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A study of generalized logistic distributions

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Abstract Because of their flexibility, recently, much attention has been given to the study of generalized distributions. A complete study of the beta generalized logistic distribution (type IV) is proposed, introducing an approximate form for the median and deducing the mean deviation from the mean and the median. A complete parameter estimation using the method of maximum likelihood and the method of moments is presented. Some characteristic properties of the generalized logistic distribution type I are discussed. Also, a highlight to some properties of an analog distribution to the generalized logistic distribution type IV, discussed by Zografos and Balakrishnan [1], is presented.

1. Introduction

More than 170 years ago, Verhulst [2,3] used the logistic function for economic demographic purposes. Gumbel [4] found that the logistic distribution arises in a purely statistical manner as the limiting distribution (as $n \to \infty$) of the standardized midrange (average of largest and smallest values) of random samples of size n from a symmetric distribution of exponential type. Gumbel and Keeney [5] showed that a logistic distribution is obtained as the limiting distribution of an appropriate multiple of the 'extremal quotient', that is, (largest value)/(smallest value). Talacko [6] proved that the logistic distribution is the limiting distribution (as $r \to \infty$) of the standardized variable corresponding to $\sum_{j=1}^{r} j^{-1} X_j$, where X_j's are independent random variables each having a type I extreme value distribution. A number of authors discussed important applications of the logistic distribution in many fields including survival analysis, growth model and public health. Several different forms of generalizations of the logistic distribution have been proposed in the literature, and studied in Balakrishnan and Leung [7], Balakrishnan [8], and Johnson et al. [9], i.e. types I, II, III and IV. The type I generalized logistic distribution has the following density function (pdf)

$$g(x) = \frac{\alpha\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^{\alpha+1}}, \quad -\infty < x < \infty, \ \alpha > 0. \tag{1}$$

If X has type I generalized logistic distribution in (1), then-X has a type II generalized logistic distribution. The type III generalized logistic distribution has the pdf $g(x) = \frac{1}{B(\alpha,\alpha)} \frac{\lambda e^{-\lambda\alpha x}}{(1+e^{-\lambda x})^{2\alpha}}$, $-\infty < x < \infty, \alpha > 0$,

Last but not least, the type IV beta generalized logistic distribution, or BGL (α, β, λ), as introduced by Prentice [10] and Kalbfleisch and Prentice [11], is given by the pdf

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$$g(x) = \frac{1}{B(\alpha,\beta)} \frac{\lambda e^{-\lambda\beta x}}{(1+e^{-\lambda x})^{\alpha+\beta}}, \quad -\infty < x < \infty, \ \alpha,\beta > 0, \qquad (2)$$

where $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ is the complete beta function and λ is the scale parameter.

The BGL (α,β,λ) defined in (2) is also called the log-F distribution. This is just the family of logistic distributions generated from the beta distribution, proposed by Jones [12], where the class of "beta-generated distributions" has the pdf given by

$$g(x,\alpha,\beta) = \frac{1}{B(\alpha,\beta)} f(x)[F(x)]^{\alpha-1}[1-F(x)]^{\beta-1}, \quad \alpha,\beta > 0, \qquad (3)$$

and

$$F(x) = (1+e^{-\lambda x})^{-1}, \quad -\infty < x < \infty, \ \lambda > 0, \qquad (4)$$

where $F(x)$ is the cumulative distribution function (Cdf) of the standard logistic distribution.

It is well known, in general, that a generalized model is more flexible than the ordinary model and it is preferred by many data analysts in analyzing statistical data.Here, we will be concerned mostly with the beta generalized logistic distribution type IV, i.e. BGL (α,β,λ). Let us introduce the Cdf of the BGL (α,β,λ) distribution, as proposed by Jones [12], given by

$$G(x) = \frac{1}{B(\alpha,\beta)} \int_0^{(1+e^{-\lambda x})^{-1}} t^{\alpha-1}(1-t)^{\beta-1}dt = I_{\frac{1}{1+e^{-\lambda x}}}(\alpha,\beta) \qquad (5)$$

where $I_y(\alpha,\beta) = \frac{B_y(\alpha,\beta)}{B(\alpha,\beta)}$ is the incomplete beta function ratio and the incomplete beta function is $B_y(\alpha,\beta) = \int_0^y w^{\alpha-1}(1-w)^{\beta-1}dw$, while the pdf is given by Eq. (2).

We can express the Cdf in Eq. (5) in terms of the hypergeometric function, as given by Gradshteyn and Ryzhik [13], as follows

$$G(x) = \frac{1}{\alpha B(\alpha,\beta)} \frac{1}{1+e^{-\lambda x}} {}_2F_1\left(\alpha, 1-\beta, \alpha+1, \frac{1}{1+e^{-\lambda x}}\right). \qquad (6)$$

The BGL (α,β,λ) distribution generalizes the various forms of the logistic distribution. For $\alpha = \beta = 1$, we obtain the standard logistic distribution. The generalized logistic (i.e. BGL $(\alpha,1,\lambda)$) distribution type I is a special case for the choice of $\beta = 1$. As for the case $\beta = \alpha$, we have the generalized logistic distribution type III. Fig. 1 gives plots of the pdf (2) for different values of (α,β,λ).

The hazard function of the BGL (α,β,λ) distribution is

$$h(x) = \frac{1}{B(\alpha,\beta)} \frac{\lambda e^{-\lambda\beta x}}{(1+e^{-\lambda x})^{\alpha+\beta}} \frac{1}{1-G(x)}, \quad -\infty < x < \infty, \ \alpha,\beta > 0. \qquad (7)$$

Section 2 introduces some properties of the BGL (α,β,λ) as studied in the literature as well as a complete discussion in deducing an explicit form for the median and hence the mean deviation followed by the deduction of Renyi and Shannon entropies. Section 3 provides different methods of inference of the parameters in (2). Some characteristic properties of BGL $(\alpha,1,\lambda)$ distribution are introduced in Section 4. In Section 5, an analog family of BGL distribution, defined by Zografos and Balakrishnan [1], is presented. We conclude, in Section 6, some remarks on discriminating between the two families of generalized logistic distributions, and an application to real data.

2. Properties of the BGL distribution

2.1. Limiting behavior

If α and β both tend to infinity, as discussed by Aroian [14], then the appropriately normalized *log F* distribution tends to the normal distribution. If $\alpha \to \infty$ but β is fixed, then the appropriately normalized log F distribution tends to the distribution with density proportional to $e^{-\beta t}e^{-e^{-t}}, -\infty < t < \infty$. Likewise, if $\beta \to \infty$ but α is fixed, then the appropriately normalized log F distribution tends to the (log gamma) distribution with density proportional to $e^{\alpha t}e^{-e^{t}}, -\infty < t < \infty$ (see Prentice [15]). That is, the case where only $\alpha \to \infty$ results in the limiting distribution of the β th largest order statistic; likewise, when $\beta \to \infty$ the limiting distribution is that of the αth smallest order statistic. The particular form of limiting extreme value-type distribution when α or β is fixed is a consequence of the exponential tails of the logistic distribution.

2.2. Unimodality

The logistic distribution is very similar in shape to the normal distribution because its symmetric bell shaped pdf. Besides the maximum difference between the two distribution functions can be less than 0.01, as proposed by Mudholkar and George [16]. So, the logistic distribution has a close approximation to the normal distribution. This is why it makes it profitable, on suitable occasions, to replace the normal by the logistic to simplify the analysis without too many discrepancies. The BGL (α,β,λ) distribution, as discussed in Johnson et al. [9], is unimodal with mode at $\frac{1}{\lambda}\log\frac{\alpha}{\beta}$. Hence the pdf of a BGL distribution is an increasing function for $x < \frac{1}{\lambda}\log\frac{\alpha}{\beta}$ and it is a decreasing function for $x > \frac{1}{\lambda}\log\frac{\alpha}{\beta}$. One can also observe that for $\alpha > \beta$, it is positively skewed, for $\alpha < \beta$, it is negatively skewed and symmetric for $\alpha = \beta$.

2.3. Median

An important and characteristic feature for any distribution is the median m, and since it has not been discussed before, our main goal in this subsection is to deduce an approximate and easy-to-use formula for the median of the BGL (α,β,λ). To deduce it, it is well known that

$G(m) = \frac{1}{2}$. Taking into consideration that β is an integer and integrating, we obtain

$$\frac{B(\alpha,\beta)}{2} = \sum_{j=o}^{\beta-1} \binom{\beta-1}{j} \frac{(-1)^j}{\alpha+j} \left(\frac{1}{1+e^{-\lambda m}}\right)^{\alpha+j}. \qquad (8)$$

The summation on the right-hand side converges absolutely for $\left|\frac{1}{1+e^{-\lambda m}}\right| < 1$. Using the approximation technique, the median can be written as

$$m \approx \frac{1}{\lambda} \log \frac{1}{\left[\frac{2}{\alpha B(\alpha,\beta)}\right]^{1/\alpha} - 1}. \qquad (9)$$

Now, if β is not an integer, the summation is infinite, and again, converges absolutely for $\left|\frac{1}{1+e^{-\lambda m}}\right| < 1$, and hence a good

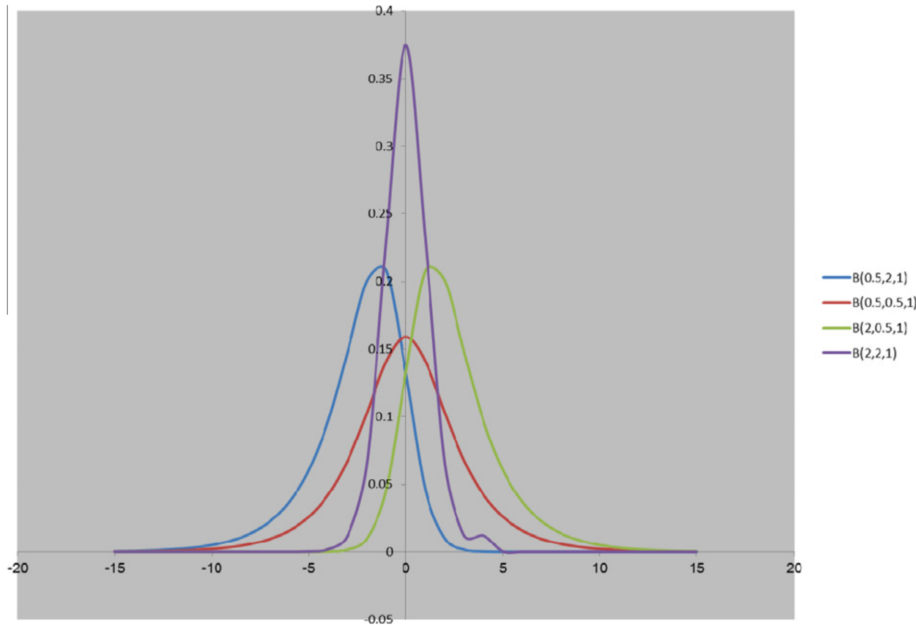

Figure 1 Plots of pdf (2) for various values of parameters.

approximation is the one given in (9). In the special case of the generalized logistic distribution type I, that is, when $\beta = 1$, we obtain

$$m = \frac{1}{\lambda}\log\frac{1}{2^{1/\alpha}-1}.$$

2.4. Characteristic function

The characteristic function (cf) of the BGL (α, β, λ) distribution can be deduced to yield

$$\zeta(t) = \frac{B\left(\alpha + \frac{it}{\lambda}, \beta - \frac{it}{\lambda}\right)}{B(\alpha, \beta)}. \quad (10)$$

If $\beta = 1$,we obtain the cf of the BGL $(\alpha, 1, \lambda)$ distribution as proposed by Ahuja and Nash [17] and Johnson et al. [9]. For $\alpha = 1$ and $\beta = 1$, the above expression reduces to the cf of the standard logistic distribution (see Johnson et al. [9]).

The mean and variance of BGL (α, β, λ), introduced by Davidson [18], can thus be written as

$$\mu = E(X) = \frac{1}{\lambda}[\psi(\alpha) - \psi(\beta)]$$

and

$$\sigma^2 = Var(X) = \frac{1}{\lambda^2}[\psi'(\alpha) + \psi'(\beta)],$$

where $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ and $\psi'(x) = \frac{d}{dx}\psi(x)$ are known as digamma and polygamma functions, respectively. In fact, $\gamma = -\psi\ (1) = 0.577215$ is called the Euler's constant.

The skewness and kurtosis are

$$\tau_1 = \frac{\psi''(a) - \psi''(b)}{[\psi'(a) + \psi'(b)]^{3/2}} \quad (11)$$

and

$$\tau_2 = \frac{\psi'''(\alpha) + \psi'''(\beta)}{[\psi'(\alpha) + \psi'(\beta)]^2}. \quad (12)$$

To discuss the behavior of skewness, we give the following result.

Theorem 1.

(i) *For any fixed α, the skewness τ_1 (α, β) is a decreasing function of β.*
(ii) *For any fixed β, the skewness τ_1 (α, β) is an increasing function of α.*

Proof. Differentiating τ_1 (α, β) with respect to β,

$$\frac{\partial\tau_1(\alpha, \beta)}{\partial\beta} = -\frac{A + B}{2[\psi'(\alpha) + \psi'(\beta)]^{5/2}}$$

where

$$A = 2\psi'''(\beta)\psi'(\alpha) + 3\psi''(\alpha)\psi''(\beta),$$
$$B = 2\psi'''(\beta)\psi'(\beta) - 3\psi''(\beta)\psi''(\beta).$$

From Proposition 9 of Uesaka [19], $B > 0$. Also, the nth derivative of the digamma function

$$\psi^{(n)}(x) = \sum_{k=0}^{\infty}\frac{(-1)^{n+1}n!}{(x+k)^{n+1}} \quad (13)$$

gives $A > 0$. Thus the derivative of skewness with respect to β is less than zero, and hence (i) is verified.

Again, differentiating with respect to α,

$$\frac{\partial\tau_1(\alpha, \beta)}{\partial\alpha} = \frac{C + D}{2[\psi'(\alpha) + \psi'(\beta)]^{5/2}}$$

where

$$C = 2\psi'''(\alpha)\psi'(\beta) + 3\psi''(\alpha)\psi''(\beta),$$
$$D = 2\psi'''(\alpha)\psi'(\alpha) - 3\psi''(\alpha)\psi''(\alpha)$$

In the same manner, using Proposition 9 of Uesaka [19], $D > 0$. Also, as discussed above, $C > 0$, thus verifying (ii).

The asymptotic behavior of skewness and kurtosis as α and β tend to zero or infinity is an important issue and is discussed in the following theorems. □

Theorem 2. *The limiting values of skewness of the BGL distribution as α and β tend to zero or infinity are given by the following*

$$\lim_{\substack{\alpha\to 0\\ \beta\to\infty}}\tau_1(\alpha,\beta) = -2, \quad \lim_{\substack{\alpha\to \infty\\ \beta\to 0}}\tau_1(\alpha,\beta) = 2, \quad \lim_{\substack{\alpha\to 0\\ \beta\to 0}}\tau_1(\alpha,\beta) = 0. \tag{14}$$

The proof can be easily deduced using Eq. (13).

Theorem 3. *The limiting values of kurtosis of the BGL distribution as α and β tend to zero or infinity are given by the following*

$$\lim_{\substack{\alpha\to 0\\ \beta\to\infty}}\tau_2(\alpha,\beta) = 6, \quad \lim_{\substack{\alpha\to \infty\\ \beta\to 0}}\tau_2(\alpha,\beta) = 6. \tag{15}$$

The proof, again, can be carried out using Eq. (13).

2.5. Mean deviation

The deviation from the mean (in the case of symmetric distributions) or the deviation from the median (in the case of skewed distributions) is an important measure of spread in a population. Let X be a random variable having pdf given in Eq. (2) with mean $\mu = E(X)$ and median m. The mean deviation from the mean and the mean deviation from the median are defined, respectively, by

$$D(\mu) = E(|X-\mu|) = \int_{-\infty}^{\infty} |X-\mu| g(x)dx$$
$$D(m) = E(|X-m|) = \int_{-\infty}^{\infty} |X-m| g(x)dx.$$

The mean deviation from the mean can be simplified as

$$D(\mu) = 2\mu G(\mu) - 2\int_{-\infty}^{\mu} xg(x)dx. \tag{16}$$

The mean deviation from the median, similarly, can be simplified as

$$D(m) = \mu - 2\int_{-\infty}^{m} xg(x)dx. \tag{17}$$

Then the mean deviation from the mean and the mean deviation from the median are, respectively, given by

$$D(\mu) = 2\mu G(\mu) - \frac{2}{\lambda B(\alpha,\beta)}\left[\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{\beta-1-k}\left(\log v - \frac{1}{\alpha+\beta-k-1}\right)\frac{v^{\alpha+\beta-k-1}}{\alpha+\beta-k-1} + \sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{\alpha-1-j}\left(\log(1-v) - \frac{1}{\alpha+\beta-j-1}\right)\frac{(1-v)^{\alpha+\beta-k-1}}{\alpha+\beta-j-1}\right]$$

and

$$D(m) = \mu - \frac{2}{\lambda B(\alpha,\beta)}\left[\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{\beta-1-k}\left(\log u - \frac{1}{\alpha+\beta-k-1}\right)\frac{u^{\alpha+\beta-k-1}}{\alpha+\beta-k-1} + \sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{\alpha-1-j}\left(\log(1-u) - \frac{1}{\alpha+\beta-j-1}\right)\frac{(1-u)^{\alpha+\beta-k-1}}{\alpha+\beta-j-1}\right]$$

where

$$v = \frac{1}{1+e^{-(\psi(\alpha)-\psi(\beta))}}, \quad u = \left(\frac{\alpha B(\alpha,\beta)}{2}\right)^{1/\alpha}.$$

2.6. Renyi and Shannon entropies

The notion of entropy is of fundamental importance in different areas such as physics, probability and statistics, communication theory, and economics. Since the entropy of a random variable is a measure of variation of the uncertainty, the Renyi entropy can be deduced to yield

$$I_R(\xi) = \frac{1}{1-\xi}\log\frac{\lambda^{\xi-1}B(\xi\alpha,\xi\beta)}{B^{\xi}(\alpha,\beta)}. \tag{18}$$

A special case, defined in Shannon's [20] pioneering work on the mathematical theory of communication, given by Shannon entropy – a major tool in information theory and in almost every branch of science and engineering – is

$$\hbar_{sh}(g^B) = \log B(\alpha,\beta) - \log\lambda - \alpha[\psi(\alpha) - \psi(\alpha+\beta)] - \beta[\psi(\beta) - \psi(\alpha+\beta)] \tag{19}$$

as introduced by Zografos and Balakrishnan [1].

3. Parameter estimation and inference

In the following subsections, we discuss two methods of parameter estimation thus deriving the estimators in each case.

3.1. Maximum likelihood estimators

Here, we consider the maximum likelihood estimators (MLEs) of the BGL(α,β,λ) distribution given in (2). Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from the BGL

(α, β, λ) distribution. The log-likelihood function can be written as follows:

$$\log L = -n \log B(\alpha, \beta) + n \log \lambda - \lambda\beta \sum_{i=1}^{n} x_i - (\alpha + \beta) \sum_{i=1}^{n} \log(1 + e^{-\lambda x_i}).$$

Setting $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and differentiating with respect to α,β and λ, we obtain the following normal equations

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} &= -n\psi(\alpha) + n\psi(\alpha + \beta) + D_n(\lambda) \\ \frac{\partial \log L}{\partial \beta} &= -n\psi(\beta) + n\psi(\alpha + \beta) + n\lambda\overline{x} + D_n(\lambda) \\ \frac{\partial \log L}{\partial \lambda} &= \frac{n}{\lambda} - n\beta\bar{x} - (\alpha + \beta) \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{1 + e^{-\lambda x_i}}, \end{aligned} \tag{20}$$

where $D_n(\lambda) = \sum_{i=1}^{n} \log(1 + e^{-\lambda x_i})$.

For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The Fisher information matrix $K = K(\theta)$, $\theta = (\alpha, \beta, \lambda)^T$, is

$$K = \begin{pmatrix} K_{\alpha,\alpha} & K_{\alpha,\beta} & K_{\alpha,\lambda} \\ K_{\alpha,\beta} & K_{\beta,\beta} & K_{\beta,\lambda} \\ K_{\alpha,\lambda} & K_{\beta,\lambda} & K_{\lambda,\lambda} \end{pmatrix},$$

whose elements are

$$\begin{aligned} K_{\alpha,\alpha} &= E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\psi'(\alpha) - n\psi'(\alpha + \beta), \\ K_{\alpha,\beta} &= E\left(-\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) = -n\psi'(\alpha + \beta), \\ K_{\beta,\beta} &= E\left(-\frac{\partial^2 \log L}{\partial \beta^2}\right) = n\psi'(\beta) - n\psi'(\alpha + \beta), \\ K_{\lambda,\lambda} &= E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right) = \frac{n}{\lambda^2} + n\frac{\alpha\beta}{\alpha + \beta + 1}, \\ K_{\beta,\lambda} &= E\left(-\frac{\partial^2 \log L}{\partial \beta \partial \lambda}\right) = \frac{n}{\lambda}(\psi(\alpha) - \psi(\beta)) + \frac{n\beta}{\alpha + \beta}(\psi(\alpha) \\ &\quad + \psi(\alpha + \beta + 1) - \psi(\alpha + \beta) - \psi(\beta + 1)), \\ K_{\alpha,\lambda} &= E\left(-\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right) = \frac{n\beta}{\alpha + \beta}(\psi(\alpha) + \psi(\alpha + \beta + 1) \\ &\quad - \psi(\alpha + \beta) - \psi(\beta + 1)). \end{aligned}$$

The MLE $\hat{\theta} = (\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\lambda}_{ML})^T$ of θ is numerically determined from the solution of the nonlinear system of equations given earlier. Under conditions that are fulfilled for the parameter θ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $[\sqrt{n}(\hat{\alpha}_{ML} - \alpha), \sqrt{n}(\hat{\beta}_{ML} - \beta), \sqrt{n}(\hat{\lambda}_{ML} - \lambda)]^T$ is $N_3(0, K^{-1}(\alpha,\beta,\lambda)^T)$. The asymptotic normal $N_3(0, K^{-1}(\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\lambda}_{ML})^T)$ distribution of $\hat{\theta} = (\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\lambda}_{ML})^T$ can be used to construct confidence regions for some parameters and for the hazard and survival functions. In fact, a $100(1 - \gamma)\%$ asymptotic confidence interval (*ACI*) for each parameter is given by

$$\begin{aligned} ACI_\alpha &= \left(\hat{\alpha}_{ML} - z_{\gamma/2}\sqrt{K_{11}}, \hat{\alpha}_{ML} + z_{\gamma/2}\sqrt{K_{11}}\right) \\ ACI_\beta &= \left(\hat{\beta}_{ML} - z_{\gamma/2}\sqrt{K_{22}}, \hat{\beta}_{ML} + z_{\gamma/2}\sqrt{K_{22}}\right) \\ ACI_\lambda &= \left(\hat{\lambda}_{ML} - z_{\gamma/2}\sqrt{K_{33}}, \hat{\lambda}_{ML} + z_{\gamma/2}\sqrt{K_{33}}\right) \end{aligned}$$

where K_{ii} denotes the ith diagonal element of $K^{-1}(\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\lambda}_{ML})^T$ for i = 1, 2, 3 and $z_{\gamma/2}$ is the $1 - \gamma/2$ of the standard normal distribution.

3.2. Moment estimators

Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from the BGL (α, β, λ) distribution. Under the method of moments, equating $E(X)$, $Var(X)$ and $E(X - E(X))^3$, respectively, with the corresponding sample estimates $s_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$, $s_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - s_1)^2$ and $s_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - s_1)^3$ respectively, one obtains the system of equations

$$\begin{aligned} s_1 &= \frac{1}{\lambda}[\psi(\alpha) - \psi(\beta)] \\ s_2 &= \frac{1}{\lambda^2}[\psi'(\alpha) + \psi'(\beta)] \\ s_3 &= \frac{1}{\lambda^3}[\psi''(\alpha) - \psi''(\beta)] \end{aligned} \tag{21}$$

Combining the three equations in (21), one obtains

$$\frac{\psi'(\alpha) + \psi'(\beta)}{[\psi(\alpha) - \psi(\beta)]^2} = \frac{s_2}{s_1^2}$$

and

$$\frac{\psi''(\alpha) - \psi''(\beta)}{[\psi(\alpha) - \psi(\beta)]^3} = \frac{s_3}{s_1^3}$$

This can be solved simultaneously to give the estimates for α and β:

$\hat{\alpha}_{MM}$ *and* $\hat{\beta}_{MM}$.

The estimate for λ can then be obtained directly from

$$\hat{\lambda}_{MM} = \frac{1}{s_1}[\psi(\hat{\alpha}_{MM}) - \psi(\hat{\beta}_{MM})].$$

4. Some characteristic properties of GL distribution

Let X be a random variable with Cdf $G(x)$ and characteristic function cf $\zeta(t)$, takinginto consideration that β = 1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the BGL$(\alpha, 1, \lambda)$, and denote the corresponding order statistics by $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$. The distribution function and characteristic function of $X_{k,n}$ are denoted by $G_{k,n}$ and $\zeta_{k,n}$, respectively. For the special case of an odd sample size $n = 2m - 1$, denote the sample median $X_{m,2m-1}$ by $X_{(m)}$ and its Cdf and cf by $G_{(m)}$ and $\zeta_{(m)}$, respectively. As usual, define $G^{-1}(t) = \inf\{x{:}G(x) \geqslant t\}, t \in (0,1)$ to be the quantile function of G. Moreover, let R be the real line and denote equality in distribution by $\underline{\underline{d}}$. We now give some basic properties.

Theorem 4. *Let X have the BGL* $(\alpha, 1, \lambda)$, *i.e.* $G(x) = (1 + e^{-\lambda x})^{-\alpha}$, $x \in R$, *and let U follow the uniform distribution on [0,1]. Then the following properties hold.*

(a) *The quantile function of G is* $G^{-1}(t) = \frac{1}{\lambda}\log\left[\frac{t^{1/\alpha}}{1-t^{1/\alpha}}\right], t \in (0,1)$, *and hence* $X \underline{\underline{d}} \frac{1}{\lambda}\log\left[\frac{U^{1/\alpha}}{1-U^{1/\alpha}}\right]$.

(b) *The moment generating function of X is*

$$M(s) = \Gamma\left(\alpha + \frac{s}{\lambda}\right)\Gamma\left(1 - \frac{s}{\lambda}\right)/\Gamma(\alpha), s \in (-1, 1).$$

(c) *The cf* $\zeta(t) = M(it) = \frac{B\left(\alpha + \frac{it}{\lambda}, 1 - \frac{it}{\lambda}\right)}{B(\alpha, 1)}, t \in R$

(d) *The cf of the smallest order statistic* $X_{1:n}$ *is*

$$\zeta_{1:n}(t) = n\zeta(t)\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{n-1-j}\frac{\Gamma\left(\alpha(n-j) + \frac{it}{\lambda}\right)\Gamma(\alpha+1)}{\Gamma(\alpha(n-j)+1)\Gamma\left(\alpha + \frac{it}{\lambda}\right)}, \quad t \in R. \tag{22}$$

In particular,

$$\zeta_{1:2}(t) = 2\zeta(t)\left[1 - \frac{\Gamma\left(2\alpha + \frac{it}{\lambda}\right)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)\Gamma\left(\alpha + \frac{it}{\lambda}\right)}\right], \quad t \in R$$

and

$$\zeta_{1:3}(t) = 3\zeta(t)\left[1 - 2\frac{\Gamma\left(2\alpha + \frac{it}{\lambda}\right)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)\Gamma\left(\alpha + \frac{it}{\lambda}\right)} + \frac{\Gamma\left(3\alpha + \frac{it}{\lambda}\right)\Gamma(\alpha+1)}{\Gamma(3\alpha+1)\Gamma\left(\alpha + \frac{it}{\lambda}\right)}\right], \quad t \in R.$$

For the special case $\alpha = 1, \lambda = 1$, *we obtain the result due to Lin and Hu* [21]

$$\zeta_{1:2}(t) = \zeta(t)[1 - it] \text{ and } \zeta_{1:3}(t) = \zeta(t)[1 - it]\left[1 - \frac{it}{2}\right], t \in R.$$

(e) *For* $n = 2m - 1 \geqslant 3$, *the cf of the sample median* $X_{(m)}$ *is*

$$\zeta_{m:2m-1}(t) = \frac{2m-1!}{m-1!^2}\zeta(t)\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^{m-1-j} \times \frac{\Gamma\left(\alpha(2m-1-j) + \frac{it}{\lambda}\right)\Gamma(\alpha+1)}{\Gamma(\alpha(2m-1-j)+1)\Gamma\left(\alpha + \frac{it}{\lambda}\right)}, \quad t \in R. \tag{23}$$

In particular,

$$\zeta_{2:3}(t) = 6\zeta(t)\left[\frac{\Gamma\left(2\alpha + \frac{it}{\lambda}\right)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)\Gamma\left(\alpha + \frac{it}{\lambda}\right)} - \frac{\Gamma\left(3\alpha + \frac{it}{\lambda}\right)\Gamma(\alpha+1)}{\Gamma(3\alpha+1)\Gamma\left(\alpha + \frac{it}{\lambda}\right)}\right], \quad t \in R$$

For the special case $\alpha = 1, \lambda = 1$, *this reduces to Lin and Hu's result* [21]

$$\zeta_{2:3}(t) = \zeta(t)(1 + t^2).$$

(f) *The moments of order statistics are given, for* $1 \leqslant j \leqslant n$ *and* $k \geqslant 1$, *by the following*

$$E(X_{j;n}^k) = \frac{1}{\lambda^k}\left[\sum_{r=0}^{n-j}\psi^{(k-1)}((n-r)\alpha) + (-1)^k\psi^{(k-1)}(1)\right] \tag{24}$$

$$E(X_{j;n}^k) - E(X_{j;n-1}^k) = \frac{1}{\lambda^k}\psi^{(k-1)}(n\alpha) \tag{25}$$

$$E(X_{j-1;n}^k) - E(X_{j;n}^k) = \frac{1}{\lambda^k}\psi^{(k-1)}((j-1)\alpha). \tag{26}$$

(g) *The moment generating function of* $X_{k,n}$ *is*

$$M_{k,n}(t) = \alpha k\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}B\left(\alpha(n-l) + \frac{t}{\lambda}, 1 - \frac{t}{\lambda}\right). \tag{27}$$

Results 4.1(a)–(g) *are generalizations of those given by* Gupta and Balakrishnan [22]. *Applying the Muntz–Szasz theorem, we are able to characterize the BGL* $(\alpha, 1, \lambda)$ *by using the recurrence relations* (24)–(27) (*see Lin* [23]).

5. Gamma-generated logistic distribution

Recently, a family of continuous distributions has been introduced through a particular case of Stacy's generalized gamma distribution-in the same spirit as Jones' family defined through the beta distribution – given by the pdf

$$g(x) = \frac{\tau}{\Gamma(\delta)}\{-\log(1 - F(x))\}^{\tau\delta-1}e^{-[-\ln(1-F(x))]^\tau}\frac{f(x)}{1 - F(x)}, \quad x \in R, \ \tau, \delta > 0. \tag{28}$$

For $\tau = 1$, the above pdf takes the form

$$g(x) = \frac{1}{\Gamma(\delta)}[-\log(1 - F(x))]^{\delta-1}f(x), \quad x \in R, \ \delta > 0,$$

with $F(x)$ given by (4) as follows:

$$g(x, \delta, \lambda) = \frac{1}{\Gamma(\delta)}\left[-\log\left(1 - \frac{1}{1 + e^{-\lambda x}}\right)\right]^{\delta-1}\frac{\lambda e^{-\lambda x}}{(1 + e^{-\lambda x})^2}, \quad -\infty < x < \infty, \delta > 0. \tag{29}$$

Let us denote the gamma-generated Logistic distribution in (29) by GGL $(\delta, 1, \lambda)$. In the case $\delta = n$, where n is a positive integer, the density function in (29) is the pdf of the nth upper record value arising from a sequence of independent and identically distributed random variables from a population with Cdf $F(x)$ given by (4). First of all, the Cdf of the GGL $(\delta, 1, \lambda)$ is given by the following

$$G(x) = \frac{1 + e^{-\lambda x}}{\lambda}\sum_{k=\delta+1}^{\infty}g(x, k, \lambda). \tag{30}$$

Fig. 2 gives plots of the pdf (29) for different values of δ and λ.

The most important characteristic features for the distribution (29) are discussed in the following subsections.

5.1. Mode

The mode of the GGL $(\delta,1,\lambda)$ is deduced by differentiating the pdf $g(x)$. This leads to the approximate mode $\frac{1}{\lambda}\log(2\delta)$. Hence the pdf of the GGL $(\delta,1,\lambda)$ is an increasing function for $x < \frac{1}{\lambda}\log(2\delta)$ and it is a decreasing function for $x > \frac{1}{\lambda}\log(2\delta)$. One can observe that for $\delta > \frac{1}{2}$, it is positively skewed, for $0 < \delta < \frac{1}{2}$, it is negatively skewed and symmetric for $\delta = \frac{1}{2}$.

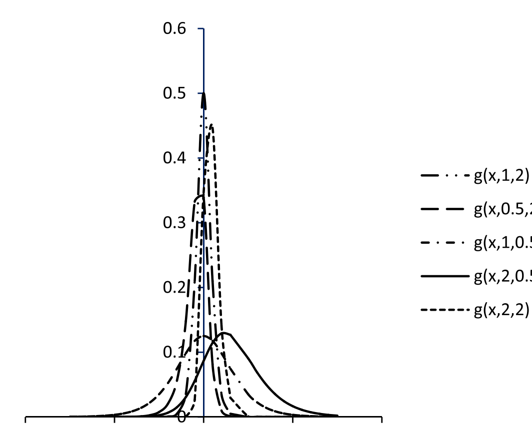

Figure 2 Plots of pdf (29) for various values of parameters.

5.2. Median

The second important feature is the median m' which can be derived using (30) and the approximation technique to obtain the approximate form for the median

$$m' \approx \frac{1}{\lambda}\log\left\{\left(\frac{\Gamma(\delta+1)}{2}\right)^{\frac{1}{\delta-1}} - 1\right\}. \tag{31}$$

5.3. Mean

The rth moment of the GGL $(\delta, 1, \lambda)$, as proposed by Zografos and Balakrishnan [1], is given by

$$E(X^r) = \frac{(-1)^{\delta-1}}{\lambda\Gamma(\delta)}\int_0^1 \left(\log\frac{1-y}{y}\right)^r [\log(1-y)]^{\delta-1} dy.$$

For $\delta > 1$ a natural number, Zografos and Balakrishnan [1] expected that it cannot be simplified further. This is true for $r > 1$, but, for the case $r = 1$, the mean of the GGL $(\delta, 1, \lambda)$ is given by

$$E(X) = \frac{\delta}{\lambda}\left[1 + \sum_{k=1}^{\infty}\frac{1}{k(k+1)^{\delta+1}}\right]. \tag{32}$$

5.4. Shannon entropy

Last but not least is Shannon entropy of the GGL $(\delta, 1, \lambda)$, as introduced by Zografos and Balakrishnan [1],

$$\hbar_{sh}(g^G) = \log\Gamma(\delta) - (\delta-1)\psi(\delta) - \log\lambda + \delta + \sum_{k=1}^{\infty}\frac{1}{k(k+1)^{\delta}}. \tag{33}$$

5.5. Maximum likelihood estimation

The log-likelihood function of the distribution (29) can be written as follows:

$$\log L = n\log\lambda - n\log\Gamma(\delta) + (\delta-1)\sum_{i=1}^{n}\log[\log(1+e^{\lambda x_i})] - \lambda\sum_{i=1}^{n}x_i - 2\sum_{i=1}^{n}\log(1+e^{-\lambda x_i}).$$

Differentiating with respect to λ and δ, we obtain the following equations,

$$\frac{\partial \log L}{\partial\lambda} = \frac{n}{\lambda} - (\delta-1)\sum_{i=1}^{n}\frac{1}{\log(1+e^{\lambda x_i})}\cdot\frac{x_i e^{\lambda x_i}}{(1+e^{\lambda x_i})} - \sum_{i=1}^{n}x_i + 2\sum_{i=1}^{n}\frac{x_i}{(1+e^{\lambda x_i})}$$

$$\frac{\partial \log L}{\partial\delta} = -n\psi(\delta) + \sum_{i=1}^{n}\log\log(1+e^{\lambda x_i}).$$

The MLE of the parameters can be determined numerically from the solution of the nonlinear system of equations given above.

6. Concluding remarks

The problem of testing whether some given observations can be considered as coming from one of two probability distributions is an old problem in statistics, as given by Kundu et al. [24] and Zografos and Balakrishnan [1]. Our interest is to identify the specific model that is most appropriate to describe the data under consideration. In the spirit of maximum entropy principle, we have to decide in favor of one of the two probability distributions using the following difference

$$\begin{aligned} D_{B,G} &= \hbar_{sh}(g^B) - \hbar_{sh}(g^G) \\ &= \log B(\alpha,\beta) - \alpha[\psi(\alpha) - \psi(\alpha+\beta)] - \beta[\psi(\beta) - \psi(\alpha+\beta)] \\ &\quad - \log\Gamma(\delta) + (\delta-1)\psi(\delta) - \delta - \sum_{k=1}^{\infty}\frac{1}{k(k+1)^{\delta}}. \end{aligned}$$

Observe that $D_{B,G}$ does not depend on the parent distribution F. It is clear because g^B and g^G are based on the same parent F.

In this paper, we studied the generalized logistic distribution and provided detailed mathematical treatment for this distribution. As an application, consider short-and long-term outcomes of constraint-induced movement therapy after stroke investigated in a randomized controlled feasibility trial by Dahl et al. [25]. The 30 patients were assessed at baseline, post treatment, and a 6-month follow-up using the Wolf Motor Function Test as primary outcome measure. The test consists of 17 tasks with two strength and 15 timed tasks which vary from gross shoulder movements to complex finger grips. The measurement was done by the analysis of videotapes. The 30 observations were 0.5, 1.0, 1.0, 1.5, 1.0, 1.5, 2.0, 1.0, 0.5, 1.0, 0.5, 1.0, 1.0, 1.5, 1.0, 0.5, 1.0, 1.5, 1.0, 1.0, 0.5, 1.0, 1.0, 1.5, 1.5, 1.0, 1.0, 0.5, 1.0, and 1.0, measured in seconds.The MLE estimates for the BGL distribution are $\hat{\alpha} = 17.8374$,

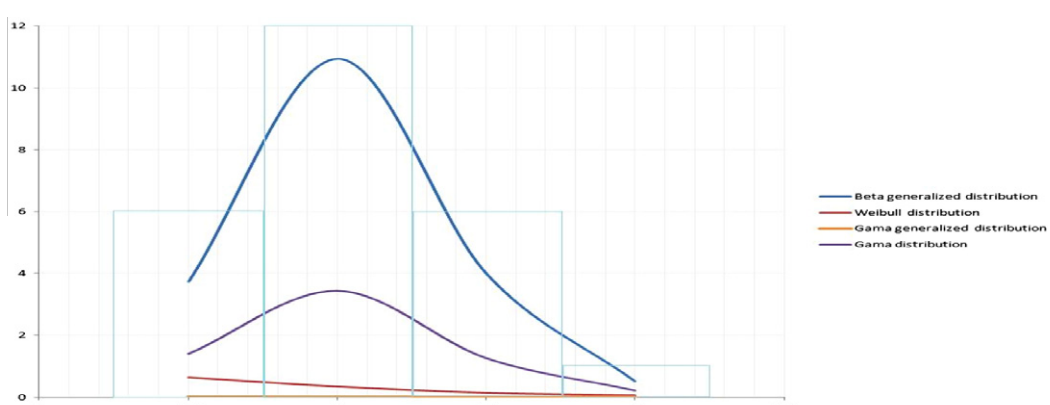

Figure 3 Fitted densities to the data set given in Dahl et al. [25].

$\hat{\beta} = 3.29048, \hat{\lambda} = 1.76279$. Also, the maximized log-likelihood determined by fitting the BGL, GGL, gamma and Weibull distributions are, respectively,

$$\begin{aligned}\log\ L(\mathrm{BGL}) &= 181.8719208,\ \log\ L(\mathrm{GGL}) \\ &= -32.39702109,\ \log\ L(\mathrm{G}) \\ &= 28.84992755,\ \log\ L(\mathrm{W}) = 119.958053.\end{aligned}$$

The fitted BGL, GGL, gamma and Weibull densities are displayed in Fig. 3. They show that the BGL distribution yields a better fit to these data than theGGL, gamma and Weibull distributions.

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