Dilip and Mathew *J Egypt Math Soc* (2021) 29:10
https://doi.org/10.1186/s42787-021-00119-6

Journal of the Egyptian Mathematical Society

ORIGINAL RESEARCH

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Dynamics of a second-order nonlinear difference system with exponents

D. S. Dilip[1,2*] and Smitha Mary Mathew[1,2]

*Correspondence: dilip@stjohns.ac.in
[1] Department of Mathematics, St. John's College, Anchal, Kerala, India
Full list of author information is available at the end of the article

Abstract

In this paper, we study the persistence, boundedness, convergence, invariance and global asymptotic behavior of the positive solutions of the second-order difference system $\begin{array}{l} x_{n+1} = \alpha_1 + ae^{-x_{n-1}} + by_n e^{-y_{n-1}}, \\ y_{n+1} = \alpha_2 + ce^{-y_{n-1}} + dx_n e^{-x_{n-1}} \quad n = 0, 1, 2, \ldots \end{array}$ where $\alpha_1, \alpha_2, a, b, c, d$ are positive real numbers and the initial conditions x_{-1}, x_0, y_{-1}, y_0 are arbitrary nonnegative numbers.

Keywords: Local behavior, Global behavior, Invariance, Persistence, Boundedness

AMS Subject Classification: 39A22

Introduction

The theory of discrete dynamical system has many applications in applied sciences. Mathematical modeling of a physical, biological or ecological problem mostly leads to a nonlinear difference system. (See [1–10].)

In [4], Papachinopoulos et al. proposed a system of equation with exponents as

$$f_{n+1} = a + bf_{n-1}e^{-g_n}, g_{n+1} = c + dg_{n-1}e^{-f_n}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where a, b, c, d and the initial conditions f_{-1}, f_0, g_{-1}, g_0 are positive real values. They studied the existence, boundedness and asymptotic behavior of the positive solutions of (1).

In [5], G.Papaschinopoulos and C.J.Schinas together modified the system as

$$\begin{aligned} f_{n+1} &= a + bg_{n-1}e^{-f_n}, g_{n+1} = c + \mathrm{d}f_{n-1}e^{-g_n}, \\ f_{n+1} &= a + bg_{n-1}e^{-g_n}, g_{n+1} = c + \mathrm{d}f_{n-1}e^{-f_n}, \end{aligned} \tag{2}$$

and put forward conditions for the positive solutions to be asymptotic.

In [11], authors multiplied f_n and g_n with a and c, respectively, in (2) and formed a new system of difference equations

$$f_{n+1} = af_n + bg_{n-1}e^{-f_n}, g_{n+1} = cg_n + \mathrm{d}f_{n-1}e^{-g_n}, n = 0, 1, \ldots$$

and described the existence of a unique positive equilibrium, the boundedness, persistence and global attractivity of the positive solutions.

Parallelly in [12], the authors worked on the asymptotic behavior of the positive solutions of a similar difference system

$$f_{n+1} = ag_n + bf_{n-1}e^{-g_n}, g_{n+1} = cf_n + \mathrm{d}g_{n-1}e^{-f_n}, n = 0, 1, \ldots.$$

N.Psarros and G.Papaschinopoulos in [13] proposed a new first-order model

$$f_{n+1} = ag_n + bf_n e^{-f_n - g_n}, g_{n+1} = cf_n + \mathrm{d}g_n e^{-f_n - g_n},$$

and studied the asymptotic behavior of the positive solutions of the system.

Motivated by the above research articles, we propose a new second order difference system

$$\begin{aligned} x_{n+1} &= \alpha_1 + ae^{-x_{n-1}} + by_n e^{-y_{n-1}}, \\ y_{n+1} &= \alpha_2 + ce^{-y_{n-1}} + dx_n e^{-x_{n-1}} \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3}$$

where $\alpha_1, \alpha_2, a, b, c, d$ are positive real numbers and the initial conditions x_{-1}, x_0, y_{-1}, y_0 are arbitrary nonnegative numbers, and investigate the persistence, boundedness, convergence, invariance, and global asymptotic behavior of the positive solutions of the system.

Methods

We use Theorem 1.16 of [14] to prove the lemma which we use to derive a condition for the existence, uniqueness of equilibrium solutions and the convergence of positive solutions to the equilibrium solution. We also use Remark 1.3.1 of [15] to obtain conditions for global asymptotic stability of the unique equilibrium point.

Results and discussion

The following theorem proposes conditions for persistence and boundedness for the positive solution (x_n, y_n) of (3).

Theorem 1 *Every positive solution (x_n, y_n) of (3) is bounded and persists whenever $bde^{-\alpha_1-\alpha_2} < 1$.*

Proof

$x_n \geq \alpha_1, y_n \geq \alpha_2, n = 3, 4, \ldots.$

Hence, (x_n, y_n) of system (3) persists.

Also, (3) becomes

$$\begin{aligned} x_{n+1} &\leq \alpha_1 + ae^{-\alpha_1} + be^{-\alpha_2}[\alpha_2 + \mathrm{d}x_{n-1}e^{-x_{n-2}} + ce^{-y_{n-2}}]. \\ &\leq A + bdx_{n-1}e^{-\alpha_1-\alpha_2} \end{aligned} \tag{4}$$

where $A = \alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2}$.

Similarly,

$$y_{n+1} \leq C + bdy_{n-1}e^{-\alpha_1-\alpha_2} \tag{5}$$

where $C = \alpha_2 + ce^{-\alpha_2} + d\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}$.

Now, consider the difference equations

$$\begin{aligned} z_{n+1} &= A + Bz_{n-1}. \\ v_{n+1} &= C + Dv_{n-1}, \end{aligned} \tag{6}$$

where $B = D = bde^{-\alpha_1-\alpha_2} < 1$. Therefore, an arbitrary solution (z_n, v_n) of (6) can be written as

$$z_n = r_1 B^{n/2} + r_2(-1)^n B^{n/2} + \frac{A}{1-B}, \quad n = 0, 1, 2, \ldots \tag{7}$$

$$v_n = s_1 B^{n/2} + s_2(-1)^n B^{n/2} + \frac{C}{1-B}, \quad n = 0, 1, 2, \ldots \tag{8}$$

where r_1, r_2 rely on the initial conditions z_{-1}, z_0 and s_1, s_2 rely on the initial conditions v_{-1}, v_0. Hence, (z_n, v_n) is bounded.

Let us examine the solution (z_n, v_n) such that $z_{-1} = x_{-1}, z_0 = x_0, v_{-1} = y_{-1}, v_0 = y_0$.

Hence by induction, $x_n \leq z_n$ and $y_n \leq v_n, n = 0, 1, 2, \ldots$.

Therefore, we get (x_n, y_n) is bounded. □

The following two theorems confirm the existence of invariant boxes of (3).

Theorem 2 *Let* $bde^{-\alpha_1-\alpha_2} < 1$. *Let* (x_n, y_n) *denote a positive solution of* (3). *Then* $[\alpha_1, \frac{\alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2}}{(1 - bde^{-\alpha_1-\alpha_2})}] \times [\alpha_2, \frac{\alpha_2 + ce^{-\alpha_2} + d\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}}{(1 - bde^{-\alpha_1-\alpha_2})}]$ *is an invariant set for* (3).

Proof

Let $I_1 = [\alpha_1, \frac{\alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2}}{(1 - bde^{-\alpha_1-\alpha_2})}]$ *and* $I_2 = [\alpha_2, \frac{\alpha_2 + ce^{-\alpha_2} + d\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}}{(1 - bde^{-\alpha_1-\alpha_2})}]$.

Let $x_{-1}, x_0 \in I_1$ and $y_{-1}, y_0 \in I_2$.

Then

$$\begin{aligned} x_1 &\leq \alpha_1 + ae^{-\alpha_1} + be^{-\alpha_2} y_0 \\ &\leq \alpha_1 + ae^{-\alpha_1} + be^{-\alpha_2} \left[\frac{\alpha_2 + ce^{-\alpha_2} + d\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}}{1 - bde^{-\alpha_1-\alpha_2}} \right]. \end{aligned}$$

Hence, we get $x_1 \leq \frac{\alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2}}{1 - bde^{-\alpha_1-\alpha_2}}$., i.e., $x_1 \in I_1$. Similarly, we get $y_1 \in I_2$.

Hence, the proof follows by applying the method of induction. □

Theorem 3 *Let* $bde^{-\alpha_1-\alpha_2} < 1$. *Consider the intervals*

$$I_3 = \left[\alpha_1, \frac{\alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2} + \epsilon}{1 - bde^{-\alpha_1-\alpha_2}}\right]$$

and

$$I_4 = \left[\alpha_2, \frac{\alpha_2 + ce^{-\alpha_2} + d\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1} + \epsilon}{1 - bde^{-\alpha_1-\alpha_2}}\right]$$

where ϵ *is an arbitrary positive number. If* (x_n, y_n) *is any arbitrary solution of* (3), *then there exists an* $N \in \mathbb{N}$ *such that* $x_n \in I_3$ *and* $y_n \in I_4, n \geq N$.

Proof
Let (x_n, y_n) *denote an arbitrary solution of (3).*

Then by Theorem 1, $\limsup_{n\to\infty} x_n = M < \infty$ and $\limsup_{n\to\infty} y_n = L < \infty$.

Hence from Theorem 1, $x_{n+1} \leq A + bdx_{n-1}e^{-\alpha_1-\alpha_2}$ and $y_{n+1} \leq C + bdy_{n-1}e^{-\alpha_1-\alpha_2}$

Hence $M \leq \frac{A}{1 - bde^{-\alpha_1-\alpha_2}}$, and $L \leq \frac{C}{1 - bde^{-\alpha_1-\alpha_2}}$.

Hence, there exists an $N \in \mathbb{N}$ such that the theorem holds. □

Now we prove a lemma which is an alteration of Theorem 1.16 of [14].

Lemma 4 *Let* $[a, b]$ *and* $[c, d]$ *denote intervals of real numbers. Let* $f : [a, b] \times [c, d] \times [c, d] \to [a, b]$ *and* $g : [a, b] \times [a, b] \times [c, d] \to [c, d]$ *be continuous functions. Consider the difference system*

$$\begin{aligned} x_{n+1} &= f(x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} &= g(x_n, x_{n-1}, y_{n-1}), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{9}$$

such that the initial values $x_{-1}, x_0 \in [a, b]$ *and* $y_{-1}, y_0 \in [c, d]$. (*or* $x_{n_0}, x_{n_0+1} \in [a, b]$, $y_{n_0}, y_{n_0+1} \in [c, d], n_0 \in \mathbb{N}$). *Suppose the following are true.*

1. *If f(x, y, z) is nonincreasing in x, f(x, y, z) is nondecreasing in y and f(x, y, z) is nonincreasing in z.*
2. *If g(x, y, z) is nondecreasing in x, g(x, y, z) is nonincreasing in y and g(x, y, z) is nonincreasing in z.*

3. *If* $(m_1, M_1, m_2, M_2) \in [a,b]^2 \times [c,d]^2$ *satisfies the systems* $m_1 = f(M_1, m_2, M_2)$, $M_1 = f(m_1, M_2, m_2)$ *and* $m_2 = g(m_1, M_1, M_2), M_2 = g(M_1, m_1, m_2)$ *then* $M_1 = m_1$ *and* $M_2 = m_2$,

then there exists a unique equilibrium solution $(\bar{x}, \bar{y})$ *of* (9) *with* $\bar{x} \in [a,b]$, $\bar{y} \in [c,d]$. *Also every solution of* (9) *converges to* $(\bar{x}, \bar{y})$.

Proof

Set $m_1^{-1} = a, m_1^0 = a, m_2^{-1} = c, m_2^0 = c.$

$$M_1^{-1} = b, M_1^0 = b, M_2^{-1} = d, M_2^0 = d.$$

For each $i \geq 0$, let $m_1^{i+1} = f(M_1^{i-1}, m_2^i, M_2^{i-1}), M_1^{i+1} = f(m_1^{i-1}, M_2^i, m_2^{i-1})$ and

$$m_2^{i+1} = g(M_1^i, m_1^{i-1}, m_2^{i-1}), M_2^{i+1} = g(m_1^i, M_1^{i-1}, M_2^{i-1}).$$

Hence $m_1^1 = f(M_1^{-1}, m_2^0, M_2^{-1}) \leq f(m_1^{-1}, M_2^0, m_2^{-1}) = M_1^1$, and

$$m_2^1 = g(m_1^0, M_1^{-1}, M_2^{-1}) \leq g(M_1^0, m_1^{-1}, m_2^{-1}) = M_2^1.$$

Therefore,

$$M_1^{-1} \geq M_1^0 \geq M_1^1 \geq m_1^1 \geq m_1^0 \geq m_1^{-1} \quad \text{and}$$
$$M_2^{-1} \geq M_2^0 \geq M_2^1 \geq m_2^1 \geq m_2^0 \geq m_2^{-1}.$$

Also $m_1^0 = a \leq x_n \leq b = M_1^0, n \geq 0$ and $m_2^0 = c \leq y_n \leq d = M_2^0, n \geq 0$.

For all $n \geq 0$, we have

$$m_1^1 = f(M_1^{-1}, m_2^0, M_2^{-1}) \leq f(x_{n-1}, y_n, y_{n-1}) \leq f(m_1^{-1}, M_2^0, m_1^{-1}) = M_1^1.$$
$$m_2^1 = g(m_1^0, M_1^{-1}, M_2^{-1}) \leq g(x_n, x_{n-1}, y_{n-1}) \leq g(M_1^0, m_1^{-1}, M_2^0) = M_2^1.$$

Hence $m_1^1 \leq x_n \leq M_1^1, n \geq 1$ and $m_2^1 \leq y_n \leq M_2^1, n \geq 1$.

We then obtain by induction that for $i \geq 0$, the following are true.

1. $a = m_1^{-1} \leq m_1^0 \leq m_1^1 \ldots \leq m_1^{i-1} \leq m_1^i \leq M_1^i \ldots \leq M_1^1 \leq M_1^0 \leq M_1^{-1} = b.$
2. $c = m_2^{-1} \leq m_2^0 \leq m_2^1 \ldots \leq m_2^{i-1} \leq m_2^i \leq M_2^i \ldots \leq M_2^1 \leq M_2^0 \leq M_2^{-1} = d.$
3. $m_1^i \leq x_n \leq M_1^i, n \geq 1$ and $m_2^i \leq y_n \leq M_2^i, n \geq 1$.

Set $m_1 = \lim_{i\to\infty} m_1^i, m_2 = \lim_{i\to\infty} m_2^i$ and $M_1 = \lim_{i\to\infty} M_1^i, M_2 = \lim_{i\to\infty} M_2^i$. Since f and g are continuous, we get $m_1 = f(M_1, m_2, M_2), M_1 = f(m_1, M_2, m_2)$ and $m_2 = g(m_1, M_1, M_2), M_2 = g(M_1, m_1, m_2)$.

Hence $M_1 = m_1 = \bar{x}$ and $M_2 = m_2 = \bar{y}$, from which we get the proof. □

The following theorem proposes conditions for the convergence of the equilibrium solution of (3).

Theorem 5 *Suppose*

$$bde^{-\alpha_1-\alpha_2} < 1, ce^{-\alpha_2} < 1, ae^{-\alpha_1} < 1 \tag{10}$$

and

$$\frac{bde^{-\alpha_1-\alpha_2}}{[1-bde^{-\alpha_1-\alpha_2}]^2}\frac{[1-bde^{-\alpha_1-\alpha_2}+\alpha_2+ce^{-\alpha_2}+\mathrm{d}\alpha_1e^{-\alpha_1}+ade^{-\alpha_1-\alpha_1}]}{[1-ae^{-\alpha_1}]} \times \frac{[1-bde^{-\alpha_1-\alpha_2}+\alpha_1+ae^{-\alpha_1}+b\alpha_2e^{-\alpha_2}+bce^{-\alpha_2-\alpha_2}]}{[1-ce^{-\alpha_2}]} < 1. \tag{11}$$

Then (3) *has a unique positive equilibrium* $E(\bar{x},\bar{y})$. *Also, every solution of* (3) *converges to* $E(\bar{x},\bar{y})$.

Proof
Let $f:\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+, g:\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ *be continuous functions such that* $f(x,y,z)=\alpha_1+ae^{-x}+bye^{-z}$, $g(x,y,z)=\alpha_2+ce^{-z}+dxe^{-y}$.

Let M_1, m_1, M_2, m_2 be positive real numbers satisfying

$$m_1=\alpha_1+ae^{-M_1}+bm_2e^{-M_2}, M_1=\alpha_1+ae^{-m_1}+bM_2e^{-m_2}$$

and

$$m_2=\alpha_2+ce^{-M_2}+\mathrm{d}m_1e^{-M_1}, M_2=\alpha_2+ce^{-m_2}+\mathrm{d}M_1e^{-m_1}. \tag{12}$$

Therefore, $M_1-m_1=a[e^{-m_1}-e^{-M_1}]+b[M_2e^{-m_2}-m_2e^{-M_2}]$.

$$M_1-m_1=a[e^{-m_1}-e^{-M_1}]+be^{-m_2-M_2}[M_2e^{M_2}-m_2e^{m_2}]. \tag{13}$$

Also, there exists a ζ , $m_2\le\zeta\le M_2$ satisfying

$$M_2e^{M_2}-m_2e^{m_2}=(1+\zeta)e^{\zeta}(M_2-m_2). \tag{14}$$

From (13) and (14), we get

$$M_1-m_1=a[e^{-m_1}-e^{-M_1}]+be^{-m_2-M_2+\zeta}(1+\zeta)[M_2-m_2]. \tag{15}$$

Now, $a[e^{-m_1}-e^{-M_1}]=ae^{-m_1-M_1}[e^{M_1}-e^{m_1}]$.

Also there exists a λ, $m_1\le\lambda\le M_1$ satisfying

$$a[e^{-m_1}-e^{-M_1}]=ae^{-m_1-M_1+\lambda}[M_1-m_1]. \tag{16}$$

Since $M_1, m_1\ge\alpha_1$ and $\lambda\le M_1$,

$$a[e^{-m_1}-e^{-M_1}]\le ae^{-\alpha_1}[M_1-m_1]. \tag{17}$$

Thus, from (15) and (17) we get,

$$M_1 - m_1 \leq ae^{-\alpha_1}[M_1 - m_1] + be^{-m_2-M_2+\zeta}(1+\zeta)[M_2 - m_2]. \tag{18}$$

Since $M_2, m_2 \geq \alpha_2$ and $\zeta \leq M_2$, (18) becomes

$$M_1 - m_1 \leq ae^{-\alpha_1}[M_1 - m_1] + be^{-\alpha_2}(1+\zeta)[M_2 - m_2]. \tag{19}$$

, i.e.,

$$[1 - ae^{-\alpha_1}][M_1 - m_1] \leq be^{-\alpha_2}(1+\zeta)[M_2 - m_2]. \tag{20}$$

Also, (12) can be written as

$$M_2 = \alpha_2 + ce^{-m_2} + \mathrm{d}[\alpha_1 + ae^{-m_1} + bM_2e^{-m_2}]e^{-m_1}. \tag{21}$$

$$M_2 \leq \frac{\alpha_2 + ce^{-\alpha_2} + \mathrm{d}\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}}{1 - bde^{-\alpha_1-\alpha_2}}. \tag{22}$$

Since $\zeta \leq M_2$ we get,

$$\zeta \leq \frac{\alpha_2 + ce^{-\alpha_2} + \mathrm{d}\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}}{1 - bde^{-\alpha_1-\alpha_2}}. \tag{23}$$

Therefore, (20) becomes

$$\begin{aligned}&[1 - ae^{-\alpha_1}][M_1 - m_1]\\ &\quad\leq be^{-\alpha_2}\left[\frac{1 - bde^{-\alpha_1-\alpha_2} + \alpha_2 + ce^{-\alpha_2} + \mathrm{d}\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}}{1 - bde^{-\alpha_1-\alpha_2}}\right][M_2 - m_2].\end{aligned} \tag{24}$$

Similarly, we get

$$\begin{aligned}&[1 - ce^{-\alpha_2}][M_2 - m_2]\\ &\quad\leq \mathrm{d}e^{-\alpha_1}\left[\frac{1 - bde^{-\alpha_1-\alpha_2} + \alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2}}{1 - bde^{-\alpha_1-\alpha_2}}\right][M_1 - m_1].\end{aligned} \tag{25}$$

From (24) and (25), we get

$$\begin{aligned}&[M_1 - m_1]\\ &\quad\leq \frac{bde^{-\alpha_1-\alpha_2}}{[1 - (bde^{-\alpha_1-\alpha_2})]^2}\frac{[1 - bde^{-\alpha_1-\alpha_2} + \alpha_2 + ce^{-\alpha_2} + \mathrm{d}\alpha_1 e^{-\alpha_1} + ade^{-\alpha_1-\alpha_1}]}{[1 - ae^{-\alpha_1}]}\\ &\qquad\times\frac{[1 - bde^{-\alpha_1-\alpha_2} + \alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2}]}{[1 - ce^{-\alpha_2}]}[M_1 - m_1].\end{aligned} \tag{26}$$

Therefore from (11) and (26), we get $M_1 = m_1$ and $M_2 = m_2$.

Therefore by applying Lemma 4, the result is obtained. □

In the next theorem, we derive conditions for the global asymptotic stability of the equilibrium solution of (3).

Theorem 6 *Assume* (10) *and* (11) *holds.*

1. *Let* $(a+ac+c)<1$. *If* $(1+\bar{x})(1+\bar{y})<\dfrac{1-(a+ac+c)}{bd}$, *then the unique equilibrium* $E(\bar{x},\bar{y})$ *is globally asymptotically stable.*
2. *If* $(a+c+ac+bd)+bd[\frac{A}{1-B}+\frac{C}{1-B}+\frac{AC}{(1-B)^2}]<1$, *where* A, B *and* C *are defined as in* (4) *and* (5), *then the unique equilibrium* $E(\bar{x},\bar{y})$ *is globally asymptotically stable.*

Proof

First we show that $E(\bar{x},\bar{y})$ *is locally asymptotically stable in both the cases. The Jacobian* $JF(\bar{x},\bar{y})$ *about the equilibrium point* $E(\bar{x},\bar{y})$ *is given by*

$$\begin{bmatrix} 0 & -ae^{-\bar{x}} & be^{-\bar{y}} & -b\bar{y}e^{-\bar{y}} \\ 1 & 0 & 0 & 0 \\ \mathrm{d}e^{-\bar{x}} & -d\bar{x}e^{-\bar{x}} & 0 & -ce^{-\bar{y}} \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Hence the characteristic equation of the Jacobian $JF(\bar{x},\bar{y})$ about the equilibrium point $E(\bar{x},\bar{y})$ is given by

$$\begin{aligned}&-\lambda^4+\lambda^2(-ce^{-\bar{y}}+bde^{-\bar{x}}e^{-\bar{y}}-ae^{-\bar{x}})\\ &\quad+\lambda(-bd\bar{y}e^{-\bar{x}}e^{-\bar{y}}-bd\bar{x}e^{-\bar{x}}e^{-\bar{y}})+bd\bar{x}\bar{y}e^{-\bar{x}}e^{-\bar{y}}-ace^{-\bar{x}}e^{-\bar{y}}=0.\end{aligned}$$

Then

$$\begin{aligned}&|-ce^{-\bar{y}}|+|bde^{-\bar{x}}e^{-\bar{y}}|+|ae^{-\bar{x}}|\\ &\quad+|bd\bar{y}e^{-\bar{x}}e^{-\bar{y}}|+|bd\bar{x}e^{-\bar{x}}e^{-\bar{y}}|+|bd\bar{x}\bar{y}e^{-\bar{x}}e^{-\bar{y}}|+|ace^{-\bar{x}}e^{-\bar{y}}|<1\end{aligned}$$

is satisfied whenever

$$|c|+|bd|+|a|+|bd\bar{y}|+|bd\bar{x}|+|bd\bar{x}\bar{y}|+|ac|<1. \tag{27}$$

1. From (27), we get

$$(1+\bar{x})(1+\bar{y})<\frac{1-(a+ac+c)}{bd}. \tag{28}$$

   Hence, by (28) and Remark 1.3.1 of [15], we get the result.
2. Since $E(\bar{x},\bar{y})$ is the equilibrium point of (3), we get

$$\bar{x}\le\alpha_1+ae^{-\alpha_1}+be^{-\alpha_2}[\alpha_2+d\bar{x}e^{-\alpha_1}+ce^{-\alpha_2}].$$

   , i.e.,

$$\bar{x}\le\frac{A}{(1-bde^{-\alpha_1-\alpha_2})}. \tag{29}$$

   Similarly

$$\bar{y} \leq \frac{C}{(1 - bde^{-\alpha_1 - \alpha_2})}. \tag{30}$$

Substituting (29), (30) in (27), we get

$$(a + c + ac + bd) + bd\left[\frac{A}{1-B} + \frac{C}{1-B} + \frac{AC}{(1-B)^2}\right] < 1.$$

Hence by Remark 1.3.1 of [15], we get the result.

Therefore by using Theorem 5, we obtain the conditions for global asymptotic stability. □

Conclusions

In this paper, we analyzed the persistence, boundedness, convergence, invariance and global asymptotic behavior of the positive solutions of a second-order difference system. Here we expressed all the conditions in terms of the parameters occurring in the system. We also obtained two conditions for the occurrence of global stability where in the first one the condition was given in terms of the equilibrium point and in the second one the condition was given in terms of parameters of the system.

Acknowledgements
The authors would like to thank the referees for their valuable suggestions.

Authors' contributions
DSD wrote the title, abstract, introduction and references. SMM wrote the main results. Both authors read and approved the final manuscript.

Funding
This research work is not funded.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author details
[1]Department of Mathematics, St. John's College, Anchal, Kerala, India. [2]Department of Mathematics, Mar Ivanios College, Research Centre, University of Kerala, Thiruvananthapuram, India.

Received: 23 September 2020 Accepted: 5 April 2021
Published online: 20 April 2021

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