)} u_{(i+1)}) = p2^{t(n-1)+k}. \tag{2}$$

From Eqs. (1), (2), it is observed that the edges of $SSD(P_n)$ receive only $2t$ distinct Zumkeller numbers $p2^{t(n-2)+k}, p2^{t(n-1)+k}$ for $1 \leq k \leq t$. Hence $SSD(P_n)$ graph admits a $2t$-Zumkeller labeling. □

Illustration 3.1 *Figure 1 illustrates the super subdivision of P_5 and its 8-Zumkeller labeling, where $t = 4$ and $p = 3$.*

Theorem 3.2 *$SSD(C_n)$ admits a 3t-Zumkeller labeling for all $n \geq 2$.*

Proof Let C_n be a cycle with vertices set $V = \{u_i : 1 \leq i \leq n\}$ and edge set $E = \{e_i = u_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{e_n = u_nu_1\}$. Let $SSD(C_n)$ be a super subdivision of the cycle C_n. Let $u^k_{i(i+1)}$ and u^k_{n1} $(1 \leq i \leq n-1, 1 \leq k \leq t)$ be the vertices of the t-vertices part of $K_{2,t}$ which are used for super subdivision of C_n. Thus $SSD(C_n)$ has $p = n(1+t)$ vertices and $q = 2nt$ edges. We define $f : V(SSD(C_n)) \to N$ as follows:

For $1 \leq k \leq t$.

$$f(u_i) = 2^{ti+(1-t)}, 1 \leq i \leq n$$
$$f(u^k_{i(i+1)}) = p2^{tn-ti+(k-1)}, 1 \leq i \leq n-1$$
$$f(u^k_{n1}) = p2^{k-1}.$$

where $p < 10, p \neq 2$ is a prime number. Then the labels of the edges of $SSD(C_n)$ are given as follows:

For $1 \leq i \leq n-1$.

$$f^*(u_iu^k_{i(i+1)}) = f^*(u_nu^k_{n1}) = p2^{t(n-1)+k}, \tag{3}$$

$$f^*(u^k_{i(i+1)}u_{(i+1)}) = p2^{tn+k}, \tag{4}$$

$$f^*(u^k_{n1}u_1) = p2^k. \tag{5}$$

From Eqs. (3) to (5), it is observed that the edges of $SSD(C_n)$ receive only $3t$ distinct Zumkeller numbers $p2^{t(n-1)+k}, p2^{tn+k}$ and $p2^k$ for $1 \leq k \leq t$. Hence $SSD(C_n)$ graph admits a $3t$-Zumkeller labeling. □

Illustration 3.2 *Figure 2 illustrates the super subdivision of C_8 and its 9-Zumkeller labeling, where $t = 3$ and $p = 3$.*

Theorem 3.3 *$SSD(P_n \odot K_1)$ admits a 3t-Zumkeller labeling for all $n \geq 3$.*

Proof Let $V = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and $E = \{e_i = u_iu_{i+1}, \hat{e}_i = u_iv_i : 1 \leq i \leq n-1\} \bigcup \{\hat{e}_n = u_nv_n\}$ be the edge set of the comb graph. Suppose that $u^k_{i(i+1)}, w^k_i (1 \leq i \leq n-1, 1 \leq k \leq t)$ and w^k_n are the vertices of the t-vertices part of $K_{2,t}$ which are used for super subdivision of $P_n \odot K_1$. Hence, we observe

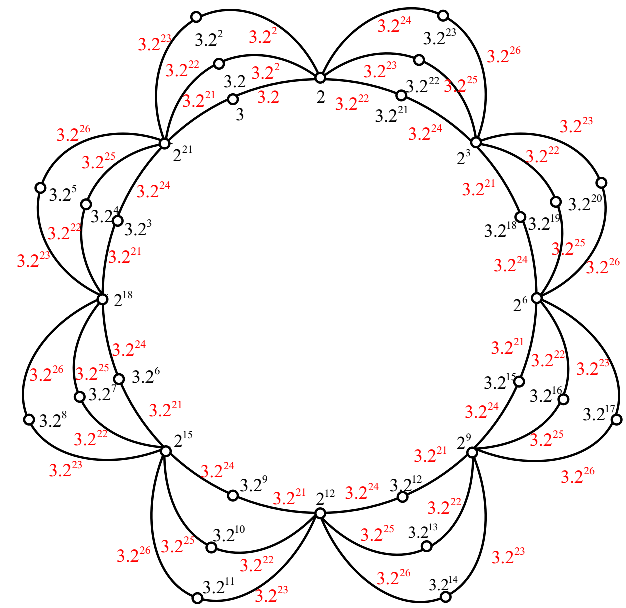

Fig. 2 9-Zumkeller labeling of $SSD(C_8)$

that the graph $SSD(P_n \odot K_1)$ has $2tn + (2n - t)$ vertices and $4tn - 2t$ edges. Define labeling $f : V(SSD(P_n \odot K_1)) \rightarrow N$ as follows:

For $1 \leq k \leq t$.

$$f(u_i) = 2^{2ti+(1-t)}, 1 \leq i \leq n$$
$$f(v_i) = 2^{2ti+(1-2t)}, 1 \leq i \leq n$$
$$f(u^k_{i(i+1)}) = p2^{2t(n-i)-(1+t)+k}, 1 \leq i \leq n-1$$
$$f(w^k_i) = p2^{2t(n-i)+(k-1)}, 1 \leq i \leq n-1.$$

Where $p < 10, p \neq 2$ is a prime number. So the labels of the edges of $SSD(P_n \odot K_1)$ are given as follows:

For $1 \leq i \leq n-1, 1 \leq k \leq t$

$$f^*(u_i u^k_{i(i+1)}) = f^*(w^k_i v_i) = f^*(w^k_n v_n) = p2^{2t(n-1)+k}, \tag{6}$$

$$f^*(u^k_{i(i+1)} u_{(i+1)}) = p2^{2tn+k}, \tag{7}$$

$$f^*(u_i w^k_i) = p2^{t(2n-1)+k}. \tag{8}$$

From Eqs. (6) to (8), it is noticed that the edges of $SSD(P_n \odot K_1)$ receive only $3t$ distinct Zumkeller numbers $p2^{2t(n-1)+k}, p2^{2tn+k}$ and $p2^{t(2n-1)+k}$ for $1 \leq k \leq t$. Hence $SSD(P_n \odot K_1)$ graph admits a $3t$-Zumkeller labeling. □

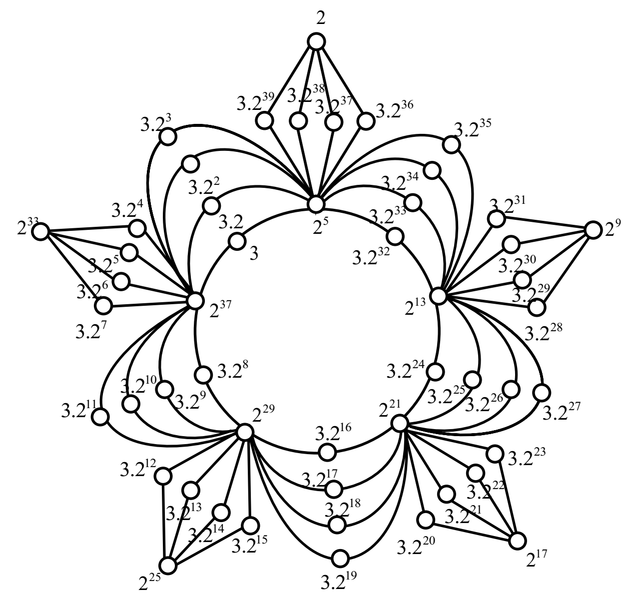

Fig. 3 15-Zumkeller labeling of $SSD(P_5 \odot K_1)$

Illustration 3.3 *Figure 3 illustrates the super subdivision of* $P_5 \odot K_1$ *and its* 15*-Zumkeller labeling, where* $t = 4$ *and* $p = 3$.

Theorem 3.4 $SSD(L_n)$ *admits a* $4t$*-Zumkeller labeling for all* $n \geq 3, t \geq 1$.

Proof We get the ladder graph L_n by attaching the vertices v_i and v_{i+1} for $1 \leq i \leq n-1$ of the comb graph $P_n \odot K_1$ in Theorem 3.3, with additional edges. Let $V = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and $E = \{e_i = u_i u_{i+1}, \hat{e}_i = u_i v_i, \bar{e}_i = v_i v_{i+1} : 1 \leq i \leq n-1\} \bigcup \{\hat{e}_n = u_n v_n\}$ be the edge set of the ladder graph. By the definition of super subdivision of L_n, we can consider $u^k_{i(i+1)}, v^k_{i(i+1)}, w^k_i (1 \leq i \leq n-1, 1 \leq k \leq t)$ and w^k_n are the vertices of the $t-$vertices part of $k_{2,t}$. Then, it is clear that $SSD(L_n)$ has $3tn + 2(n-t)$ vertices and $6tn - 4t$ edges. Define labeling $f : V(SSD(L_n)) \rightarrow N$ as follows:

For $1 \leq i \leq n\,, 1 \leq k \leq t$.

$$f(u_i) = \begin{cases} 2^{3ti+(1-3t)}, & i \text{ is odd} \\ 2^{3ti+(1-2t)}, & i \text{ is even.} \end{cases}$$

$$f(v_i) = \begin{cases} 2^{3ti+(1-2t)}, & i \text{ is odd} \\ 2^{3ti+(1-3t)}, & i \text{ is even.} \end{cases}$$

$f(w^k_i) = p2^{3t(n-i)+k-1}$.

For $1 \leq i \leq n-1$.

$$f(u_{i(i+1)}^{k}) = \begin{cases} p2^{3t(n-i)-(2t+1)+k}, & i \text{ is odd} \\ p2^{3t(n-i)-(t+1)+k}, & i \text{ is even.} \end{cases}$$
$$f(v_{i(i+1)}^{k}) = \begin{cases} p2^{3t(n-i)-(t+1)+k}, & i \text{ is odd} \\ p2^{3t(n-i)-(2t+1)+k}, & i \text{ is even.} \end{cases}$$

Where p is a prime number greater than 2 but less than 10. Therefore, the labels of the edges of $SSD(L_n)$ are given as follows:

For $1 \leq i \leq n-1$.

$$f^{*}(u_i u_{i(i+1)}^{k}) = \begin{cases} p2^{3tn-5t+k}, & i \text{ is odd} \\ p2^{3tn-3t+k}, & i \text{ is even.} \end{cases} \tag{9}$$

$$f^{*}(v_i v_{i(i+1)}^{k}) = \begin{cases} p2^{3tn-3t+k}, & i \text{ is odd} \\ p2^{3tn-5t+k}, & i \text{ is even.} \end{cases} \tag{10}$$

$$f^{*}(u_{i(i+1)}^{k} u_{(i+1)})* = f^{*}(v_{i(i+1)}^{k} v_{(i+1)}) = p2^{3tn-t+k}. \tag{11}$$

For $1 \leq i \leq n$.

$$f^{*}(u_i w_i^{k}) = \begin{cases} p2^{3tn-3t+k}, & i \text{ is odd} \\ p2^{3tn-2t+k}, & i \text{ is even.} \end{cases} \tag{12}$$

$$f^{*}(v_i w_i^{k}) = \begin{cases} p2^{3tn-2t+k}, & i \text{ is odd} \\ p2^{3tn-3t+k}, & i \text{ is even.} \end{cases} \tag{13}$$

From Eqs. (9) to (13), it is observed that the edges of $SSD(L_n)$ receive only $4t$ distinct Zumkeller numbers $p2^{3tn-3t+k}, p2^{3tn-5t+k}, p2^{3tn-2t+k}$

and $p2^{3tn-t+k}$ for $1 \leq k \leq t$. Hence $SSD(L_n)$ graph admits a $4t-$Zumkeller labeling. □

Illustration 3.4 *Figure 4 illustrates the super subdivision of $SSD(L_5)$ and its 12-Zumkeller labeling where, $t = 3$ and $p = 3$.*

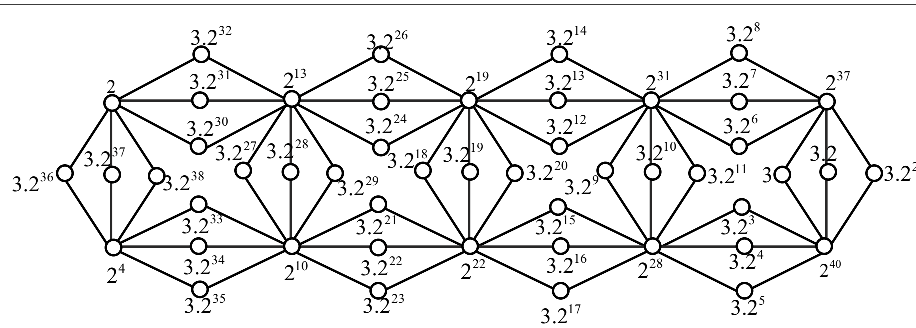

Fig. 4 12-Zumkeller labeling of $SSD(L_5)$

Theorem 3.5 *$SSD(C_n \odot K_1)$ admits a $4t$-Zumkeller labeling for all $n \geq 2$.*

Proof Let $V = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and $E = \{e_i = u_i u_{i+1}, \hat{e}_i = u_i v_i : 1 \leq i \leq n-1\} \bigcup \{e_n = u_n u_1, \hat{e}_n = u_n v_n\}$ be the edge set of the crown graph. Suppose that $u^k_{i(i+1)}, w^k_i$ $(1 \leq i \leq n-1, 1 \leq k \leq t)$,u^k_{n1} and w^k_n are the vertices of the t-vertices part of $K_{2,t}$ which are used for super subdivision of $C_n \odot K_1$. Then, it obvious that the graph $SSD(C_n \odot K_1)$ has $2tn + 2n$ vertices and $4tn$ edges. The labels of vertices u_i, v_i for $1 \leq i \leq n$ are given as in Theorem 3.4. For the vertices $u^k_{i(i+1)}, w^k_i$ and u^k_{n1}, we define the vertex function f as follows:

For $1 \leq k \leq t$

$$\begin{aligned} f(u^k_{i(i+1)}) &= p2^{2t(n-i)+k-1}, \qquad 1 \leq i \leq n-1 \\ f(w^k_i) &= p2^{2t(n-i)+(t-1)+k}, \qquad 1 \leq i \leq n \\ f(u^k_{n1}) &= p2^{k-1}. \end{aligned}$$

Where $p < 10, p \neq 2$ is a prime number. Now the labels of the edges of $SSD(C_n \odot K_1)$ are given as follows:

For $1 \leq i \leq n-1$

$$f^*(u_i u^k_{i(i+1)}) = f^*(u_n u^k_{n1}) = p2^{2tn-t+k} \tag{14}$$

$$f^*(w^k_i v_i) = f^*(w^k_n v_n) = p2^{2tn-t+k} \tag{15}$$

$$f^*(u^k_{i(i+1)} u_{(i+1)}) = p2^{2tn+t+k} \tag{16}$$

$$f^*(u^k_{n1} u_1) = p2^{t+k} \tag{17}$$

$$f^*(u_i w^k_i) = p2^{2tn+k}. \tag{18}$$

From Eqs. (14) to (18), it is noticed that the edges of $SSD(C_n \odot K_1)$ receive only $4t$ distinct Zumkeller numbers $p2^{2tn-t+k}, p2^{2tn+k-1}, p2^{t+k}$ and $p2^{2tn+k}$ for $1 \leq k \leq t$. Hence $SSD(C_n \odot K_1)$ graph admits a $4t$-Zumkeller labeling. □

Illustration 3.5 Figure 5 illustrates the super subdivision of $SSD(C_5 \odot K_1)$ and its 16-Zumkeller labeling where, $t = 4$ and $p = 3$.

Theorem 3.6 For $n \geq 2$, *$SSD(CL_n)$ admits a $6t$-Zumkeller labeling if n is odd and $5t$-Zumkeller labeling if n is even.*

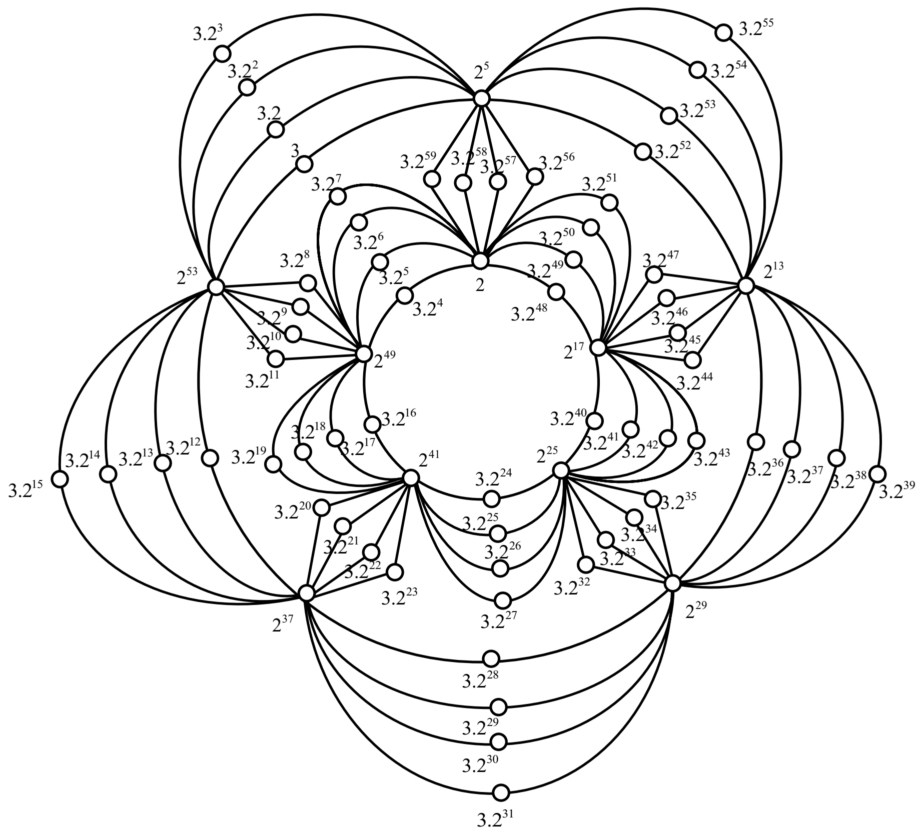

Fig. 5 16-Zumkeller labeling of $SSD(C_8 \odot K_1)$

Proof By joining the vertices u_n, u_1 and v_n, v_1 of the ladder graph L_n in Theorem 3.5, with additional edges we get the circular ladder graph CL_n. Let $V = \{u_i, v_i : 1 \leq i \leq n\}$be the vertex set and $E = \{e_i = u_i u_{i+1}, \bar{e_i} = v_i v_{i+1}, \hat{e}_i = u_i v_i : 1 \leq i \leq n-1\} \bigcup \{e_n = u_n u_1, \bar{e_n} = u_n u_1, \hat{e}_n = u_n v_n\}$. be the edge set of the circular graph. Let $SDD(CL_n)$ be the graph obtained by super subdivision of CL_n by a complete bipartite graph $K_{2,t}$. Thus, $SDD(CL_n)$ has $3tn + 2n$ vertices and $6tn$ edges. The labels of vertices u_i, v_i for $1 \leq i \leq n$ are given as in Theorem 3.5. For the vertices $u^k_{i(i+1)}, v^k_{i(i+1)}$ and w^k_i we define the vertex function *f* as follows:

Case (I): If n is odd.

For $1 \leq i \leq n-1$,$1 \leq k \leq t$.

$$f(u^k_{i(i+1)}) = \begin{cases} p2^{3t(n-i)+k-1}, & i \text{ is odd} \\ p2^{3t(n-i)+(t-1)+k}, & i \text{ is even.} \end{cases}$$
$$f(v^k_{i(i+1)}) = \begin{cases} p2^{3t(n-i)+(t-1)+k}, & i \text{ is odd} \\ p2^{3t(n-i)+k-1}, & i \text{ is even.} \end{cases}$$
$$f(w^k_i) = p2^{3t(n-i)+(2t-1)+k}$$
$$f(u^k_{n1}) = p2^{k+t-1}$$
$$f(v^k_{n1}) = p2^k.$$

where p is a prime number greater than 2 but less than 10. Then the labels of the edges of $SSD(CL_n)$ are given as follows:

$$f^*(u_iu^k_{i(i+1)}) = \begin{cases} p2^{3tn-3t+k}, & i \text{ is odd} \\ p2^{3tn-t+k}, & i \text{ is even.} \end{cases} \tag{19}$$

$$f^*(v_iv^k_{i(i+1)}) = \begin{cases} p2^{3tn-t+k}, & i \text{ is odd} \\ p2^{3tn-3t+k}, & i \text{ is even.} \end{cases} \tag{20}$$

$$f^*(u_iw^k_i) = \begin{cases} p2^{3tn-t+k}, & i \text{ is odd} \\ p2^{3tn+k}, & i \text{ is even.} \end{cases} \tag{21}$$

$$f^*(v_iw^k_i) = \begin{cases} p2^{3tn+k}, & i \text{ is odd} \\ p2^{3tn-t+k}, & i \text{ is even.} \end{cases} \tag{22}$$

$$f^*(v_nv^k_{n1}) = f^*(u_nu^k_{n1}) = p2^{3tn-2t+k} \tag{23}$$

$$f^*(u^k_{i(i+1)}u_{(i+1)}) = f^*(v^k_{i(i+1)}v_{(i+1)}) = p2^{3tn+t+k} \tag{24}$$

$$f^*(u^k_{n1}u_1) = f^*(v^k_{n1}v_1) = p2^{k+t}. \tag{25}$$

From Eqs. (19) to (25), it is noticed that the edges of $SSD(CL_n)$ receive only $6t$ distinct Zumkeller numbers $p2^{3tn-3t+k}, p2^{3tn-t+k}, p2^{3tn+k}, p2^{3tn+t+k}$,

$p2^{k+1}$ and $p2^{3tn-2t+k}$ for $1 \leq k \leq t$. Hence $SSD(CL_n)$ graph admits a $6t$-Zumkeller labeling when n is odd.

Case (II): If n is even. It is easy verified that the edge labels of $u_nu^k_{n1}$, $v_nv^k_{n1}$ equal to the edge labels of $v_iw^k_i$, $v_iv^k_{(i(i+1)}$, respectively, for i is even. Then, the edges of $SSD(CL_n)$ receive only $5t$ distinct Zumkeller numbers $p2^{3tn-3t+k}, p2^{3tn-t+k}, p2^{3tn+k}, p2^{3tn+t+k}$

and $p2^{k+1}$ for $1 \leq k \leq t$. Hence $SSD(CL_n)$ graph admits a $5t$-Zumkeller labeling when n is even. □

Illustration 3.6 *Figure 6 illustrates the super subdivision of* $SSD(CL_5)$ *and its* 24*-Zumkeller labeling where,* $t = 4$ *and* $p = 3$.

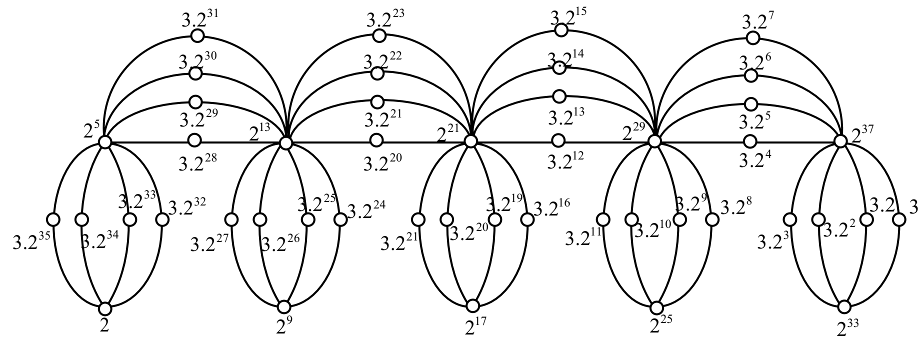

Fig. 6 24-Zumkeller labeling of $SSD(CL_5)$

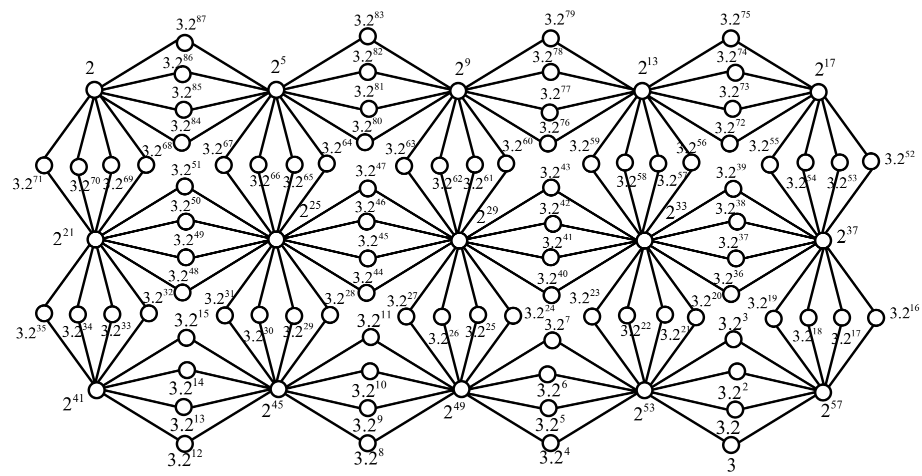

Fig. 7 15-Zumkeller labeling of $SSD(CL_4)$

Illustration 3.7 *Figure 7 illustrates the super subdivision of $SSD(CL_4)$ and its 15-Zumkeller labeling where, $t = 3$ and $p = 3$.*

Theorem 3.7 For *$SSD(P_m \times P_n)$ admits a $2mt$-Zumkeller labeling for all $m, n \geq 3$.*

Proof Let $V = \{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ be the vertex set and $E = \{e_{ij} = u_{ij}u_{i(j+1)}, \hat{e}_{ij} = u_{ij}u_{(i+1)j} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\}$ be the edge set of the planar grid $P_m \times P_n$. Let $SSD(P_m \times P_n)$ be the graph obtained by super subdivision of $P_m \times P_n$ in which the edges $e_{ij}, \hat{e}_{ij}$ of $P_m \times P_n$ are replaced by a complete bipartite graph $k_{2,t}$. Let $u^k_{i,j(j+1)}, u^k_{i(i+1),j}$ be the vertices of t-vertices part where $1 \leq i \leq m-1, 1 \leq j \leq n-1$ and $1 \leq k \leq t$. Thus, graph $G = SSD(P_m \times P_n)$ has $mn(1+2t) - (m+n)t$ vertices and $4mnt - 2t(m+n)$ edges. Define labeling $f : V(SSD(P_m \times P_n)) \to N$ as follows: For $1 \leq k \leq t$.

$$f(u_{ij}) = 2^{1+nt(i-1)+t(j-1)} \qquad 1 \le i \le m, 1 \le j \le n$$
$$f(u^k_{i,j(j+1)}) = p2^{(2mn-m+n-1)t-(2n-1)ti-tj+(k-1)}, \qquad 1 \le i \le m, 1 \le j \le n-1$$
$$f(u^k_{i(i+1),j}) = p2^{(2mn-m)t-(2n-1)ti-tj+(k-1)}, \qquad 1 \le i \le m-1, 1 \le j \le n.$$

where $p < 10, p \ne 2$ is a prime number. Thus the labels of the edges of $SSD(P_m \times P_n)$ are given as follows:

$$f^*(u_{ij}u^k_{i,j(j+1)}) = p2^{(2mn-m-2)t-(n-1)ti+k}, \qquad 1 \le i \le m \tag{26}$$

$$f^*(u^k_{i,j(j+1)}u_{i(j+1)}) = p2^{(2mn-m-1)t-(n-1)ti+k} \qquad 1 \le i \le m \tag{27}$$

$$f^*(u_{ij}u^k_{i(i+1),j}) = p2^{(2mn-m-n-1)t-(n-1)ti+k}, \qquad 1 \le i \le m-1 \tag{28}$$

$$f^*(u^k_{i(i+1),j}u_{(i+1)j}) = p2^{(2mn-m-1)t-(n-1)ti+k}, \qquad 1 \le i \le m-1. \tag{29}$$

From Eqs. (26) to (29), it is observed that the edges of $SSD(P_m \times P_n)$ receive Zumkeller numbers and the numbers $p2^{(2mn-m-n-1)t-(n-1)ti+k}, 1 \le i \le m-1$ are equal to the numbers $p2^{(2mn-m-2)t-(n-1)ti+k}, 2 \le i \le m$. From Table 1, we found that the number of Zumkeller numbers used to label the edges is $2mt$. □

Illustration 3.8 *Figure 8 illustrates the super subdivision of $SSD(P_3 \times P_5)$ and its 24-Zumkeller labeling where, $t = 4$ and $p = 3$.*

Theorem 3.8 *$SSD(C_m \times P_n)$ admits a $(2m+1)t$-Zumkeller labeling for all $m, n \ge 3$.*

Proof By joining the vertices u_{i1} and u_{in}, $1 \le i \le m$ of the planar grid $(P_m \times P_n)$ in Theorem 3.7, we obtain the prism graph $C_m \times P_n$. Let $V(C_m \times P_n) = \{u_{ij}, 1 \le i \le m, 1 \le j \le n\}$ and

Table 1 Zumkeller numbers used to label the edges of $SSD(P_m \times P_n)$

Edges	Labels	Num. of labels
$u_{1j}u^k_{1,j(j+1)}$	$p2^{(2mn-m-n-1)t+k}$	$(n-1)t$
$u_{ij}u^k_{i,j(j+1)}$, $u_{(i-1)j}u^k_{(i-1)i,j}$, $2 \le i \le m$	$p2^{(2mn-m-1)t-(n-1)ti+k}$	$(m-1)(n-1)t$
$u^k_{i,j(j+1)}u_{i(j+1)}$, $u^k_{i(i+1),j}u_{(i+1)j}$, $1 \le i \le m-1$	$p2^{(2mn-m-1)t-(n-1)ti+k}$	$(m-1)(n-1)t$
$u^k_{m,j(j+1)}u_{m(j+1)}$	$p2^{(mn-1)t+k}$	$(n-1)t$

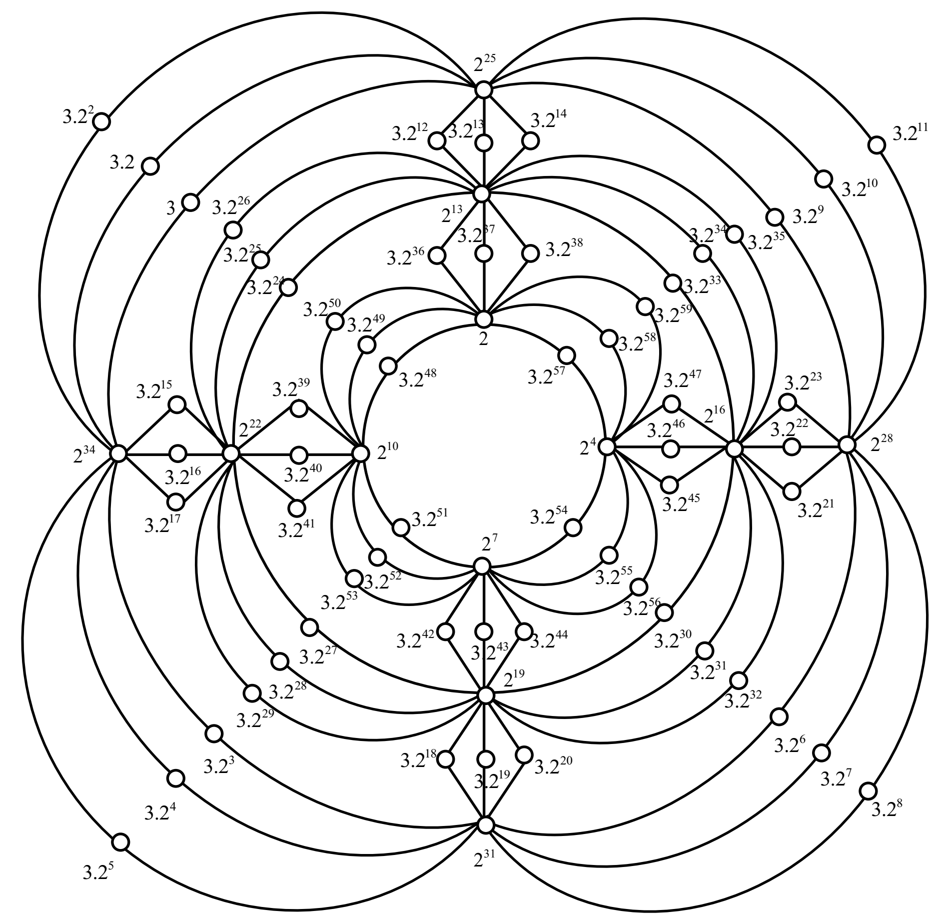

Fig. 8 24-Zumkeller labeling of $SSD(P_3 \times P_5)$

$E = \{e_{ij} = u_{ij}u_{i(j+1)}, \hat{e}_{ij} = u_{ij}u_{(i+1)j} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \bigcup \{e_{in} = u_{i1}u_{in}, 1 \leq i \leq m\}$. By the definition of super subdivision of $C_m \times P_n$, we can consider $u^k_{i,j(j+1)}, u^k_{i(i+1),j}, u^k_{i(i+1),n}$ and $u^k_{i,n1}$ are the vertices of t-vertices part where $1 \leq i \leq m-1, 1 \leq j \leq n-1$ and $1 \leq k \leq t$. Hence, it is clear that $SSD(C_m \times P_n)$ has $mn(1+2t) - (m+n)t$ vertices and $4mnt - 2nt$ edges. The labels of vertices u_{ij} for $1 \leq i \leq m, 1 \leq j \leq n$ are given as in Theorem 3.7. For the vertices $u^k_{i,j(j+1)}, u^k_{i(i+1),j}, u^k_{i(i+1),n}$ and $u^k_{i,n1}$ we define the vertex function f as follows:

For $1 \leq k \leq t$

$$\begin{aligned} f(u^k_{i,j(j+1)}) &= p2^{(2mn+n)t-2nti-tj+k-1}, \quad 1 \leq i \leq m, 1 \leq j \leq n-1 \\ f(u^k_{i,n1}) &= p2^{2mnt-2nti+k-1}, \qquad 1 \leq i \leq m \\ f(u^k_{i(i+1),1}) &= p2^{(2mn-n)t-2nti+k-1}, \qquad 1 \leq i \leq m-1 \\ f(u^k_{i(i+1),j}) &= p2^{(2mn+1)t-2nti-tj+k-1}, \quad 1 \leq i \leq m-1, 2 \leq j \leq n. \end{aligned}$$

Where $p < 10$, $p \neq 2$ is a prime number. Then the labels of the edges of $SSD(C_m \times P_n)$ are given as follows:

For $1 \leq i \leq m, 1 \leq j \leq n-1$.

$$f^*(u_{ij}u^k_{i,j(j+1)}) = f^*(u_{in}u^k_{i,n1}) = p2^{(2mn-1)t-nti+k} \tag{30}$$

$$f^*(u^k_{i,j(j+1)}u_{i(j+1)}) = p2^{2mnt-nti+k} \tag{31}$$

$$f^*(u^k_{i,n1}u_{i1}) = p2^{n(2m-1)t-nti+k}. \tag{32}$$

For $1 \leq i \leq m-1, 2 \leq j \leq n$.

$$f^*(u_{i1}u^k_{i(i+1),1}) = p2^{2mnt-nt(i+2)+k} \tag{33}$$

$$f^*(u_{ij}u^k_{i(i+1),j}) = f^*(u^k_{i(i+1),1}u_{(i+1)1}) = p2^{n(2m-1)t-nti+k} \tag{34}$$

$$f^*(u^k_{i(i+1),j}u_{(i+1)j}) = p2^{2mnt-nti+k}. \tag{35}$$

Table 2 Zumkeller numbers used to label the edges of $SSD(C_m \times P_n)$

Edges	Num. of edges	Labels	Num. of labels
$u_{ij}u^k_{i,j(j+1)}$, $u_{in}u^k_{i,n1}$ $1 \leq i \leq m$, $1 \leq j \leq n-1$	mnt	$p2^{(2mn-1)t-nti+k}$	mt
$u^k_{i,j(j+1)}u_{i(j+1)}$ $1 \leq i \leq m$, $1 \leq j \leq n-1$ $u_{(i-1)j}u^k_{(i-1)i,j}$ $2 \leq i \leq m$, $2 \leq j \leq n-1$ $u^k_{(i-1),n1}u_{(i-1)1}$, $2 \leq i \leq m$ $u^k_{i(i+1),j}u_{(i+1)j}$ $1 \leq i \leq m$, $2 \leq j \leq n-1$ $u_{(i-2)1}u^k_{(i-2)(i-1),1}$ $3 \leq i \leq m$, $1 \leq j \leq n-1$ $u^k_{(i-1)i,1}u_{i1}$, $2 \leq i \leq m$	$(n-1)mt$ $(n-1)(m-1)t$ $(m-1)t$ $(m-1)t$ $(m-2)t$ $(m-1)t$	$p2^{2mnt-nti+k}$	mt
$u^k_{m,n1}u_{m1}$, $u_{(m-1)1}u^k_{(m-1)m,1}$	$2t$	$p2^{mnt-nt+k}$	t

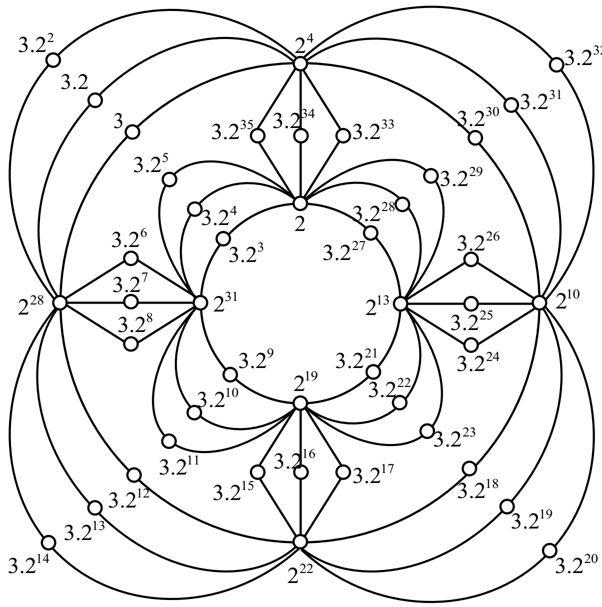

Fig. 9 21-Zumkeller labeling of $SSD(C_3 \times P_4)$

From Eqs. (30) to (35), it is observed that the edges of $SSD(C_m \times P_n)$ receive Zumkeller numbers. Now the total number of edges is $4mnt - 2nt$ and from Table 2 we found that the number of Zumkeller numbers used to label the edges is $(2m+1)t$. □

Illustration 3.9 *Figure 9 illustrates the super subdivision of* $SSD(C_3 \times P_4)$ *and its* 21-*Zumkeller labeling where,* $t = 3$ and $p = 3$.

Conclusions

From the previous sections, we studied the k-Zumkeller labeling for some graphs. For any natural number $n \geq 2$, we show that the super subdivision of the path P_n is $2t$-Zumkeller graph, while the super subdivision of the cycle C_n and the comb $P_n \odot K_1$, $n \geq 3$ are $3t$-Zumkeler graphs. Also, we prove that the super subdivision of the ladder L_n, $n \geq 3$ and the crown $C_n \odot K_1$, $n \geq 2$ admit $4t$-Zumkeller labeling. Moreover, we show that the super subdivision of the circular ladder CL_n, $n \geq 2$ has a $6t$-Zumkeller labeling when n is odd and a $5t$-Zumkeller labeling when n is even. Finally, for $m, n \geq 3$ we prove that the planer grid $P_m \times P_n$ is $2mt$-Zumkeller graph, however the prism $C_m \times P_n$ is $(2m+1)t$ -Zumkeller graph, where t is the number of vertices of t-vertices part.

Abbreviations
SSD(G): Super subdivision of graph *G*.

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MB carried out all theorems, proofs and drafted the manuscript by himself without any participation. All authors read and approved the final manuscript.

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Declarations

Competing interests
The author declare that no competing interests.

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