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ORIGINAL ARTICLE

T-proximity compatible with T-neighbourhood structure

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Abstract In this paper, we show that every T-neighbourhood space induces a T-proximity space, where T stands for any continuous triangular norm. An axiom of T-completely regular of T-neighbourhood spaces introduced by Hashem and Morsi (2003) [3], guided by that axiom we supply a Sierpinski object for category T-PS of T-proximity spaces. Also, we define the degree of functional T-separatedness for a pair of crisp fuzzy subsets of a T-neighbourhood space. Moreover, we define the Čech T-proximity space of a T-completely regular T-neighbourhood space, hence, we establishes it is the finest T-proximity space which induces the given T-neighbourhood space.

1. Introduction

In [2], Hashem and Morsi deduced the T-neighbourhood spaces, for each continuous triangular norm T. In this manuscript, we introduce for a given T-neighbourhood space, a new structure of functional T-separatedness, which generates a T-proximity space. Moreover, we show that the existence of correspondence between T-proximity and T-neighbourhood structure is fulfilled. Also, we define the Čech T-proximity space for a T-neighbourhood space, we establish that it is the finest T-proximity space which generates the given T-neighbourhood space. We divided this manuscript into four sections:

In the first section, we recapitulate on some definitions and ideas of fuzzy sets, T-proximity spaces and T-uniform spaces.

In the second section, we introduce five propositions, which well be used to supply the notion of a Sierpinski object for category T-PS of T-proximity spaces.

In the third section, we introduce the definition and properties of functional T-separatedness of crisp fuzzy subsets for a T-neighbourhood space, together with an illustrative example for this notion.

In the fourth section, we complete the proof of the compatibility between T-proximity spaces and T-neighbourhood spaces. Also, we introduce the notion of Čech T-proximity space.

2. Prerequisites

In this section we will recall some of the definitions related to fuzzy sets, T-proximity spaces, T-uniform spaces and I-topological spaces.

A triangular norm (cf. [10]) is a binary operation on the unit interval $I = [0, 1]$ that is associative, symmetric, monotone in each argument and has the neutral element 1.

A fuzzy set λ in a universe set X, introduced by Zadeh in [11], is a function $\lambda : X \to I$. The collection of all fuzzy sets

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of X is denoted by I^X. The height of a fuzzy set λ is the following real number: hgt$\lambda = sup\{\lambda(x): x \in X\}$.

If H is a subset of X, then we shall denote to its characteristic function by the symbol $\mathbf{1}_H$, said to be a crisp fuzzy subset of X. We also denote the constant fuzzy set of X with value $\alpha \in I$ by $\underline{\alpha}$.

Given a fuzzy set $\lambda \in I^X$ and a real number $\alpha \in I_1 = [0, 1[$, the strong α-cut of λ is the following subset of X: $\lambda^\alpha = \{x \in X: \lambda(x) > \alpha\}$; and the weak α-cut of λ is the subset of X: $\lambda_{\alpha^*} = \{x \in X: \lambda(x) \geqslant \alpha\}$.

For a given two fuzzy sets $\mu, \lambda \in I^X$ we denote by $\mu T \lambda$ the following fuzzy set of X: $(\mu T \lambda)(x) = \mu(x) T \lambda(x)$, $x \in X$.

We follow Lowen's definition of a fuzzy closure operator on a set X [7]. This is an operator $^-: I^X \to I^X$ that satisfies $\mu^- \geqslant \mu, (\mu \bigvee \lambda)^- = \mu^- \bigvee \lambda^-$ for all $\mu, \lambda \in I^X$, and $\underline{\alpha}^- = \underline{\alpha}$ for all $\alpha \in I$. We may define an I-topology in the usual way, namely assuming a fuzzy set μ to be closed if and only if $\mu^- = \mu$. We denote this I-topology by τ. The pair (X, τ) is called an I-topological space. A function f: $(X, ^-) = (X, \tau) \to (Y, ^{-\prime}) = (Y, \tau')$, between two I-topological spaces, is said to be continuous [7]; if $f^{\leftarrow}(\mu) \in \tau$, for all $\mu \in \tau'$, equivalently if $f(\lambda^-) \leqslant [f(\lambda)]^{-\prime}$, for all $\lambda \in I^X$.

I-filters and I-filterbases were introduced by Lowen in [8]. An I-filter in a universe X is a nonempty collection $\mathfrak{I} \subset I^X$ which satisfies: $\underline{0} \notin \mathfrak{I}$, $\mathfrak{I}$ is closed under finite meets and contains all the fuzzy supersets of its individual members. An I-filterbase in X is a nonempty collection $\mathcal{B} \subset I^X$ which satisfies: $\underline{0} \notin \mathcal{B}$ and the meet of two members of $\mathcal{B}$ contain a member of $\mathcal{B}$.

The T-neighbourhood spaces and T-proximity spaces were introduced by Hashem and Morsi, for more definitions and properties, we can refer to [1,2].

Definition 2.1 [2]. A T-neighbourhood space is an I-topological space $(X, \tau) = (X, ^-)$ whose fuzzy closure operator $^-$ is induced by some indexed family $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$ of I-filterbases on I^X, in the following manner: For all $\mu \in I^X$ and $x \in X$,

$$\mu^-(x) = \inf_{v \in \mathcal{B}} \text{hgt}(\mu T v).$$

Theorem 2.1 [1]. *A function δ: $I^X \times I^X \to I$ is a T-proximity on a set X if and only if it satisfies the following six axioms, the first five of which are properties of its restriction δ: $2^X \times 2^X \to I$. For all $H, M, N \in 2^X$:*

(TP1) $\delta(\mathbf{1}_\emptyset, \mathbf{1}_X) = 0$;
(TP2) $\delta(\mathbf{1}_H, \mathbf{1}_M) = \delta(\mathbf{1}_M, \mathbf{1}_H)$;
(TP3) $\delta(\mathbf{1}_{(H \cup M)}, \mathbf{1}_N) = \delta(\mathbf{1}_H, \mathbf{1}_N) \bigvee \delta(\mathbf{1}_M, \mathbf{1}_N)$;
(TP4) *If* $\delta(\mathbf{1}_H, \mathbf{1}_M) < (\theta T \beta)$ *for some* $\theta, \beta \in I_1$, *there is* $C \in 2^X$ *such that* $\delta(\mathbf{1}_H, \mathbf{1}_C) \leqslant \theta$ *and* $\delta(\mathbf{1}_{(X-C)}, \mathbf{1}_M) \leqslant \beta$;
(TP5) *If* $H \cap M \neq \emptyset$, *then* $\delta(\mathbf{1}_H, \mathbf{1}_M) = 1$;
(TP6) $\delta(\mu, \lambda) = \bigvee_{\theta, \beta \in I}[\theta T \beta T \delta(\mathbf{1}_{\mu_{\theta*}}, \mathbf{1}_{\lambda_{\beta*}})]$, $\mu, \lambda \in I^X$.

The real number $\delta(\mathbf{1}_H, \mathbf{1}_M)$ can be interpreted as the degree of proximity between the two crisp fuzzy subsets $\mathbf{1}_H$ and $\mathbf{1}_M$, and the number $\delta(\mu, \lambda)$ can be interpreted as the degree of nearness of the fuzzy sets μ and λ. The pair (X, δ) is said to be a T-proximity space.

A function $f: (X, \delta) \to (Y, \rho)$ between two T-proximity spaces, is said to be continuous, if

$$\delta(\mu, \lambda) \leqslant \rho(f(\mu), f(\lambda)), \quad \forall\ \mu, \lambda \in I^X. \tag{1}$$

This is shown in [1], to be equivalent to

$$\delta(\mathbf{1}_H, \mathbf{1}_M) \leqslant \rho(f(\mathbf{1}_H), f(\mathbf{1}_M)), \quad \forall\ \mathrm{H}, \mathrm{M} \in 2^X. \tag{2}$$

Given two T-proximities δ_1, δ_2 on X, δ_1 is said to be coarser than δ_2 (δ_2 is said to be finer than δ_1), if the identity function on X is a proximally continuous from (X, δ_2) to (X, δ_1), that is $\delta_2(\mathbf{1}_H, \mathbf{1}_M) \leqslant \delta_1(\mathbf{1}_H, \mathbf{1}_M)$, for every pair of crsip fuzzy sets $H, M \subset X$.

In [5], Höhle defines for every $\psi, \varphi \in I^{X \times X}$ and $\lambda \in I^X$: The T-section of ψ over λ by $(\psi\langle\lambda\rangle_T)(x) = \sup_{z \in X}$ $[\lambda(z) T \psi(z, x)]$, $x \in X$. The T-composition of ψ, φ by $(\psi\ o_T\ \varphi)(x, y) = $ -$\sup_{z \in X}[\varphi(x, z) T \psi(z, y)]$, $x, y \in X$. Also, in [5], Höhle defines the (fuzzy) T-uniform spaces and uniformly continuous of a function f: $(X, \Omega) \to (Y, \varpi)$, between T-uniform spaces, as for every $\varphi \in \varpi$ there is $\psi \in \Omega$ such that $\psi \leqslant (f \times f)^{\leftarrow}(\varphi)$.

A functor from category T-US of T-uniform spaces to category T-PS of T-proximity spaces is obtained in [1], by leaving morphisms unchanged, and by sending $(X, \Omega) \in T$-US to the T-proximity space $(X, \delta(\Omega))$ given by

$$(\delta(\Omega))(\mu, \lambda) = \inf_{\psi \in \Omega} \sup_{y \in X} (\mu T \psi\langle\lambda\rangle_T)(y), \quad \mu, \lambda \in I^X. \tag{3}$$

Another functor from category T-PS to category I-TS of I-topological spaces is obtained in [2], by leaving morphisms unchanged and by sending $(X, \delta) \in T$-PS to the I-topological space $(X, \tau(\delta))$ with the fuzzy closure operator:

$$\mu^-(x) = \delta(\mu, \mathbf{1}_x), \quad \mu \in I^X,\ x \in X. \tag{4}$$

In [2], this I-topological space $(X, \tau(\delta))$ is shown to be a T-neighbourhood space. By applying these two functors to the identity function on X, we find that if Ω_1, Ω_2 are T-uniformities on X and $\Omega_1 \subseteq \Omega_2$, then $\delta(\Omega_2)$ is coarser than $\delta(\Omega_1)$, while if δ_1, δ_2 are T-proximities on the set X, and δ_1 is coarser than δ_2, then $\tau(\delta_1) \subseteq \tau(\delta_2)$.

3. A Sierpinski object for the category T-PS

A distance distribution function (ddf) [10], is a function from the set R^+ of positive real numbers to the unit interval I, which is isotone, left continuous and has supremum 1. The set of all ddf's is denoted by $\mathcal{D}$. The partial order $\preceq$ on $\mathcal{D}$ is the opposite of the partial order of ddf's as real functions. Obviously $(\mathcal{D}, \preceq)$ is a lattice, we denote its join by $\sqcup$, and its meet by $\sqcap$. The set R^* of nonnegative real numbers, can be embedded in $(\mathcal{D}, \preceq)$ by sending every $r \geqslant 0$ onto the crisp ddf ε_r given by

$$\varepsilon_r(s) = \begin{cases} 0, & 0 < s \leqslant r \\ 1, & s > r. \end{cases}$$

In particular, $\varepsilon_0 =$ the constant function 1 on R^+, is the bottom element of $(\mathcal{D}, \preceq)$. The T-addition $\oplus_T$ and scaler multiplication by nonnegative reals are defined on $\mathcal{D}^+$ as follows: for $\eta, \zeta \in \mathcal{D}^+$ and $s > 0$:

$$(\eta \oplus_T \zeta)(s) = \sup\{\eta(b) T \zeta(s - b) : 0 < b < s\}. \tag{5}$$

$$(b\eta)(s) = \eta(s/b), \quad \text{for any } b > 0. \tag{6}$$

Definition 3.1. [10]A probabilistic T-metric (T-PM) on a set X is a function $\mp : X \times X \to \mathcal{D}^+$ that satisfies, for all $x, y, z \in X$:

(TPM1) $\mp(x, x) = 0$;
(TPM2) $\mp(x, y) = \mp(y, x)$;

(TPM3) $\mp(x, z) \preceq \mp(x,y) \oplus_T \mp(y,z)$;
(TPM4) if $x \neq y$, then $\mp(x,y) \neq \varepsilon_0$.

The pair $(X, \mp)$ is called a probabilistic T-metric space. If $\mp$ satisfies (TPM1)–(TPM3) only, then $\mp$ is called a probabilistic T-pseudometric.

We shall apply the following notation: Given two nonempty subsets H, M of a probabilistic T-metric space $(X, \mp)$, we put

$$\mp(H, M) = \{\mp(x,y) : (x,y) \in H \times M\}. \tag{7}$$

A function $F_H : X \to \mathcal{D}^+$ is defined by

$$F_H = \mp(H, x). \tag{8}$$

Also, for $\eta \in \mathcal{D}^+$, we write

$$\eta(0+) = \inf_{r>0} \eta(r). \tag{9}$$

Theorem 3.1. [3] *Let $(X, \mp)$ be a probabilistic T-metric space. Then the T-proximity $\delta = \delta(\mp)$, induced by $\mp$, is given by:*

$$\delta(\mu,\lambda) = \inf_{r>0} \sup_{x,y\in X} \{\mu(x) T(\mp(x,y))(r) T\lambda(y)\}, \quad \mu,\lambda \in I^X.$$

In particular, $\delta(\mathbf{1}_H, \mathbf{1}_M) = (\mp(H,M))(0+)$, $H, M \in 2^X$.

Consequently, the fuzzy closure operator $^-$ of the T-neighbourhood space $(X, \tau(\mp))$ is given by:

$$\mu^-(x) = \inf_{r>0} \sup_{y\in X} [\mu(y)\ T(\mp(x,y))(r)], \quad \mu \in I^X,\ x \in X.$$

In particular, $(\mathbf{1}_H)^-(x) = (\mp(H,x))(0+)$, $H \in 2^X$, $x \in X$.

For each triangular norm T, Höhle introduced in [4] a probabilistic T-metric on $\mathcal{D}^+$, which we denote by $\Im$, as follows: for all $\eta, \zeta \in \mathcal{D}^+$,

$$\Im(\eta,\zeta) = \sqcap\{\xi \in \mathcal{D}^+ : \eta \preceq \zeta \oplus_T \xi \text{ and } \zeta \preceq \eta \oplus_T \xi\}. \tag{10}$$

Obviously, it follows at once that [5]:

$$\Im(\eta, \varepsilon_0) = \eta, \quad \forall\ \eta \in \mathcal{D}^+. \tag{11}$$

In [4–6], Höhle defines for a T-PM $\mp$ on a set X, a T-uniformity $\Omega(\mp)$ on X by its T-uniform base $\{\psi_r \in I^{X\times X}: r > 0\}$, where

$$\psi_r(x,y) = (\mp(x,y))(r), \quad x,y \in X. \tag{12}$$

Proposition 3.1. [10]*In $(\mathcal{D}^+, \Omega(\Im))$, the binary operations $\sqcup$, $\sqcap$, $\oplus_T$ are uniformly continuous. Also, scaler multiplication on $\mathcal{D}^+$ by a fixed $b \geqslant 0$ is uniformly continuous. In consequence, if $f, g : X \to \mathcal{D}^+$ are two (uniformly) continuous functions, then so will be $f \sqcup g$, $f \sqcap g$, $f \oplus_T g$, bf.*

Proposition 3.2. *Let M be a nonempty subset of X and $f : X \to (\mathcal{D}^+, \tau(\Im))$ be a function such that $f(M) = \varepsilon_0$. Then for all $x \in X$, we have $[f(M)]^-(f(x)) = (f(x))(0+)$.*

Proof 1. For every $x \in X$, we have

$$\begin{aligned}(f(x))(0+) &= \Im(f(x), \varepsilon_0)(0+), && \text{by}(11)\\ &= \Im(\varepsilon_0, f(x))(0+), && \text{by (TPM2)}\\ &= (\varepsilon_0)^-(f(x)), && \text{by Theorem 3.1}\\ &= [f(M)]^-(f(x)). && \square\end{aligned}$$

Definition 3.2. [3]A T-neighbourhood space (X, Σ) is said to be T-completely regular if its I-topology Σ equals the initial I-topology for the family of all continuous functions: $(X, \Sigma) \to (\mathcal{D}^+, \tau(\Im))$.

Theorem 3.2. [3]*Let $(X, ^-) = (X, \Sigma)$ be a T-neighbourhood space. Then, the following statements are equivalent:*

(i) *$(X, ^-)$ is T-completely regular;*
(ii) *$(X, ^-)$ is T-unifomizable;*
(iii) *For every $M \subseteq X$, $x \in X$ and $\theta \in I_0$, there is a continuous function*

$$f : (X, ^-) \to (\mathcal{D}^+, \tau(\Im)) \text{ such that} f(M) = \varepsilon_0 \quad \text{and}$$
$$(f(x))(0+) < (\mathbf{1}_M)^-(x) + \theta. \tag{13}$$

In categorical terms, Theorem 3.2 says that $(\mathcal{D}^+, \tau(\Im))$ is a Sierpinski object for the full subcategory of T-NS, of T-uniformizable T-neighbourhood spaces. For a brief introduction to topological categories and Sierpinski objects, see [9]. Now, we proceed to supply a Sierpinski object for category T-PS.

Definition 3.3. A gauge for a T-uniformity Ω on a set X is a T-PM $\mp$ on X such that $\Omega(\mp) \subseteq \Omega$.

Proposition 3.3. *Let ψ be a fuzzy vicinity in a T-uniform space (X, Ω). Then there is a gauge $\mp$ for Ω such that $\psi \in \Omega(\mp)$.*

Proof 2. We can choose a decreasing sequence $(\psi_{n-1})_{n\in N}$ of symmetric members of Ω, such that $\psi_0 \leqslant \psi$ and $(\psi_n\ o_T\ \psi_n\ o_T\ \psi_n) - 2^{-n} \leqslant \psi_{n-1}$.

Define $F : X \times X \to \mathcal{D}^+$ as follows:

$$(F(x,y))(s) = \begin{cases} 0, & s = 0\\ \psi_k(x,y), & 2^{-k} < s \leqslant 2^{-(k-1)}\\ 1, & s > 1.\end{cases}$$

Next, define $\mp : X \times X \to \mathcal{D}^+$, by

$$\mp(x,y) = \bigvee \left\{\oplus_{i=1}^n F(x_{i-1}, x_i) : x_i \in X, x_0 = x, x_n = y, n \in N\right\}.$$

It is easy to see that $\mp$ is a probabilistic T-pseudometric with $\Omega(\mp) \subseteq \Omega$, also
$\psi_{n+1}(x,y) \leqslant (\mp(x,y))2^{-n} \leqslant \psi_n(x,y) + 2^{-n}$.
Now, for every $\varepsilon > 0$, choose $n \in N$ such that $\epsilon > 2^{-n}$, we get

$$(\mp(x,y))(2^{-n}) - \varepsilon \leqslant \psi_n(x,y) \leqslant \psi(x,y).$$

Which proves that $\psi \in \Omega(\mp)$. $\square$

Proposition 3.4. *Let (X, Ω) be a T-uniform space. Then for all $\psi \in \Omega$, $\theta \in I_0$, $x \in X$ and a nonempty subset M of X, there is a uniformly continuous function $g : (X, \Omega) \to (\mathcal{D}^+, \Omega(\Im))$ such that $g(M) = \varepsilon_0$ and $(g(x))$ (1) $< (\psi\langle \mathbf{1}_M\rangle_T)(x) + \theta$.*

Proof 3. By Proposition 3.3, there is a T-PM $\mp$ on X with $\psi \in \Omega(\mp) \subseteq \Omega$.

Consequently, $\psi + \frac{1}{2}\theta \geqslant \psi_1 \in \Omega(\mp)$, where $\psi_1(x,y) = (\mp(x,y))$ (1), $\forall\ x,y \in X$ (cf.(12)).

Let $F_M : X \to \mathcal{D}^+$ be a function given by

$$F_M(x) = \mp(M,x) = \sup_{y\in M} \mp(y,x), \quad x \in X.$$

This F_M is a uniformly continuous [6. Proposition 5.2], also, $F_M(M) = \varepsilon_0$, (by (TPM1)). Thus for every $x \in X$, we have

$$\begin{aligned}(\psi\langle \mathbf{1}_M\rangle_T)(x) + \theta &\geqslant (\psi_1\langle \mathbf{1}_M\rangle_T)(x) + \frac{1}{2}\theta \\ &= \left\{\sup_{y\in X}[\mathbf{1}_M(y) T \psi_1(y,x)]\right\} + \frac{1}{2}\theta \\ &= \sup_{y\in M}\psi_1(y,x) + \frac{1}{2}\theta \\ &> \sup_{y\in M} \mp(y,x)(1) \\ &= (F_M(x))(1).\end{aligned}$$

This shows that F_M satisfies all the properties stated for g. □

Proposition 3.5. *If (X,δ) is a T-proximity space, then for all nonempty subsets H, M of X and all $\alpha > \delta(\mathbf{1}_H,\mathbf{1}_M)$, there is a proximally continuous function $f : (X,\delta) \to (\mathcal{D}^+, \delta(\Im))$ such that $f(M) = \varepsilon_0$, f is constant on H and $\eta = f(H)$ satisfies $\eta(1) = \alpha$.*

Proof 4. There is a T-uniformity Ω on X that induces δ [1]. Since $\alpha > \delta(\mathbf{1}_H,\mathbf{1}_M)$, then there is, by (3), a fuzzy vicinity $\psi \in \Omega$ such that

$$\sup_{y\in X}(\mathbf{1}_H T \psi\langle \mathbf{1}_M\rangle_T)(y) = \alpha - \theta, \quad \text{for some } \theta > 0.$$

By Proposition 3.4, there is a uniformly continuous function $g : (X,\Omega) \to (\mathcal{D}^+, \Omega(\Im))$ such that $g(M) = \varepsilon_0$ and for all $x \in H$; $g(x)$ (1) $< (\psi\langle \mathbf{1}_M\rangle_T)(x) + \theta \leqslant (\alpha - \theta) + \theta = \alpha$. Define $\eta \in \mathcal{D}^+$ by

$$\eta(s) = \begin{cases} 0, & s = 0 \\ \alpha, & 0 < s \leqslant 1 \\ 1, & s > 1. \end{cases}$$

Then $\eta \preceq g(x)$ for all $x \in H$. Define $f : X \to \mathcal{D}^+$, by $f(x) = g(x) \sqcap \eta$, $\forall x \in X$. Then f is uniformly continuous (by Proposition 3.1) and hence it is a proximally continuous from (X,δ) to $(\mathcal{D}^+, \delta(\Im))$. Also, $f(M) = \varepsilon_0$ and $f(H) = \eta$.

This completes the proof. □

Theorem 3.3. *Every T-proximity δ on a set X is the initial T-proximity (= optimal lift in T-PS) for the set of all proximally continuous functions from (X,δ) into $(\mathcal{D}^+, \delta(\Im))$. Therefore, the T-proximity space $(\mathcal{D}^+, \delta(\Im))$ is a Sierpinski object for the category T-PS.*

Proof 5. Let δ_1 be the mentioned initial (coarsest) T-proximity on X. Then δ_1 is coarser than δ.

Now, we demonstrate the opposite relationship. Let nonempty subsets $H, M \in 2^X$ and $\alpha \in I$, be such that $\alpha > \delta(\mathbf{1}_H,\mathbf{1}_M)$.

Then by Proposition 3.5, there is a proximally continuous function $f : (X,\delta) \to (\mathcal{D}^+, \delta(\Im))$ that satisfies $f(M) = \varepsilon_0$, f is constant on H, and $\eta = f(H)$ has $\eta(1) = \alpha$. By definition of δ_1, f is also a proximally continuous from (X,δ_1) into $(\mathcal{D}^+, \delta(\Im))$.

Consequently,

$$\begin{aligned}\delta_1(\mathbf{1}_H,\mathbf{1}_M) &\leqslant (\delta(\Im))(f(\mathbf{1}_H),(f\mathbf{1}_M)), \quad \text{by (2)} \\ &= \delta(\Im))(\mathbf{1}_{f(H)},\mathbf{1}_{f(M)}), \quad \text{clear} \\ &= \Im(f(H),f(M))(0+), \quad \text{by Theorem 3.1} \\ &= \Im(\eta,\varepsilon_0)(0+) \\ &= \eta(0+), \quad \text{by (11)} \\ &\leqslant \eta(1), \quad \text{by isotonicity of } \eta \\ &= \alpha.\end{aligned}$$

This establishes that $\delta_1(\mathbf{1}_H,\mathbf{1}_M) \leqslant \delta(\mathbf{1}_H,\mathbf{1}_M)$ for all $H, M \in X$, that is δ is coarser than δ_1. Therefore, $\delta = \delta_1$, as required. □

4. Functional *T*-separatedness

In this section, we introduce the definition and some properties of functional T-separatedness of crisp fuzzy subsets for a given T-neighbourhood space.

Definition 4.1. Let (X,Σ) be a T-neighbourhood space. For all nonempty subsets H, M of X, let $\Re(H,M) = \Re_\Sigma(H,M)$ be the following set of functions:

$\Re(H,M) = \{f : (X,\Sigma) \to (\mathcal{D}^+, \tau(\Im)) : f$ is continuous, $f(H) = \varepsilon_0$ and f is constant on $M\}$.

(This set is nonempty, as it contains the constant function ε_0).

We define a function $\Gamma = \Gamma_\Sigma$: $2^X \times 2^X \to I$ by

$$\Gamma(\mathbf{1}_H,\mathbf{1}_M) = \sup_{f\in\Re(H,M)} [1 - f(M)(0+)], \quad H, M \in 2^X. \tag{14}$$

and

$$\Gamma(\mathbf{1}_H,\mathbf{1}_\emptyset) = \Gamma(\mathbf{1}_\emptyset,\mathbf{1}_H) = 1, \quad H \in 2^X. \tag{15}$$

The function Γ is said to be functional T-separatedness and the real number $\Gamma(\mathbf{1}_H,\mathbf{1}_M)$ is called the degree of functional T-separatedness of H and M in (X,Σ).

In the following theorem we compile those properties of the function Γ which we shall need in the next section.

Theorem 4.1. *Let (X,Σ) be a T-neighbourhood space. Then for all $H, M, N \in 2^X$, we have*

(FTS1) *$\Gamma(\mathbf{1}_H,\mathbf{1}_M) = \Gamma(\mathbf{1}_M,\mathbf{1}_H)$;*
(FTS2) *if $H \subseteq M$, then $\Gamma(\mathbf{1}_H,\mathbf{1}_N) \geqslant \Gamma(\mathbf{1}_M,\mathbf{1}_N)$;*
(FTS3) *$\Gamma(\mathbf{1}_{(H\cup M)},\mathbf{1}_N) = \Gamma(\mathbf{1}_H,\mathbf{1}_N) \bigwedge \Gamma(\mathbf{1}_M,\mathbf{1}_N)$;*
(FTS4) *If $\Gamma(\mathbf{1}_H,\mathbf{1}_M) > 1 - (\theta T \beta)$ for some $\theta, \beta \in I_0$, there is $C \in 2^X$ such that $\Gamma(\mathbf{1}_H,\mathbf{1}_C) > 1 - \theta$ and $\Gamma(\mathbf{1}_{(X-C)},\mathbf{1}_M) > 1 - \beta$;*
(TP5) *If $H \cap M \neq \emptyset$, then $\Gamma(\mathbf{1}_H,\mathbf{1}_M) = 0$;*

Proof 6. These are easily seen to hold whenever one of the entering sets is empty. So, suppose that H, M and N are nonempty subsets of X, then

(FTS1) For all $f \in \Re(H,M)$, define $g_f : X \to \mathcal{D}^+$, by $g_f(x) = \Im(f(x),f(M)), \quad x \in X.$

Then for all $x \in H$, we get

$$\begin{aligned}g_f(x) &= \Im(\varepsilon_0,f(M)) \\ &= \Im(f(M),\varepsilon_0), \quad \text{by (TPM2)} \\ &= \Im f(M), \quad \text{by (12)}\end{aligned}$$

that is, $g_f(H) = f(M)$.

Since f is constant on M, then, for all $y \in M$, we have $g_f(y) = \Im(f(y), f(M)) = \varepsilon_0$.

Moreover, g_f equals the composite function $\Im o(f \times (f(M)))$, where $f(M)$ is constant ddf. and $\times$ restricted cartesian product of functions. Since these three functions are continuous (cf. [3]), we conclude that g_f is also continuous, therefore g_f is in $\Re(M, H)$. Consequently,

$$\begin{aligned}\Gamma(\mathbf{1}_H, \mathbf{1}_M) &= \sup_{f \in \Re(H,M)} [1 - f(M)(0+)] \\ &= \sup_{f \in \Re(H,M)} [1 - g_f(H)(0+)] \\ &\leqslant \sup_{g \in \Re(M,H)} [1 - g(H)(0+)] \\ &= \Gamma(\mathbf{1}_M, \mathbf{1}_H).\end{aligned}$$

Hence equality holds by interchanging H and M.

(FTS2) if $H \subseteq M$, then evidently $\Re(H, N) \supseteq \Re(M, N)$ and so $\Gamma(\mathbf{1}_H, \mathbf{1}_N) \geqslant \Gamma(\mathbf{1}_M, \mathbf{1}_N)$.

(FTS3) For all $f \in \Re(H, N)$ and $g \in \Re(M, N)$, $f \sqcap g$ is ε_0 on $H \cup M$ and is constant on N. It is also continuous (Proposition 3.1). Therefore, $f \sqcap g$ is also in $\Re(H \cup M, N)$. Hence we obtain

$$\begin{aligned}\Gamma(\mathbf{1}_{(H\cup M)}, \mathbf{1}_N) &= \sup_{h \in \Re(H\cup M,N)} [1 - h(N)(0+)] \\ &\geqslant \sup_{(f,g) \in \Re(H,N)\times\Re(M,N)} [1 - (f \sqcap g)(N)(0+)] \\ &= \sup_{(f,g) \in \Re(H,N)\times\Re(M,N)} \{1 - [f(N)(0+) \bigvee g(N)(0+)]\} \\ &= \sup_{(f,g) \in \Re(H,N)\times\Re(M,N)} \{[1 - f(N)(0+)] \bigwedge [1 - g(N)(0+)]\} \\ &= \{\sup_{f \in \Re(H,N)} [1 - f(N)(0+)]\} \bigwedge \{\sup_{g \in \Re(M,N)} [1 - g(N)(0+)]\} \\ &= \Gamma(\mathbf{1}_H, \mathbf{1}_N) \bigwedge \Gamma(\mathbf{1}_M, \mathbf{1}_N).\end{aligned}$$

The opposite inequality follows from (FTS2). Which renders (FTS3).

(FTS4) Suppose that $\Gamma(\mathbf{1}_H, \mathbf{1}_M) > 1 - (\theta T\ \beta)$ for some $\theta, \beta \in I_0$, then there are $\alpha, \gamma \in I_0$ and $f_1 \in \Re(H, M)$ such that $[1 - f_1(M)(0+)] = \alpha > \gamma > 1 - (\theta T \beta)$.

Let $\zeta \in \mathcal{D}^+$ be the ddf defined by

$$\zeta(s) = \begin{cases} 0, & s = 0 \\ 1 - \gamma, & 0 < s \leqslant 1 \\ 1, & s > 1. \end{cases}$$

Define $f : X \to \mathcal{D}^+$ by, $f(x) = f_1(x) \sqcap \zeta$, $\forall\ x \in X$. Then $f(H) = \varepsilon_0$, $f(M) = \zeta$ and f is continuous, by Proposition 3.1.

Take C = {x ∈ X: $f(x)(1) \leqslant 1 - \alpha$}, and let $\eta \in \mathcal{D}^+$ be the ddf

$$\eta(s) = \begin{cases} 0, & s = 0 \\ 1 - \alpha, & 0 < s \leqslant 1 \\ 1, & s > 1. \end{cases}$$

Define $h : X \to \mathcal{D}^+$ by, $h(x) = f(x) \sqcap \eta$, $\forall\ x \in X$. Then $h(H) = \varepsilon_0$, f is continuous and for all $x \in$ C, we have, at $s \in]0, 1]$:

$$\begin{aligned}(h(x))(s) &= (f(x) \sqcap \eta)(s) \\ &= (f(x))(s) \bigvee \eta(s) \\ &\geqslant \eta(s) \\ &= 1 - \alpha \\ &= (f(x))(1) \bigvee (1 - \alpha) \\ &\geqslant (f(x))(s) \bigvee \eta(s), \text{ because } f(x) \text{ is isotone} \\ &= (h(x))(s).\end{aligned}$$

Moreover, at $s > 1$:

$(h(x))(s) = (f(x) \sqcap \eta)(s) = (f(x))(s) \bigvee \eta\ \ (s) = (f(x))(s) \bigvee 1 = 1 = \eta(s)$. Also, $(h(x))(0) = (f(x) \sqcap \eta)(0) = f(x)(0) \bigvee \eta(0) = (f_1(x) \sqcap \zeta)(0) \bigvee \eta(0) = f_1(x)(0) \bigvee \zeta(0) \bigvee \eta(0) = 0$. This proves that $h(C) = \eta$, which completes the proof that h is in $\Re(H, C)$. Consequently,

$$\begin{aligned}\Gamma(\mathbf{1}_H, \mathbf{1}_C) &\geqslant 1 - h(C)(0+) = 1 - \eta(0+) = \alpha > 1 - (\theta T \beta) \\ &\geqslant 1 - \theta;\end{aligned}$$

which establishes one half of (FTS4).

Now, define a function $g : X \to \mathcal{D}^+$ by, $g(x) = \Im(\eta, \eta \sqcup \Im(\zeta, \zeta \sqcap \frac{1}{2} f(x)), x \in X$.

We have g is continuous because f, $\sqcup$, $\sqcap$ and $\Im$ are continuous with respect to τ and $\tau(\Im)$.

We need the following identities, which easily follow from definitions of η, ζ and $\Im$:

$$\Im(\zeta, \eta \oplus_T \zeta) = \eta \tag{16}$$

$$\Im\left(\zeta, \frac{1}{2}\zeta\right) = \zeta \tag{17}$$

$$\Im(\eta, \zeta) = \zeta \tag{18}$$

Thus for every $y \in M$, we get

$$\begin{aligned}g(y) &= \Im(\eta, \eta \sqcup \Im\left(\zeta, \zeta \sqcap \frac{1}{2}\zeta\right) \\ &= \Im(\eta, \eta \sqcup \Im\left(\zeta, \frac{1}{2}\zeta\right), \text{ because } \frac{1}{2}\zeta \preceq \zeta \\ &= \Im(\eta, \eta \sqcup \zeta), \quad \text{by (17)} \\ &= \Im(\eta, \zeta), \quad \text{because } \eta \preceq \zeta \\ &= \zeta, \quad \text{by (18)}\end{aligned}$$

That is, $g(M) = \zeta$.

Now, for all $s \geqslant 0$, we have

$$(\eta \oplus_T \zeta)(s) = \begin{cases} 0, & s = 0 \\ (1 - \alpha) T (1 - \gamma), & 0 < s \leqslant 1 \\ 1 - \alpha, & 1 < s \leqslant 2 \\ 1, & s > 2. \end{cases}$$

Thus,

$$\zeta \preceq \eta \oplus_T \zeta. \tag{19}$$

Hence, for all $x \in X - C$, we get

$$\begin{aligned}\varepsilon_0 \preceq g(x) &= \Im\left(\eta, \eta \sqcup \Im\left(\zeta, \zeta \sqcap \frac{1}{2} f(x)\right)\right)\\ &\preceq \Im(\eta, \eta \sqcup \Im(\zeta, \eta \oplus_T \zeta)), \quad \text{by (19)and definition of } \Im\\ &= \Im(\eta, \eta \sqcup \eta), \quad \text{by (16)}\\ &= \Im(\eta, \eta) = \varepsilon_0\end{aligned}$$

that is, $g(X - C) = \varepsilon_0$.

Which completes the proof that g is in $\Re(X - C, M)$. In consequence,

$\Gamma(\mathbf{1}_{(X-C)}, \mathbf{1}_M) \geqslant 1 - g(M)(0+) = 1 - \zeta(0+) = \gamma > 1 - (\theta T\beta) \geqslant 1 - \beta$,

which establishes the other half of (FTS4).

(FTS5) If $H \cap M \neq \emptyset$, then evidently, every f in $\Re(H, M)$ must be equal to ε_0 on M. Hence,

$\Gamma(\mathbf{1}_H, \mathbf{1}_M) = \sup_{f \in \Re(H,M)}[1 - f(M)(0+)] = 1 - \varepsilon_0(0+) = 1 - 1 = 0$.

Which completes the proof. □

Lemma 4.1. *If $(X, ^-)$ is a T-neighbourhood space, then for all $\mu \in I^X$ and $H \subseteq X$, we have $\mu\ T\ (\mathbf{1}_H)^- \leqslant (\mu\ T\ \mathbf{1}_H)^-$.*

Proof 7. Let $\mu \in I^X$ and $H \subseteq X$. Then for every $x \in X$, we have

$$\begin{aligned}[\mu T(\mathbf{1}_H)^-](x) &= \mu(x) T(\mathbf{1}_H)^-(x)\\ &= \mu(x) T \inf_{v \in \Sigma(x)} \text{hgt}(\mathbf{1}_H T v)\\ &= \inf_{v \in \Sigma(x)} \text{hgt}[\underline{\mu(x)} T \mathbf{1}_H T v], \quad \text{by continuity and isotonicity of } T\\ &\leqslant \inf_{v \in \Sigma(x)} \text{hgt}[(\mu T \mathbf{1}_H) T v]\\ &= (\mu T \mathbf{1}_H)^-(x).\end{aligned}$$

That is, $\mu T(\mathbf{1}_H)^- \leqslant (\mu T \mathbf{1}_H)^-$. □

Example 4.1. Let $(X, ^-) = (X, \Sigma)$ be a T-neighbourhood space and define a function Γ: $2^X \times 2^X \to I$ by

$\Gamma(\mathbf{1}_H, \mathbf{1}_M) = 1 - hgt[(\mathbf{1}_H)^- T(\mathbf{1}_M)^-], \quad H, M \in 2^X$.

It is easy to verify that the function Γ is a functional T-separatedness, it is enough to check (FTS3) and (FTS4) of Theorem 4.1, since the other axioms are trivially hold.

(FTS3) Let $H, M, N \in 2^X$. Then

$$\begin{aligned}\Gamma(\mathbf{1}_{(H\cup M)}, \mathbf{1}_N) &= 1 - \text{hgt}[(\mathbf{1}_{(H\cup M)})^- T(\mathbf{1}_N)^-]\\ &= 1 - hgt\{[(\mathbf{1}_H)^- \bigvee (\mathbf{1}_M)^-] T(\mathbf{1}_N)^-\}\\ &= 1 - \text{hgt}\{[(\mathbf{1}_H)^- T(\mathbf{1}_N)^-] \bigvee [(\mathbf{1}_M)^- T(\mathbf{1}_N)^-]\}\\ &= 1 - \{\text{hgt}[(\mathbf{1}_H)^- T(\mathbf{1}_N)^-] \bigvee \text{hgt}[(\mathbf{1}_M)^- T(\mathbf{1}_N)^-]\}\\ &= \{1 - \text{hgt}[(\mathbf{1}_H)^- T(\mathbf{1}_N)^-]\} \bigwedge \{1 - \text{hgt}[(\mathbf{1}_M)^- T(\mathbf{1}_N)^-]\}\\ &= \Gamma(\mathbf{1}_H, \mathbf{1}_N) \bigwedge \Gamma(\mathbf{1}_M, \mathbf{1}_N).\end{aligned}$$

(FTS4) Let $H, M \in 2^X$, with $\Gamma(\mathbf{1}_H, \mathbf{1}_M) > 1 - (\theta T\beta)$ for some $\theta, \beta \in I_0$. Then hgt $[(\mathbf{1}_H)^-\ T\ (\mathbf{1}_M)^-] < \theta T\beta$. So, there are $\theta_1, \beta_1 \in I$, such that $\theta_1 < \theta$ and $\beta_1 < \beta$, for which hgt $[(\mathbf{1}_H)^-\ T\ (\mathbf{1}_M)^-] < \theta_1 T\beta_1$, hence

$$\begin{aligned}\emptyset &= [(\mathbf{1}_H)^- T(\mathbf{1}_M)^-]_{(\theta_1 T\beta_1)*}\\ &= \bigcup_{\alpha T\gamma \geqslant \theta_1 T\beta_1} \{[(\mathbf{1}_H)^-]_{\alpha*} \bigcap [(\mathbf{1}_M)^-]_{\gamma*}\}, \quad \text{by[4, Lemma 1.2]}\\ &\supseteq [(\mathbf{1}_H)^-]_{\theta_1*} \bigcap [(\mathbf{1}_M)^-]_{\beta_1*}.\end{aligned}$$

By taking $C = [(\mathbf{1}_M)^-]_{\beta_1*} \in 2^X$, we have

$$\begin{aligned}\Gamma(\mathbf{1}_H, \mathbf{1}_C) &= 1 - \text{hgt}[(\mathbf{1}_H)^- T(\mathbf{1}_C)^-]\\ &= 1 - \text{hgt}\{(\mathbf{1}_H)^- T[((\mathbf{1}_M)^-)_{\beta_1*}]^-\}\\ &\geqslant 1 - \text{hgt}\{(\mathbf{1}_H)^- T[\mathbf{1}_X - ((\mathbf{1}_H)^-)_{\theta_1*}]^-\}\\ &\geqslant 1 - \text{hgt}\{(\mathbf{1}_H)^- T[\mathbf{1}_X - (\mathbf{1}_H)^-]_{\theta_1*}\}^-, \quad \text{by Lemma 4.1}\\ &\geqslant 1 - \text{hgt}(\underline{\theta_1})^-\\ &= 1 - \theta_1\\ &> 1 - \theta,\end{aligned}$$

and

$$\begin{aligned}\Gamma(\mathbf{1}_{(X-C)}, \mathbf{1}_M) &= 1 - \text{hgt}[(\mathbf{1}_{X-C})^- T(\mathbf{1}_M)^-]\\ &= 1 - \text{hgt}\{[\mathbf{1}_X - ((\mathbf{1}_M)^-)_{\beta_1*}]^- T(\mathbf{1}_M)^-\}\\ &\geqslant 1 - \text{hgt}\{[\mathbf{1}_X - ((\mathbf{1}_M)^-)_{\beta_1*}]^- T(\mathbf{1}_M)^-\}^-, \quad \text{by Lemma 4.1 again}\\ &\geqslant 1 - \text{hgt}(\underline{\beta_1})^-\\ &= 1 - \beta_1\\ &> 1 - \beta.\end{aligned}$$

5. *T*-proximity induced by *T*-neighbourhood structure

In this section, we show that every T-neighbourhood space generates a T-proximity space, also, we introduce the notion of Čech T-proximity space. In [1], we have seen that every T-uniformity Ω on a set X, induces a T-proximity $\delta(\Omega)$, we prove that, the I-topologies generated by the two structures Ω and $\delta(\Omega)$ are coincide.

Theorem 5.1. *Let (X, Σ) be a T-neighbourhood space and define δ_Σ: $2^X \times 2^X \to I$ by*

$$\delta_\Sigma(\mathbf{1}_H, \mathbf{1}_M) = 1 - \Gamma_\Sigma(\mathbf{1}_H, \mathbf{1}_M), \quad M, H \in 2^X. \tag{20}$$

Then δ_Σ is a T-proximity on X, also $\tau(\delta_\Sigma) \subseteq \Sigma$ and equality holds if and only if (X, Σ) is T-completely regular.

Proof 8. From definition of Γ_Σ, we get δ_Σ satisfies (TP1), and the other axioms follows immediately from properties of Γ_Σ established in Theorem 4.1. Therefore, δ_Σ is a T-proximity on X.

Now, let $M \in 2^X$, $x \in X$ and denote the fuzzy closure operators associated with Σ, $\tau(\delta_\Sigma)$ and $\tau(\Im)$ respectively by $^{-1}$, $^{-2}$, $^{-3}$. Then, we have

$$\begin{aligned}[(\mathbf{1}_M)^{-2}](x) &= \delta_\Sigma(\mathbf{1}_M,\ \mathbf{1}_x), \quad \text{by (4)}\\ &= 1 - \Gamma_\Sigma(\mathbf{1}_M, \mathbf{1}_x)\\ &= 1 - \{\sup_{f \in \Re(M,x)} [1 - (f(x))(0+)]\}\\ &= \inf_{f \in \Re(M,x)} (f(x))(0+)\\ &= \inf_{f \in \Re(M,x)} ([f(\mathbf{1}_M)]^{-3})(f(x)), \quad \text{by Proposition 3.2}\\ &\geqslant \inf_{f \in \Re(M,x)} [f((\mathbf{1}_M)^{-1})](f(x)), \quad \text{by continuity of } f\\ &= \inf_{f \in \Re(M,x)} [f^{\leftarrow}(f((\mathbf{1}_M)^{-1}))](x)\\ &\geqslant [(\mathbf{1}_M)^{-1}](x), \quad \text{clear}\end{aligned}$$

Which yields,

$$(\mathbf{1}_M)^{-2} \geqslant (\mathbf{1}_M)^{-1}, \quad \forall\ M \in 2^X. \tag{21}$$

This establishes (cf. [5. Corollary 2.1]) that, $\tau(\delta_\Sigma)$ is coarser than Σ.

On the other hand, if (X,Σ) is T-completely regular, then by Theorem 3.2, we get for every $M \in 2^X$, $x \in X$ and $\theta \in I_0$, there is a continuous function

$g : (X, ^-) \to (\mathcal{D}^+, \tau(\mathfrak{I}))$, for which $g(M) = \varepsilon_0$ and $(g(x))(0+) < (\mathbf{1}_M)^-(x) + \theta$, (that is $g \in \mathfrak{R}(M,x)$). Consequently

$$\begin{aligned}
[(\mathbf{1}_M)^{-2}](x) &= \delta_\Sigma(\mathbf{1}_M, \mathbf{1}_x)\\
&= 1 - \Gamma_\Sigma(\mathbf{1}_M, \mathbf{1}_x)\\
&= 1 - \{\sup_{f\in\mathfrak{R}(M,x)} [1 - (f(x))(0+)]\}\\
&= \inf_{f\in\mathfrak{R}(M,x)} (f(x))(0+)\\
&\leqslant (g(x))(0+)\\
&\leqslant [(\mathbf{1}_M)^{-1}](x) + \theta.
\end{aligned}$$

This yields,

$$(\mathbf{1}_M)^{-2} \leqslant (\mathbf{1}_M)^{-1}, \quad \forall\, M \in 2^X.$$

This establishes the opposite inequality (21), which renders $\tau(\delta_\Sigma) = \Sigma$.

Conversely, if $\tau(\delta_\Sigma) = \Sigma$ then Σ is T-proximizable, and hence T-completely regular.

As in [2], since the I-topological space $(X, \tau(\delta))$ induced by the T-proximity space (X,δ) is a T-neighbourhood space, then from this fact together with Theorem 5.1, we have there is a one to one corresponding between T-proximity and T-neighbourhood structures. □

Definition 5.1. If the T-neighbourhood space (X,Σ) is a T-completely regular, then the T-proximity δ_Σ on X, defined by (20), is called Čech T-proximity of (X,Σ).

To justify this terminology, we proceed to establish a maximality property for Čech T-proximities.

Theorem 5.2. *The Čech T-proximity δ_Σ, of a T-completely regular T-neighbourhood space (X,Σ), is the finest T-proximity on X that induces Σ.*

Proof 9. By Theorem 5.1, we have δ_Σ induces Σ. Now, let δ be another T-proximity on X that induces Σ. For all nonempty subsets H, M of X, and all $\alpha > \delta(\mathbf{1}_H, \mathbf{1}_M)$, there is, by Proposition 3.5, a function $f \in \mathfrak{R}(H,M)$ with $(f(M))$ (1) $= \alpha$. Consequently, $\delta_\Sigma(\mathbf{1}_H,\mathbf{1}_M) = 1 - \Gamma_\Sigma(\mathbf{1}_H,\mathbf{1}_M) \leqslant (f(M))(0+) \leqslant (f(M))$ (1) $= \alpha$.

This establishes $\delta(\mathbf{1}_H,\mathbf{1}_M) \geqslant \delta_\Sigma(\mathbf{1}_H,\mathbf{1}_M)$, which proves that δ is coarser than δ_Σ. □

Theorem 5.3. *Let f: $(X,\delta) \to (Y,\rho)$ be a proximally continuous function. Then it is continuous with respect to the I-topologies generated by δ and ρ, respectively.*

Proof 10. We denote the fuzzy closure operators associated with $\tau(\delta)$ and $\tau(\rho)$ respectively by $^{-1}$, $^{-2}$. Then, for every $\lambda \in I^X$ and all $y \in Y$, we have

$$\begin{aligned}
[f(\lambda^{-1})](y) &= \sup_{x\in f^{\leftarrow}(y)} (\lambda^{-1})(x)\\
&= \sup_{x\in f^{\leftarrow}(y)} \delta(\lambda, \mathbf{1}_x), \quad \text{by (4)}\\
&\leqslant \sup_{x\in f^{\leftarrow}(y)} \rho(f(\lambda), f(\mathbf{1}_x)), \quad \text{by hypothesis}\\
&= \sup_{x\in f^{\leftarrow}(y)} \rho(f(\lambda), \mathbf{1}_{f(x)})\\
&= \sup_{x\in f^{\leftarrow}(y)} [f(\lambda)]^{-2}(f(x))\\
&= [f(\lambda)]^{-2}(y),
\end{aligned}$$

that is, $f(\lambda^{-1}) \leqslant [f(\lambda)]^{-2}$

Which proves the continuity of f: $(X, \tau(\delta)) \to (Y, \tau(\rho))$. □

Proposition 5.1. *If the function f: $(X,\Sigma) \to (Y,\Sigma')$ between T-neighbourhood spaces is continuous, then it is a proximally continuous from (X,δ_Σ) to $(Y,\delta_{\Sigma'})$. The converse holds when its codomain (Y,Σ') is T-completely regular.*

Proof 11. For all nonempty $H, M \in 2^X$ and all $g \in \mathfrak{R}_{\Sigma'}(f(H),(f(M))$, the composite function g o f is in $\mathfrak{R}_\Sigma$(H, M). This entails that

$$\begin{aligned}
\delta_\Sigma(\mathbf{1}_H,\mathbf{1}_M) &= 1 - \Gamma_\Sigma(\mathbf{1}_H,\mathbf{1}_M)\\
&\leqslant 1 - \Gamma_{\Sigma'}(\mathbf{1}_{f(H)},\mathbf{1}_{f(M)})\\
&= \delta_{\Sigma'}(\mathbf{1}_{f(H)},\mathbf{1}_{f(M)})\\
&= \delta_{\Sigma'}(f(\mathbf{1}_H), f(\mathbf{1}_M)).
\end{aligned}$$

Hence, by (2), we have f is a proximally continuous with respect to δ_Σ and $\delta_{\Sigma'}$.

Conversely, suppose that $h : (X,\delta_\Sigma) \to (Y,\delta_{\Sigma'})$ is a proximally continuous, then, by Theorem 5.3, we get $h : (X,\tau(\delta_\Sigma)) \to (Y,\tau(\delta_{\Sigma'}))$ is continuous.

But from Theorem 5.1, we have $\tau(\delta_\Sigma) \subseteq \Sigma$ and $\tau(\delta_{\Sigma'}) = \Sigma'$, consequently,

h is also continuous: $(X,\Sigma) \to (Y,\Sigma')$.

Now, we define a function $\delta^\sim$, from category of T-neighbourhood spaces and continuous functions to category of T-proximity spaces and proximally continuous functions, as:

On object (X,Σ) in T-NS, by $\delta^\sim(X,\Sigma) = (X,\delta_\Sigma)$ an objects in T-PS. On morphisms, $\delta^\sim$ is the identity function. Then an obvious conclusion from the above theorems is that these $\delta^\sim$ is well defined functor. □

Proposition 5.2. [2]*Let (X,Ω) be a T-uniform space. Then the fuzzy closure operator of the T-neighbourhood space $(X,\tau(\Omega))$ is given by:*

$$\mu^- = \inf_{\psi\in\Omega} \psi\langle\mu\rangle_T, \quad \mu \in I^X.$$

Theorem 5.4. *If Ω is a T-uniformity on a set X, and $\delta(\Omega)$ is the T-proximity induced by the T-uniformity Ω, then the I-topology $\tau(\Omega)$ coincide with $\tau(\delta(\Omega))$.*

Proof 12. Let $\mu \in I^X$ and $x \in X$, and denote the fuzzy closure operators associated with $\tau(\Omega)$ and $\tau(\delta(\Omega))$ respectively by $^{-1}$, $^{-2}$. Then, we have

$$
\begin{aligned}
\mu^{-2}(x) &= (\delta(\Omega))(\mu, \mathbf{1}_x), \quad \text{by (4)} \\
&= (\delta(\Omega))(\mathbf{1}_x, \mu), \quad \text{by (TP2)} \\
&= \inf_{\psi \in \Omega} \sup_{y \in X} (\mathbf{1}_x, T\psi\langle\mu\rangle_T)(y), \quad \text{by (3)} \\
&= \inf_{\psi \in \Omega} (\psi\langle\mu\rangle_T)(x) \\
&= \mu^{-1}(x), \quad \text{by Proposition 5.2}
\end{aligned}
$$

This proves our assertion. □

Proposition 5.3 1. *If Ω_δ is the T-uniformity induced by a T-proximity δ on a set X, then $\delta(\Omega_\delta) = \delta$. By combining Theorem* 5.4 *and Proposition* 5.3*, we arrive to the fact that, a T-proximizability is equivalent to T-uniformizability. Hence, from Theorem* 3.2*, we get a T-neighbourhood space is T-proximizable (i.e., induced by a T-proximity) if and only if it is T-completely regular.*

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