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ORIGINAL RESEARCH

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An efficient algorithm to solve damped forced oscillator problems by Bernoulli operational matrix of integration

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Abstract

An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives was obtained in Yang and Srivastava (Commun Nonlinear Sci Numer Simul 29(1–3):499–504, 2015). In this paper, we obtain the numerical solution of damped forced oscillator problems by employing the operational matrix of integration of Bernoulli orthonormal polynomials. The operational matrix of integration is determined with the help of the integral operator on Bernoulli orthonormal polynomials. Numerical examples of two different problems of spring are given to delineate the performance and perfection of this approach and compared the results with the exact solution.

Keywords: Orthonormal Bernoulli polynomials, Operational matrix, Damped forced oscillator problem

Mathematics Subject Classification: 65T60, 34A35

Introduction

The Bernoulli numbers were probably first appeared in the book of Jakob Bernoulli which was published in 1713. There are numerous applications of Bernoulli numbers in innumerable fields such as algebraic topology, number theory, combinatorics and the calculus of finite differences [11–14]. Many authors[3, 7, 9, 13, 15, 21, 25] studied the solutions of differential equations by Bernoulli polynomials. Dilcher [5] discussed the sums of products of Bernoulli numbers. Later, Tuenter [22] discussed a symmetry of power sum polynomials and Bernoulli numbers.

Srivastava and Todorov [16] gave an explicit formula for the generalized Bernoulli polynomials while Granville and Sun [6] investigated values of polynomials in the context of a problem posed by Emma Lehmer in 1938. A simple property of the Bernoulli and the Euler polynomials was studied by Cheon [2]. Also, a new approach to Bernoulli polynomials was investigated by Costabile et al. [4]. An identity related to symmetry for the Bernoulli polynomials was discussed by Yang [24]. Recently, Boutiche et al. [1] obtained explicit Formulas associated with some families of generalized Bernoulli and Euler Polynomials. Srivastava et al. [19] studied parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and He et al. [8] investigated Higher-Order Convolutions for Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi Polynomials.

Very recently, Srivastava et al. [17, 19] obtained some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and a study on identities and relations involving the modified degenerate hermite-based Apostol-Bernoulli. Srivastava et al. [18] introduced the notions of modified degenerate Hermite-based Apostol-Bernoulli, the modified degenerate Hermite-based Apostol-Euler and the modified Hermite-based Apostol-Genocchi polynomials.

The classifications of oscillating systems are expressed in Thomsen [20] and Rama and Dukkipati [10]. The mechanism that outcomes in dissipation of the energy of an oscillator is named damping. In mechanical oscillator, the damping may be due to (a) viscous drag (b) friction and (c) structure. An oscillator to which a continuous excitation is allowed by some outside agency is called forced oscillator.

Assume a mass M involved to the end of a spring of stiffness constant. A rigid support attached on one end of the spring. Let the driven force acting on the system be $F(t)$. At any instant of time, the system will work under the influence of following forces:

(a) Restoring force, $F_1 = -Sx$, where x is the displacement of the mass from the equilibrium position,
(b) Damping force, $F_2 = -r\frac{\mathrm{dx}}{\mathrm{dt}}$, where r is damping constant,
(c) Driven force, $F_3 = F(t)$.

The negative sign in the first two above expressions shows that both the restoring and damping forces opposes the displacement. By Newton second law of motion yields

$$M\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{dt}^2} = -Sx - r\frac{\mathrm{dx}}{\mathrm{dt}} + F(t).$$

An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives was undertaken by Yang and Srivastava [23].

Motivated by the work of Yang and Srivastava [23], we investigated the numerical solutions of damped forced oscillator problems by operational matrix of integration on Bernoulli orthonormal polynomials. In fact, used the operational matrix of integration of the Bernoulli orthonormal polynomials to find the approximate solutions of damped forced oscillator and spring problems and compared these solutions with their exact solutions.

Bernoulli polynomials

The Bernoulli polynomials of nth degree are defined on the closed interval [0, 1] as

$$\mathcal{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{1}$$

where $B_k = B_k(0)$ is the Bernoulli number for each $k = 0, 1, \ldots, n$.

Leopold Kronecker expressed the Bernoulli number $\mathcal{B}_n$ in the following form:

$$\begin{cases} \mathcal{B}_n & = -\sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \sum_{j=1}^{k} j^n, \quad \textit{for } n \geq 0,\ n \neq 1 \text{ and} \\ \mathcal{B}_1 & = -\frac{1}{2} \end{cases} \tag{2}$$

Thus the first few Bernoulli numbers are given by

$$\mathcal{B}_0 = 1, \mathcal{B}_1 = -\frac{1}{2}, \mathcal{B}_2 = \frac{1}{6}, \mathcal{B}_4 = -\frac{1}{30}, \mathcal{B}_6 = \frac{1}{42}, \mathcal{B}_8 = -\frac{1}{30}, \text{ and } \mathcal{B}_n = 0, \text{ for all odd } n \geq 3.$$

Substituting value of $\mathcal{B}_n$ from relation (2) in Eq. (1), the first ten Bernoulli polynomials are given by

$$\begin{aligned} &\mathcal{B}_0(x) = 1, \quad \mathcal{B}_1(x) = x - \frac{1}{2}, \quad \mathcal{B}_2(x) = x^2 - x + \frac{1}{6}, \\ &\mathcal{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad \mathcal{B}_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad \mathcal{B}_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\ &\mathcal{B}_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \quad \mathcal{B}_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x, \\ &\mathcal{B}_8(x) = x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}, \quad \mathcal{B}_9(x) = x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x. \end{aligned}$$

Bernoulli polynomials satisfy the following well-known properties

$$\frac{\mathrm{d}}{\mathrm{dx}}\mathcal{B}_n(x) = n\mathcal{B}_{n-1}(x), \tag{3}$$

$$\int_0^1 \mathcal{B}_n(x)\,\mathrm{dx} = 0, \text{ for all } n \geq 1. \tag{4}$$

The integral formula for Bernoulli polynomials $\mathcal{B}_n(x)$ is obtained by integrating equation (3) as

$$\mathcal{B}_n(x) = n\int_0^x \mathcal{B}_{n-1}(t)\,\mathrm{dt} + \mathcal{B}_\mathrm{n} \tag{5}$$

Orthonormal Bernoulli polynomials

Using Gram–Schmidt process on Bernoulli polynomials $\mathcal{B}_m(x)$ and normalizing them, a class of orthonormal Bernoulli polynomials of order m, denoted by $w_{0m}, w_{1m}, \ldots, w_{mm}$ has been obtained [15].

The first ten Bernoulli orthonormal polynomials are given by

$$\left.\begin{aligned} w_{09}(x) &= 1, \\ w_{19}(x) &= \sqrt{3}(-1 + 2x), \\ w_{29}(x) &= \sqrt{5}(1 - 6x + 6x^2), \\ w_{39}(x) &= \sqrt{7}(-1 + 12x - 30x^2 + 20x^3), \\ w_{49}(x) &= 3(1 - 20x + 90x^2 - 140x^3 + 70x^4), \\ w_{59}(x) &= \sqrt{11}(-1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5), \\ w_{69}(x) &= \sqrt{13}(1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6), \\ w_{79}(x) &= \sqrt{15}(-1 + 56x - 756x^2 + 4200x^3 - 11550x^4 + 16632x^5 - 12012x^6 + 3432x^7), \\ w_{89}(x) &= \sqrt{17}(1 - 72x + 1260x^2 - 9240x^3 + 34650x^4 - 72072x^5 + 84084x^6 - 51480x^7 + 12870x^8), \\ w_{99}(x) &= \sqrt{19}(-1 + 90x - 1980x^2 + 18480x^3 - 90090x^4 + 252252x^5 - 420420x^6 + 411840x^7 \\ &\quad - 218790x^8 + 48620x^9) \end{aligned}\right\} \tag{6}$$

A function $g \in L^2[0,1]$ may be written as

$$g(t) = \lim_{n\to\infty} \sum_{i=0}^{n} c_{in} w_{in}(t), \tag{7}$$

where $c_{in} = \langle g, w_{in} \rangle$ and $\langle , \rangle$ is the standard inner product on $L^2[0,1]$.

If the series (7) is truncated at $n = m$, we get an approximation $\tilde{g}$ of g as,

$$g \cong \tilde{g} = \sum_{i=0}^{m} c_{im} w_{im}(t) = C^T \mathcal{B}(t), \tag{8}$$

where

$$C = [c_{0m}, c_{1m}, \ldots, c_{mm}]^T, \tag{9}$$

and

$$\mathcal{B}(t) = [w_{0m}(t), w_{1m}(t), \ldots, w_{mm}(t)]^T. \tag{10}$$

The operational matrix of integration

Integrating Bernoulli orthonormal polynomials given in Eq. (6) w.r.t. x from 0 to t can be written as

$$\begin{aligned} \int_0^t w_{im}(x)\,\mathrm{dx} &= \zeta_i(t), \qquad 0 \le t < 1, \qquad i = 0, 1, \ldots, m. \\ &= \sum_{j=0}^{m} c^i_{jm} w_{jm}(t), \quad \text{where } c^i_{jm} = \langle \zeta_i, w_{jm} \rangle, \\ \text{or } \int_0^t w_{im}(x)\,\mathrm{dx} &= [c^i_{0m}, c^i_{1m}, \ldots, c^i_{mm}]\mathcal{B}(t)\,, \text{ for } 0 \le i \le m. \end{aligned} \tag{11}$$

Equation (11) can be written as

$$\int_0^t \mathcal{B}(x)\,\mathrm{dx} = \mathrm{P_{m+1}}\mathcal{B}(\mathrm{t}), \tag{12}$$

where P_{m+1} is the tridiagonal operational matrix of integration of order $(m+1) \times (m+1)$ associated with orthonormal Bernoulli polynomials and is given as

$$\mathcal{P}_{m+1} = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\sqrt{(1).(3)}} & 0 & \ldots & 0 & 0 \\ \frac{-1}{\sqrt{(1).(3)}} & 0 & \frac{1}{\sqrt{(3).(5)}} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & \frac{-1}{\sqrt{(3).(5)}} & 0 & \ldots & 0 & \frac{1}{\sqrt{(2m-1)(2m+1)}} \\ 0 & 0 & \frac{-1}{\sqrt{(2m-1)(2m+1)}} & \ldots & 0 & 0 \end{bmatrix}_{(m+1)\times(m+1)}, \text{ for } m \ge 1. \tag{13}$$

and $P_1 = \left[\frac{1}{2}\right]$.

Numerical examples

In this section, we obtain solutions (approximate) to two differential equations for damped motion and forced motion using operational matrix of integration of Bernoulli orthonormal polynomial and examine its capability and exactness. First, we discuss the differential equation for damped motion.

Example 1 Consider the basic differential equation for the damped motion

$$m\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2}+a\frac{\mathrm{dy}}{\mathrm{dt}}+ky=F(t) \tag{14}$$

where m is the mass of spring, $a>0$ is the damping constant, k is the spring constant, and $F(t)$ is any external impressed force that acts upon the mass [12].

Taking, $m=\frac{w}{g}=\frac{16}{32}=\frac{1}{2}, a=2, k=10$ and $F(t)=5\cos 2t$ Eq. (14) becomes

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2}+4\frac{\mathrm{dy}}{\mathrm{dt}}+20y=10\cos 2t \tag{15}$$

The initial conditions are

$$y(0)=y'(0)=0 \tag{16}$$

The exact solution of this problem is

$$y(t)=e^{-2t}\left(-\frac{3}{8}\sin 4t-\frac{1}{2}\cos 4t\right)+\frac{1}{2}\cos 2t+\frac{1}{4}\sin 2t.$$

Let us now find an approximate solution for $m=2$ of Eq. (15). Consider

$$y''(t)=C^T\mathcal{B}(t) \tag{17}$$

and

$$10\cos 2t=d^T\mathcal{B}(t). \tag{18}$$

Integrating Eq. (17) two times and using the initial conditions, we have

$$y'(t)=C^T\mathcal{P}_3\mathcal{B}(t), \tag{19}$$

$$y(t)=C^T\mathcal{P}_3^2\mathcal{B}(t), \tag{20}$$

where $C=[c_{02},c_{12},\cdots,c_{22}]^T$ is to be determined and $\mathcal{P}_3=\begin{bmatrix}\frac{1}{2} & \frac{1}{2\sqrt{3}} & 0\\ -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}}\\ 0 & -\frac{1}{2\sqrt{15}} & 0\end{bmatrix}$.

Substitution of Eqs. (17)–(20) in Eq. (15) gives

$$C^T\mathcal{B}(t)+4C^T\mathcal{P}_3\mathcal{B}(t)+20C^T\mathcal{P}_3^2\mathcal{B}(t)=d^T\mathcal{B}(t). \tag{21}$$

Since Eq. (21) holds for all $t\in[0,1]$, it reduces to

Table 1 Numerical results of solutions *y*(*t*) corresponding to Example 1

t	Exact solution	Approximate numerical solution				
		m = 2	*m* = 3	*m* = 5	*m* = 7	*m* = 9
0.0	0.0000000000	0.0791342855	− 0.0867203112	0.0017122077	0.0000136462	− 0.0000009350
0.1	0.0430893512	0.1919158386	0.0630874540	0.0424955458	0.0430864782	0.0430890650
0.2	0.1440550329	0.2756498587	0.1858065079	0.1445339827	0.1440607936	0.1440552691
0.3	0.2625776389	0.3303363458	0.2814368505	0.2629329416	0.2625741070	0.2625775397
0.4	0.3658259586	0.3559752999	0.3499784820	0.3654360586	0.3658218164	0.3658258381
0.5	0.4316228960	0.3525667209	0.3914314023	0.4311192308	0.4316278591	0.4316231361
0.6	0.4489459081	0.3201106090	0.4057956114	0.4490846654	0.4489487670	0.4489457778
0.7	0.4165429404	0.2586069640	0.3930711092	0.4171549281	0.4165371068	0.4165428505
0.8	0.3404893353	0.1680557861	0.3532578958	0.3405669919	0.3404902893	0.3404895683
0.9	0.2314079965	0.0484570751	0.2863559712	0.2306662855	0.2314107365	0.2314077114
1.0	0.1018897407	− 0.1001891689	0.1923653355	0.1036007422	0.1019033887	0.1018888057

Table 2 Absolute errors for Example 1 given by $\|y_e - y_a\|$

t	*m* = 2	*m* = 3	*m* = 5	*m* = 7	*m* = 9
0.0	7.91342855×10^{-2}	8.67203112×10^{-2}	1.7122077×10^{-3}	1.36462×10^{-5}	9.350×10^{-7}
0.1	1.488264874×10^{-1}	1.99981028×10^{-2}	5.938054×10^{-4}	2.8730×10^{-6}	2.862×10^{-7}
0.2	1.315948258×10^{-1}	4.17514750×10^{-2}	4.789498×10^{-4}	5.7607×10^{-6}	2.362×10^{-7}
0.3	6.77587069×10^{-2}	1.88592116×10^{-2}	3.553027×10^{-4}	3.5319×10^{-6}	9.92×10^{-8}
0.4	9.8506587×10^{-3}	1.58474766×10^{-2}	3.899000×10^{-4}	4.1422×10^{-6}	1.205×10^{-7}
0.5	7.90561751×10^{-2}	4.01914937×10^{-2}	5.036652×10^{-4}	4.9631×10^{-6}	2.401×10^{-7}
0.6	1.288352991×10^{-1}	4.31502967×10^{-2}	1.387573×10^{-4}	2.8589×10^{-6}	1.303×10^{-7}
0.7	1.579359764×10^{-1}	2.34718312×10^{-2}	6.119877×10^{-4}	5.8336×10^{-6}	8.99×10^{-8}
0.8	1.724335492×10^{-1}	1.27685605×10^{-2}	7.76566×10^{-5}	9.540×10^{-7}	2.330×10^{-7}
0.9	1.829509214×10^{-1}	5.49479747×10^{-2}	7.417110×10^{-4}	2.7400×10^{-6}	2.851×10^{-7}
1.0	2.020789096×10^{-1}	9.04755948×10^{-2}	1.7110015×10^{-3}	1.36480×10^{-5}	9.350×10^{-7}

$$C^T = d^T[(I + 4\mathcal{P}_3 + 20\mathcal{P}_3^2)]^{-1}, \tag{22}$$

where I is a unit matrix of order 3×3.

Taking $m = 2$ and using Eq. (10) in Eq. (18), we have

$$d^T = \left[\, 4.54649, -4.38944, -0.749478 \,\right]. \tag{23}$$

Substituting the values of $\mathcal{P}_3$ and d^T in Eq. (22) gives

$$C^T = \left[\, -1.45209, -2.89489, 2.74164 \,\right]. \tag{24}$$

The approximate solution $y_a(t)$ is obtained by using Eqs. (10), (13) and (24) in Eq. (20) for $m = 2$ as

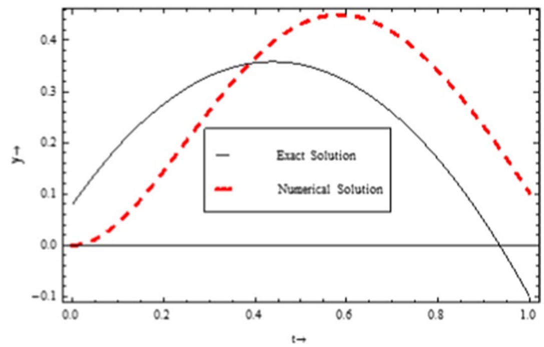


Fig. 1 The exact and the approximate values of Eq. (15) for $m = 2$ in Example 1

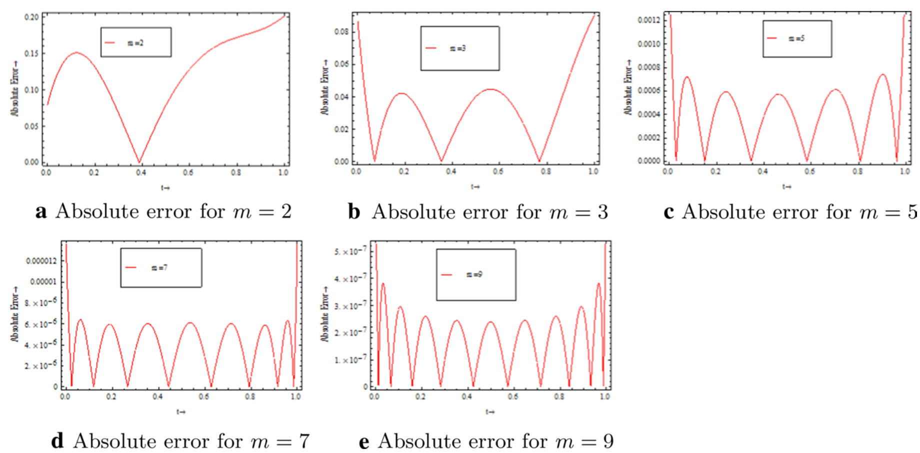


Fig. 2 The absolute errors for $m = 2, 3, 5, 7$ and 9

$$y_a(t) = 0.0791343 + 1.27305\,t - 1.45238\,t^2. \tag{25}$$

In order to choose suitable value of m, convergence study has been accomplished for different values of $y(t)$ for $t = 0.0, 0.1,$ to 1.0 and for $m = 2, 3, 5, 7,$ and 9. The results of this study are presented in Table 1. It is observed that the value of the approximate convergence is nearing to the exact solution as we increase the value of m. Note that m is fixed at 9 as no further rectification or improvement was found. The calculations are worked out using Mathematica 7.0 by Wolfram.

Now, we give the calculations for the absolute error calculated using the formula $\|y_e - y_a\|$ and presented in Table 2.

Next, we give Fig. 1, which depicts the exact and approximate numerical solutions of Eq. (15). Table 1 illustrates that value of exact and approximate solution of Eq. (15) for different values of t and m .

The graphs of absolute errors for $m = 2, 3, 5, 7$ and 9 can be seen in the Fig. 2.

In the following example, we shall discuss the approximate solution of the differential equation for forced motion using Bernoulli orthonormal polynomials compare the result with the exact solution of the equation.

Example 2 Consider the following forced motion equation given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 20y = 1. \tag{26}$$

The initial conditions are

$$y(0) = y'(0) = 0 \tag{27}$$

The exact solution of this problem is

$$y(t) = -e^{-2t}\left[\frac{1}{20}\cos 4t + \frac{1}{40}\sin 4t\right] + \frac{1}{20}.$$

Let us now find an approximate solution for m=2 of Eq. (26). Consider

$$y''(t) = C^T \mathcal{B}(t) \tag{28}$$

and

$$1 = d^T \mathcal{B}(t). \tag{29}$$

Integrating (28) two times and using the initial conditions, we have

$$y'(t) = C^T \mathcal{P}_3 \mathcal{B}(t), \tag{30}$$

$$y(t) = C^T \mathcal{P}_3^2 \mathcal{B}(t), \tag{31}$$

where $C = [c_{02}, c_{12}, \cdots, c_{22}]^T$ is to be determined and $\mathcal{P}_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} \\ 0 & -\frac{1}{2\sqrt{15}} & 0 \end{bmatrix}$.

Substitution of Eqs. (28)–(31) in Eq. (26) gives

$$C^T \mathcal{B}(t) + 4C^T \mathcal{P}_3 \mathcal{B}(t) + 20C^T \mathcal{P}_3^2 \mathcal{B}(t) = d^T \mathcal{B}(t). \tag{32}$$

Since Eq. (32) holds for all $t \in [0, 1]$, it reduces to

$$C^T = d^T [(I + 4\mathcal{P}_3 + 20\mathcal{P}_3^2)]^{-1}, \tag{33}$$

where I is a unit matrix of order 3×3.

Taking $m = 2$ and using Eq. (10) in Eq. (29), we have

$$d^T = [1, 0, 0]. \tag{34}$$

The approximate solution $y_a(t)$ is obtained by using relations (10), (13) and (34) in Eq. (31) for $m = 2$ as

Table 3 Numerical results of solutions *y*(*t*) corresponding to Example 2

t	Exact solution	Approximate numerical solution				
		$m = 2$	$m = 3$	$m = 5$	$m = 7$	$m = 9$
0.0	0.0000000000	− 0.0073937153	− 0.0040027165	0.0001187872	0.0000022238	− 0.0000000886
0.1	0.0043242326	0.0060536044	0.0055340385	0.0042852305	0.0043240016	0.0043242054
0.2	0.0146277221	0.0181146026	0.0162203655	0.0146643552	0.0146284149	0.0146277447
0.3	0.0272688450	0.0287892791	0.0273042888	0.0272894001	0.0272682392	0.0272688349
0.4	0.0394275753	0.0380776340	0.0380338327	0.0393956406	0.0394272213	0.0394275646
0.5	0.0492917976	0.0459796673	0.0476570217	0.0492590084	0.0492925214	0.0492918201
0.6	0.0560187959	0.0524953789	0.0554218801	0.0560347104	0.0560189543	0.0560187830
0.7	0.0595522819	0.0576247689	0.0605764323	0.0595958489	0.0595515045	0.0595522742
0.8	0.0603722504	0.0613678373	0.0623687027	0.0603720403	0.0603725898	0.0603722720
0.9	0.0592403619	0.0637245841	0.0600467155	0.0591880351	0.0592405489	0.0592403351
1.0	0.0569836042	0.0646950092	0.0528584952	0.0571023372	0.0569858282	0.0569835156

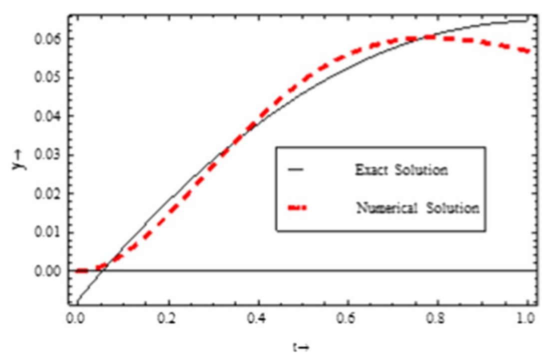


Fig. 3 The exact and the approximate solution of Eq. (26) for $m = 2$

$$y_a(t) = -0.00739372 + 0.141405\, t - 0.0693161\, t^2.$$

The convergence study suitable value of m has been done for different values of $y(t)$ for $t = 0.0, 0.1,$ to 1.0 and for $m = 2, 3, 5, 7,$ and 9. The outcome of this study are given in Table 3. It has been observed that the value of the approximate convergence is sufficiently near to the exact solution for the increasing value of m. Also,the value of m is fixed at 9 as no further improvement is found. The calculations are worked out using Mathematica 7.0 by Wolfram.

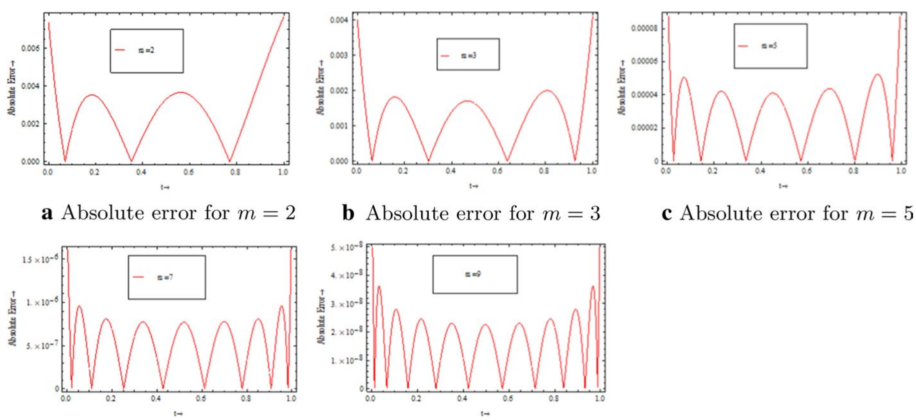
Table 3 shows that value of exact and approximate numerical solution of Eq. (26) for different values of t and m.

Figure 3 illustrates the exact and approximate numerical solutions of Eq. (26).

The calculations for the absolute error calculated using the formula $\|y_e - y_a\|$ and presented in Table 4.

Table 4 Absolute errors for Example 2 given by $\|y_e - y_a\|$ are

t	$m = 2$	$m = 3$	$m = 5$	$m = 7$	$m = 9$
0.0	7.3937153×10^{-3}	4.0027165×10^{-3}	1.187872×10^{-4}	2.2238×10^{-6}	8.86×10^{-8}
0.1	1.7293718×10^{-3}	1.2098059×10^{-3}	3.90021×10^{-5}	2.310×10^{-7}	2.72×10^{-8}
0.2	3.4868805×10^{-3}	1.5926434×10^{-3}	3.66331×10^{-5}	6.928×10^{-7}	226×10^{-8}
0.3	1.5204341×10^{-3}	3.54438×10^{-5}	2.05551×10^{-5}	6.058×10^{-7}	1.01×10^{-8}
0.4	1.3499413×10^{-3}	1.3937426×10^{-3}	3.19347×10^{-5}	3.540×10^{-7}	1.07×10^{-8}
0.5	3.3121303×10^{-3}	1.6347759×10^{-3}	$3.2789/2 \times 10^{-5}$	7.238×10^{-7}	2.25×10^{-8}
0.6	3.5234170×10^{-3}	5.969158×10^{-4}	1.59145×10^{-5}	1.584×10^{-7}	1.29×10^{-8}
0.7	1.9275130×10^{-3}	10241504×10^{-5}	4.35670×10^{-5}	7.774×10^{-7}	7.7×10^{-9}
0.8	9.955869×10^{-4}	1.9964523×10^{-3}	2.101×10^{-7}	3.394×10^{-7}	2.16×10^{-8}
0.9	4.4842222×10^{-3}	8.063536×10^{-4}	5.23268×10^{-5}	1.870×10^{-7}	2.68×10^{-8}
1.0	7.7114050×10^{-3}	4.1251090×10^{-3}	1.187330×10^{-5}	2.2240×10^{-6}	8.86×10^{-8}



a Absolute error for $m = 2$ **b** Absolute error for $m = 3$ **c** Absolute error for $m = 5$ **d** Absolute error for $m = 7$ **e** Absolute error for $m = 9$

Fig. 4 The absolute errors for $m = 2, 3, 5, 7$ and 9

Let us now see the graphs of absolute errors for $m = 2, 3, 5, 7$ and 9 in the figures given below (Fig. 4):

Conclusions

Damped forced oscillatory differential equations have a very important role in physics, mathematics and engineering. Through this work, we use the operational matrix of integration of Bernoulli orthonormal polynomials to find approximate solutions of damped forced oscillator and spring problems. A simple procedure of forming the operational matrix of integration of the Bernoulli orthonormal polynomials is given. It is observed that the exact and approximate solutions of these problems are approximately coinciding. This method is more precise, easy to use and stable as shown in the given numerical examples.

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Authors' contributions

SR carried out the study of Bernoulli orthonormal polynomials and performed the statistical analysis. Further, SR participated in the design of the study and was a major contributor in writing the manuscript. MS participated in the sequence alignment. SS conceived of the study, and participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

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Not applicable.

Competing interests

The authors declare that they have no competing interests.

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